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ON DERIVATIONS OF SUBTRACTION ALGEBRAS

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Abstract

The aim of this paper is to introduce the notion of derivations of subtraction algebras. We define a derivation of a subtraction algebra X as a function d on X satisfying $d(x - y) = (d(x) - y) \wedge (x - d(y))$ for all $x, y \in X$. Then it is characterized as a function d satisfying $d(x - y) = d(x) - y$ for all $x, y \in X$. Also we define a simple derivation as a function d_a on X satisfying $d_a(x) = x - a$ for all $x \in X$. Then every simple derivation is a derivation and every derivation can be partially a simple derivation on intervals. For any derivation d of a subtraction algebra X, $\mathrm{Ker}(d)$ and $\mathrm{Im}(d)$ are ideals of X, and $X/\mathrm{Ker}(d) \cong \mathrm{Im}(d)$ and $X/\mathrm{Im}(d) \cong \mathrm{Ker}(d)$. Finally, we show that every subtraction algebra X is embedded in $\mathrm{Im}(d) \times \mathrm{Ker}(d)$ for any derivation d of X.

Keywords: Subtraction algebra, Derivation, Simple derivation, Non-expansive map, Dual closure operator, Boolean algebra.

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1. Introduction

B. M. Schein [2] considered systems of the form $(\Phi; \circ, \setminus)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and set theoretic subtraction "$\setminus$" (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [1]. He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [4] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called atomic subtraction algebras. The notion of derivation of lattices was introduced and studied in [3].

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In this paper, we define a derivation of a subtraction algebra and introduce the notion of derivations. In Section 2, we introduce some basic results of subtraction algebras. In Section 3, we define a derivation as a function d on X satisfying $d(x-y) = (d(x)-y)\wedge(x-d(y))$ for all $x, y \in X$, and characterize it as a function d satisfying $d(x-y) = d(x) - y$ for all $x, y \in X$. Also we define a simple derivation as a function d_a on X satisfying $d_a(x) = x - a$ for all $x \in X$, and we show that every simple derivation is a derivation and conversely, every derivation is partially a simple derivation on intervals. In Section 4 we show that for any derivation d of a subtraction algebra X, $\mathrm{Ker}(d)$ and $\mathrm{Im}(d)$ are ideals of X and $X/\mathrm{Ker}(d) \cong \mathrm{Im}(d)$ and $X/\mathrm{Im}(d) \cong \mathrm{Ker}(d)$. Also the map $\mu : x \mapsto x - d(x)$ is a derivation of X, hence the sequence of derivations and subtraction algebras :

$$0 \longrightarrow \mathrm{Im}(d) \xrightarrow{i} X \xrightarrow{\mu^\circ} \mathrm{Ker}(d) \longrightarrow 0$$

is similar to a split exact sequence. Finally, we show that every subtraction algebra X is embedded in $\mathrm{Im}(d) \times \mathrm{Ker}(d)$ for any derivation d of X.

2. Subtraction algebras

We first recall some basic concepts which are used to present the paper.

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \le b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \le)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true:

(p1) $(x - y) - y = x - y$.
(p2) $x - 0 = x$ and $0 - x = 0$.
(p3) $x - y \le x$.
(p4) $x - (x - y) \le y$.
(p5) $(x - y) - (y - x) = x - y$.
(p6) $x - (x - (x - y)) = x - y$.
(p7) $(x - y) - (z - y) \le x - z$.
(p8) $x \le y$ if and only if $x = y - w$ for some $w \in X$.
(p9) $x \le y$ implies $x - z \le y - z$ and $z - y \le z - x$ for all $z \in X$.
(p10) $x, y \le z$ implies $x - y = x \wedge (z - y)$.
(p11) $(x \wedge y) - (x \wedge z) \le x \wedge (y - z)$.
(p12) $(x - y) - z = (x - z) - (y - z)$.

Let X and Y be subtraction algebras. A mapping f from X to Y is called a *homomorphism* if $f(x - y) = f(x) - f(y)$ for all $x, y \in X$. Especially, f is *monomorphism* (resp. *epimorphism*) if f is one-to-one (resp. onto) homomorphism, and f is an *isomorphism* if

f is a monomorphism and epimorphism. In this case, we say X is isomorphic to Y, and denote this by $X \cong Y$.

A function f of a semilattice ($\wedge$-semilattice) X into itself is a *dual closure* if f is monotone, non-expansive (i.e., $f(x) \leq x$ for all $x \in X$) and idempotent(i.e., $f \circ f = f$),

3. Derivations and simple derivations

3.1. Definition. Let X be a subtraction algebra. By a *derivation* of X we mean a self-map d of X satisfying the identity $d(x - y) = (d(x) - y) \wedge (x - d(y))$ for all $x, y \in X$.

3.2. Example. (1) Let $X = \{0, a, b, 1\}$ in which "$-$" is defined by

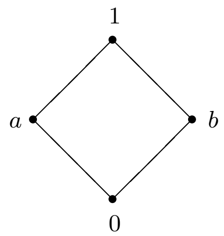
$-$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

It is easy to check that $(X; -)$ is a subtraction algebra. Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 0 & \text{if } x = 0,\ a, \\ b & \text{if } x = b,\ 1. \end{cases}$$

Then d is a derivation of the subtraction algebra X.

Figure 1. The Hasse diagram of Example 3.2 (1)



(2) Let $X = \{0, a, b\}$ be a subtraction algebra with the following Cayley table

$-$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 0 & \text{if } x = 0,\ b, \\ b & \text{if } x = a. \end{cases}$$

Then it is easily checked that d is a derivation of subtraction algebra X.

3.3. Example. Let X be a subtraction algebra. We define a function d by $d(x) = 0$ for all $x \in X$. Then d is a derivation on X, which is called the *zero derivation.*

3.4. Example. Let d be the identity function on a subtraction algebra X. Then d is a derivation on X, which is called the *identity derivation.*

3.5. Proposition. *Let d be a derivation of a subtraction algebra X. Then $d(0) = 0$.*

Proof. Let d be a derivation of a subtraction algebra of X. Then

$$d(0) = d(0 - x) = (d(0) - x) \wedge (0 - d(x)) = (d(0) - x) \wedge 0 = 0.$$

□

3.6. Proposition. *Let d be a derivation of a subtraction algebra X. Then $d(x - d(x)) = 0$ for every $x \in X$.*

Proof. Let d be a derivation of a subtraction algebra of X and let $x \in X$. Then

$$d(x - d(x)) = (d(x) - d(x)) \wedge (x - d(d(x))) = 0 \wedge (x - d(d(x))) = 0.$$

□

3.7. Proposition. *Let d be a derivation of a subtraction algebra X. Then we have $d(x) = d(x) \wedge x$.*

Proof. Let d be a derivation of X. Then

$$d(x) = d(x - 0) = (d(x) - 0) \wedge (x - d(0)) = d(x) \wedge (x - 0) = d(x) \wedge x.$$

□

3.8. Corollary. *Let d be a derivation of subtraction algebra X. Then we have $d(x) \leq x$. That is, d is a non-expansive map.* □

3.9. Theorem. *Let d be a derivation of a subtraction algebra X. If $x \leq y$ for $x, y \in X$, then $d(x) \leq d(y)$.*

Proof. Let $x \leq y$ for $x, y \in X$. Then by (p8), $x = y - w$ for some $w \in X$. Hence we have

$$d(x) = d(y - w) = (d(y) - w) \wedge (y - d(w)) \leq d(y) - w \leq d(y).$$

□

3.10. Theorem. *Let d be a derivation of a subtraction algebra X. Then we have $d^2 = d \circ d = d$.*

Proof. Let d be a derivation of X. Then by definition of the derivation d and Proposition 3.6, we have

$$\begin{aligned} d^2(x) &= d(d(x)) = d(x \wedge d(x)) \\ &= d(x - (x - d(x))) \\ &= (d(x) - (x - d(x))) \wedge (x - d(x - d(x))) \\ &= d(x) \wedge (x - 0) \\ &= d(x) \wedge x \\ &= d(x) \end{aligned}$$

□

3.11. Corollary. *Let d be a derivation of a subtraction algebra X. Then d is a dual closure operator on X.*

Proof. Clear from Corollary 3.8 and Theorems 3.9 and 3.10. □

3.12. Proposition. *Let f is a non-expansive map on a subtraction algebra X, i.e., $f(x) \leq x$ for all $x \in X$. Then $f(x) - y \leq x - f(y)$ for all $x, y \in X$.*

Proof. Suppose that f is a non-expansive map on X and $x, y \in X$. Then $f(x) \leq x$ and $f(y) \leq y$. Hence $f(x) - y \leq x - y$ and $x - y \leq x - f(y)$ by (p9). It follows that $f(x) - y \leq x - f(y)$. □

3.13. Theorem. *Let d be a map on a subtraction algebra X. Then the following are equivalent :*

(1) *d is a derivation of X;*
(2) *$d(x-y) = d(x) - y$ for all $x, y \in X$.*

Proof. Suppose that d is a derivation of X. Then d is non-expansive by Corollary 3.8. Hence for any $x, y \in X$, $d(x) - y \leq x - d(y)$ by Proposition 3.12, and

$$d(x-y) = (d(x) - y) \wedge (x - d(y)) = d(x) - y.$$

Suppose that d is a map satisfying $d(x - y) = d(x) - y$ for all $x, y \in X$. Then $d(0) = d(0 - d(0)) = d(0) - d(0) = 0$, hence we have

$$0 = d(0) = d(x - x) = d(x) - x$$

for any $x \in X$. It follows that $d(x) \leq x$ for any $x \in X$. That is, d is non-expansive. Hence by Proposition 3.12, $d(x) - y \leq x - d(y)$ and

$$d(x-y) = d(x) - y = (d(x) - y) \wedge (x - d(y))$$

for any $x, y \in X$. □

3.14. Theorem. *Let X be a subtraction algebra. The every derivation of X is an homomorphism.*

Proof. Suppose that d is a derivation of X and $x, y \in X$. Then $d(y) \leq y$. It implies

$$d(x-y) = d(x) - y \leq d(x) - d(y)$$

by (p9). Also we have

$$\begin{aligned}
&(d(x) - d(y)) - (d(x) - y) \\
&\quad = (dd(x) - d(y)) - (d(x) - y) \quad \text{(by Theorem 3.10)} \\
&\quad = (dd(x) - (d(x) - y)) - d(y) \quad \text{(by (S3))} \\
&\quad = d(d(x) - (d(x) - y)) - d(y) \quad \text{(by Theorem 3.13)} \\
&\quad = d(y - (y - d(x))) - d(y) \quad \text{(by (S2))} \\
&\quad \leq d(y) - d(y) \quad \text{(by (p3), Theorem 3.9 and (p9))} \\
&\quad = 0
\end{aligned}$$

It follows that $(d(x) - d(y)) - (d(x) - y) = 0$ and $d(x) - d(y) \leq d(x) - y = d(x - y)$. Hence $d(x) - d(y) = d(x - y)$. □

The converse of Theorem 3.14 is not true in general.

3.15. Example. Let $X = \{0, a, b, 1\}$ be the subtraction algebra of Example 3.2(1). Define a map $f : X \to X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0,\ a, \\ 1 & \text{if } x = b,\ 1. \end{cases}$$

Then f is a endomorphism of X which is not a derivation because of $f(b - a) = f(b) = 1 \neq b = 1 - a = f(b) - a$.

Let X be a subtraction algebra. Then, for each $a \in X$, we will define a map $d_a : X \to X$ by

$$d_a(x) = x - a$$

for all $x \in X$.

3.16. Proposition. *Let X be a subtraction algebra. Then for each $a \in X$, the map d_a is a derivation of X.*

Proof. Suppose that d_a is the map defined by $d_a(x) = x - a$ for each $x \in X$. Then for any $x, y \in X$, we have

$$d_a(x - y) = (x - y) - a = (x - a) - y = d_a(x) - y$$

by (S3). Hence d_a is a derivation of X by Theorem 3.13. □

We will call the derivation d_a of Proposition 3.16 a *simple derivation*.

3.17. Proposition. *Let d be a derivation of a subtraction algebra X. Then for each $x \in X$, there exists a unique $\hat{x} \in [0, x]$ such that $d(x) = x - \hat{x}$ and $d(\hat{x}) = 0$.*

Proof. Suppose that d is a derivation of X and $x \in X$. Then $d(x) \leq x$ since d is non-expansive.

Let $\hat{x} = x - d(x)$. Then $\hat{x} \in [0, x]$ and $d(\hat{x}) = 0$ by Proposition 3.6, and we have

$$x - \hat{x} = x - (x - d(x)) = x \wedge d(x) = d(x).$$

If $x - \hat{x} = d(x) = x - w'$ for some $w' \in [0, x]$, then

$$\begin{aligned}\hat{x} - w' &= (x \wedge \hat{x}) - w' \\ &= (x - (x - \hat{x})) - w' \\ &= (x - w') - (x - \hat{x}) \text{ (by (S3))} \\ &= 0.\end{aligned}$$

It follows that $\hat{x} \leq w'$. Similarly, we can show that $w' \leq \hat{x}$. Hence $\hat{x} = w'$, and $\hat{x}$ is the unique element in $[0, x]$ such that $d(x) = x - \hat{x}$. □

3.18. Lemma. *Let d be a derivation of a subtraction algebra X. Then* $\mathrm{Ker}(d) = \{\hat{x} \mid x \in X\}$.

Proof. It is clear that $\{\hat{x} \mid x \in X\} \subseteq \mathrm{Ker}(d)$ by Theorem 3.17.

If $x \in \mathrm{Ker}(d)$, then $x = x - 0 = x - d(x) = \hat{x}$. It implies $\mathrm{Ker}(d) \subseteq \{\hat{x} \mid x \in X\}$. □

3.19. Theorem. *Let d be a derivation of a subtraction algebra X. The for each interval $[0, a]$ in X,*

$$d(x) = d_{\hat{a}}(x)$$

for all $x \in [0, a]$, that is, the restriction $d|_{[0,a]} : [0, a] \to X$ of d is a simple derivation $d_{\hat{a}}$, where $\hat{a} \in [0, a]$ is the unique element of Theorem 3.17.

Proof. Suppose that d is a derivation of X and $a \in X$. Then by Theorem 3.17 there is a unique $\hat{a} \in [0, a]$ such that $d(a) = a - \hat{a}$, and for any $x \in [0, a]$ we have

$$\begin{aligned}d(x) &= d(a \wedge x) = d(a - (a - x)) = d(a) - (a - x) = (a - \hat{a}) - (a - x) \\ &= (a - (a - x)) - \hat{a} = (a \wedge x) - \hat{a} = x - \hat{a}.\end{aligned}$$

Hence $d(x) = x - \hat{a} = d_{\hat{a}}(x)$ for all $x \in [0, a]$. □

3.20. Corollary. *Let X be a subtraction algebra with greatest element* 1*. Then every derivation d of X is a simple derivation $d_{\hat{1}}$.*

Proof. Suppose that $1 \in X$ and d is a derivation of X. Then $X = [0,1]$ and by Theorem 3.19,

$$d(x) = x - \hat{1} = d_{\hat{1}}(x)$$

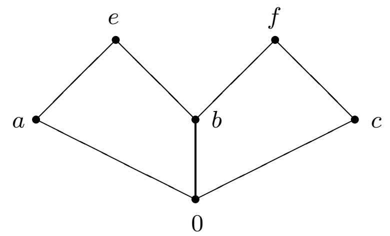
for all $x \in [0,1] = X$. Hence d is the simple derivation $d_{\hat{1}}$. □

There can be a derivation on a subtraction algebra which is not simple.

3.21. Example. Let $X = \{0, a, b, c, e, f\}$ be a subtraction algebra with "$-$" defined by

$-$	0	a	b	c	e	f
0	0	0	0	0	0	0
a	a	0	a	a	0	a
b	b	b	0	b	0	0
c	c	c	c	0	c	0
e	e	b	a	e	0	a
f	f	f	c	b	c	0

Figure 2. The Hasse diagram of Example 3.21



Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 0 & \text{if } x = 0,\ a,\ c, \\ b & \text{if } x = b, e, f. \end{cases}$$

Then d is a derivation of X which is not simple, because there is no $x \in X$ satisfying either $d(e) = b = e - x$ or $d(f) = b = f - x$. For the interval $A = [0, e]$ and $B = [0, f]$, $\hat{e} = e - d(e) = e - b = a$ and $\hat{f} = c$. Hence the restrictions $d|_A$ and $d|_B$ are simple, being given by

$$d|_A(x) = x - a = d(x) \quad (x \in A) \text{ and } d|_B(x) = x - c = d(x) \quad (x \in B),$$

respectively.

4. Derivations and ideals of subtraction algebras

A nonempty subset I of a subtraction algebra X is called an *ideal* of X if it satisfies

(I1) $0 \in I$,

(I2) for any $x, y \in X$, $y \in I$ and $x - y \in I$ implies $x \in I$.

For an ideal I of a subtraction algebra X, it is clear that $x \le y$ and $y \in I$ imply $x \in I$ for any $x, y \in X$.

4.1. Proposition. *Let d be a derivation of a subtraction algebra X. Then* $\mathrm{Ker}d = \{x \in X \mid d(x) = 0\}$ *is an ideal of X.*

Proof. Let $y \in \mathrm{Ker}d$ and $x \in X$ with $x - y \in \mathrm{Ker}d$. Then $d(y) = 0$ implies

$$d(x) = d(x) - 0 = d(x) - d(y) = d(x - y) = 0.$$

Hence $x \in \mathrm{Ker}d$. □

4.2. Proposition. *Let d be a derivation of a subtraction algebra X. If $\mathrm{Ker}d = \{0\}$, then d is the identity derivation.*

Proof. Let $x \in X$. Then $d(x) \leq x$, and $x - d(x) \in \mathrm{Ker}d = \{0\}$ by Proposition 3.6. It implies $x - d(x) = 0$ and $x \leq d(x)$. Hence $d(x) = x$. □

Let X be a subtraction algebra and A a non-empty subset of X. Then we will write $A^* = \{x \in X \mid x \wedge a = 0 \text{ for all } a \in A\}$.

4.3. Proposition. *Let X be a subtraction algebra and A non-empty subset of X. Then A^* is an ideal of X.*

Proof. Let $y \in A^*$ and $x - y \in A^*$ for any $x \in X$. Then $y \wedge a = 0$ and $(x - y) \wedge a = 0$ for all $a \in A$. By (p11), we have

$$x \wedge a = (x \wedge a) - 0 = (x \wedge a) - (y \wedge a) \leq (x - y) \wedge a = 0$$

for all $a \in A$. It implies $x \wedge a = 0$ for all $a \in A$, and $x \in A^*$. Hence A^* is an ideal of X. □

In particular, for any singleton subset $A = \{a\}$ of a subtraction algebra X, $\{a\}^* = A^* = \{x \in X \mid x \wedge a = 0\}$ is an ideal of X.

4.4. Proposition. *Let X be a subtraction algebra and d_y a simple derivation with $y \in X$. Then $d_y(x) = x$ if and only if $x \in \{y\}^*$.*

Proof. Suppose that $x, y \in X$ and $d_y(x) = x$. Then $x \wedge y = x - (x - y) = x - d_y(x) = x - x = 0$. Hence $x \in \{y\}^*$.

Conversely, suppose that $x \in \{y\}^*$. Then $y - (y - x) = x - (x - y) = x \wedge y = 0$. Hence we have

$$\begin{aligned} d_y(x) &= x - y \\ &= (x - y) - (y - x) \quad \text{(by (p5))} \\ &= (x - (y - x)) - (y - (y - x)) \quad \text{(by (p12))} \\ &= x - 0 \quad \text{(by (S1))} \\ &= x. \end{aligned}$$

□

4.5. Corollary. *Let X be a subtraction algebra and d_y a simple derivation with respect to $y \in X$. Then $d_y(X) = \{y\}^*$, that is, $\mathrm{Im}(d_y)$ is an ideal of X.*

Proof. Let $x \in d_y(X)$. Then $x = d_y(z)$ for some $z \in X$, and by Theorem 3.10

$$x = d_y(z) = d_y(d_y(z)) = d_y(x).$$

It implies $x \in \{y\}^*$ by Proposition 4.4. Hence $d_y(X) \subseteq \{y\}^*$. Also it is clear that $\{y\}^* \subseteq d_y(X)$ from Proposition 4.4. □

4.6. Proposition. *Let d be a derivation of a subtraction algebra X. If I is an ideal of X, then we have $d(I) \subseteq I$.*

Proof. For all $x \in I$, we have $d(x) \leq x$, and $d(x) = x - w$ for some $w \in X$ by (p8). Hence by the definition of an ideal, we have $d(x) \in I$. □

4.7. Theorem. *Let d be a derivation of a subtraction algebra X. Then $d(X) = \mathrm{Im}(d)$ is an ideal of X.*

Proof. Let $y \in d(X)$ and $x - y \in d(X)$ with $x \in X$. Then $d(y) = y$ and $d(x - y) = x - y$ by Theorem 3.10, there exists $\hat{x} \in [0, x]$ satisfying $d(x) = x - \hat{x}$ and $d(\hat{x}) = 0$, and $d_{\hat{x}}(z) = d(z)$ for all $z \in [0, x]$ by Theorems 3.17 and 3.19. Since $x - y \le x$, we have

$$d_{\hat{x}}(x - y) = d(x - y) = x - y.$$

It implies $x - y \in \{\hat{x}\}^*$ by Proposition 4.4, i.e., $(x - y) \wedge \hat{x} = 0$. Since $\hat{x} \le x$, we have

$$\begin{aligned}\hat{x} - y &= (x \wedge \hat{x}) - y \\ &= (x - (x - \hat{x})) - y \\ &= (x - y) - ((x - \hat{x}) - y) \quad \text{(by (p12))} \\ &= (x - y) - ((x - y) - \hat{x}) \\ &= (x - y) \wedge \hat{x} \\ &= 0.\end{aligned}$$

Hence $\hat{x} \le y$ and we have

$$\begin{aligned}\hat{x} &= y \wedge \hat{x} \\ &= y - (y - \hat{x}) \\ &= y - (d(y) - \hat{x}) \\ &= y - d(y - \hat{x}) \\ &= y - (d(y) - d(\hat{x})) \quad \text{(by Theorem 3.14)} \\ &= y - (d(y) - 0) \\ &= y - y = 0.\end{aligned}$$

It implies $x = x - 0 = x - \hat{x} = d(x) \in d(X)$, and so $d(X)$ is an ideal of X. □

Let X be a subtraction algebra and I an ideal of X. If $\sim_I$ is the binary relation on X given by

$$x \sim_I y \text{ if and only if } x - y \in I \text{ and } y - x \in I,$$

then $\sim_I$ is a congruence relation and the quotient set X/I is a subtraction algebra with the binary operation defined by

$$[x] - [y] = [x - y]$$

for all $[x], [y] \in X/I$, where $[x]$ is an equivalence class of x with respect to $\sim_I$.

4.8. Theorem. *Let d be a derivation of a subtraction algebra X. Then there exists a monomorphism $\bar{d} : X/\mathrm{Ker}(d) \to X$ such that $\bar{d}([x]) = d(x)$. Hence $X/\mathrm{Ker}(d)$ is isomorphic to $\mathrm{Im}(\bar{d}) = \mathrm{Im}(d)$.*

Proof. Suppose that d is a derivation on X. Then d is a homomorphism of X by Theorem 3.14.

Define a map $\bar{d} : X/\mathrm{Ker}(d) \to X$ by $\bar{d}([x]) = d(x)$ for all $[x] \in X/\mathrm{Ker}(d)$. If $[x] = [y]$, then $x \sim_{\mathrm{Ker}(d)} y$ implies $x - y, y - x \in \mathrm{Ker}(d)$. Hence we have

$$d(x) - d(y) = d(x - y) = 0 \text{ and } d(y) - d(x) = d(x - y) = 0.$$

It follow that $d(x) \le d(y)$ and $d(y) \le d(x)$, that is, $\bar{d}([x]) = d(x) = d(y) = \bar{d}([y])$. Therefore $\bar{d}$ is well-defined.

Let $[x], [y] \in X/\mathrm{Ker}(d)$. Then we have

$$\bar{d}([x]-[y]) = \bar{d}([x-y]) = d(x-y) = d(x)-d(y) = \bar{d}([x]) - \bar{d}([y]).$$

Hence $\bar{d}$ is a homomorphism.

To show that $\bar{d}$ is a monomorphism, let $d(x) = d(y)$. Then $d(x-y) = d(x)-d(y) = 0$ and $d(y-x) = d(y)-d(x) = 0$. Hence $x-y, y-x \in \mathrm{Ker}(d)$. It follows that $x \sim_{\mathrm{Ker}(d)} y$, and $[x] = [y]$. Therefore $\bar{d}$ is a monomorphism. □

4.9. Theorem. *Let X be a subtraction algebra and d a derivation of X. If $\mu : X \to X$ is the map defined by*

$$\mu(x) = \hat{x} = x - d(x)$$

for all $x \in X$, then μ is a derivation with $\mathrm{Ker}(\mu) = \mathrm{Im}(d)$.

Proof. Suppose that $\mu : X \to X$ is the map defined by $\mu(x) = \hat{x} = x - d(x)$ for all $x \in X$. Since $\hat{x} = x - d(x)$ is unique for each $x \in X$, μ is well-defined.

Let $x, y \in X$. The

$$\begin{aligned}\mu(x-y) &= (x-y) - d(x-y)\\ &= (x-y) - (d(x)-y)\\ &= (x-d(x)) - y \quad \text{(by (p12))}\\ &= \mu(x) - y.\end{aligned}$$

Hence μ is a derivation.

If $d(x) \in \mathrm{Im}(d)$, then $\mu(d(x)) = d(x) - d(x) = 0$, and $d(x) \in \mathrm{Ker}(\mu)$, hence $\mathrm{Im}(d) \subseteq \mathrm{Ker}(\mu)$. If $x \in \mathrm{Ker}(\mu)$, then $0 = \mu(x) = x - d(x)$, and $x = x - 0 = x - (x - d(x)) = x \wedge d(x) = d(x) \in \mathrm{Im}(d)$, and so $\mathrm{Ker}(\mu) \subseteq \mathrm{Im}(d)$. Hence it follows that $\mathrm{Ker}(\mu) = \mathrm{Im}(d)$. □

4.10. Corollary. *Let X be a subtraction algebra and d a derivation of X. Then the corestriction $\mu^\circ : X \to \mathrm{Ker}(d)$ of μ is an epimorphism.*

Proof. By Theorem 4.9, $\mu : X \to X$ is a derivation, hence μ is a homomorphism, and it is clear that $\mathrm{Im}(\mu) = \mathrm{Ker}(d)$ by Lemma 3.18. □

4.11. Theorem. *Let X be a subtraction algebra and d a derivation of X. If $\bar{\mu} : X/\mathrm{Im}(d) \to X$ is the map defined by*

$$\bar{\mu}([x]) = \mu(x)$$

for all $[x] \in X/\mathrm{Im}(d)$, then $\bar{\mu}$ is a monomorphism. In particular, $X/\mathrm{Im}(d) \cong \mathrm{Ker}(d)$.

Proof. Suppose that $\bar{\mu} : X/\mathrm{Im}(d) \to X$ is the map defined by

$$\bar{\mu}([x]) = \mu(x)$$

for all $[x] \in X/Im(d)$. If $[x] = [y]$, then $x \sim_{\mathrm{Im}(d)} y$, which implies $x-y, y-x \in \mathrm{Im}(d)$, hence $d(x-y) = x-y$ and $d(y-x) = y-x$. It follows that

$$\bar{\mu}([x]) - \bar{\mu}([y]) = \mu(x) - \mu(y) = \mu(x-y) = (x-y) - d(x-y) = 0,$$

and $\bar{\mu}([y]) - \bar{\mu}([x]) = 0$ in a similar way. Hence $\bar{\mu}([x]) = \bar{\mu}([y])$, and $\bar{\mu}$ is well-defined.

Let $[x], [y] \in X/\mathrm{Im}(d)$. Then we have

$$\bar{\mu}([x]-[y]) = \bar{\mu}([x-y]) = \mu(x-y) = \mu(x) - \mu(y) = \bar{\mu}([x]) - \bar{\mu}([y]),$$

and $\bar{\mu}$ is a homomorphism.

To show that $\bar{\mu}$ is a monomorphism, let $\bar{\mu}([x]) = \bar{\mu}([y])$. Then $\mu(x) = \mu(y)$, and

$$\begin{aligned} 0 &= \mu(x) - \mu(y) = \mu(x - y) = (x - y) - d(x - y), \\ 0 &= \mu(y) - \mu(x) = \mu(y - x) = (y - x) - d(y - x), \end{aligned}$$

hence $x - y \leq d(x - y)$ and $y - x \leq d(y - x)$. Since d is non-expansive, $x - y = d(x - y) \in \mathrm{Im}(d)$ and $y - x = d(y - x) \in \mathrm{Im}(d)$. Therefore, $x \sim_{\mathrm{Im}(d)} y$. This implies $[x] = [y]$. Hence $\bar{\mu}$ is a monomorphism.

It is clear that $\mathrm{Im}(\bar{\mu}) = \mathrm{Im}(\mu)$, and $\mathrm{Im}(\mu) = \mathrm{Ker}(d)$ by Corollary 4.10. Hence $X/\mathrm{Im}(d) \cong \mathrm{Ker}(d)$. □

Now consider the sequence

$$0 \longrightarrow \mathrm{Im}(d) \xrightarrow{i} X \xrightarrow{\mu^\circ} \mathrm{Ker}(d) \longrightarrow 0,$$

of homomorphisms of subtraction algebras, where i is the inclusion map. We note that it is similar to a split exact sequence, since i is a monomorphism, μ° is an epimorphism and $\mathrm{Ker}(\mu^\circ) = \mathrm{Im}(i)$ by Corollary 4.10 and Theorem 4.9.

4.12. Proposition. *Let d be a derivation of a subtraction algebra X. Then for each $x \in X$, $x = d(x) \vee \hat{x}$ with $d(x) \in \mathrm{Im}(d)$ and $\hat{x} \in \mathrm{Ker}(d)$.*

Proof. Let X be a subtraction algebra and $x \in X$. Then the interval $[0, x]$ is a Boolean algebra with respect to the induced partial order and $\hat{x} = x - d(x)$ is the complement of $d(x)$ in $[0, x]$. Hence $d(x) \vee \hat{x} = d(x) \vee (x - d(x)) = x$. □

Let d be a derivation of a subtraction algebra X. Then $\mathrm{Im}(d)$ and $\mathrm{Ker}(d)$ are subtraction subalgebras. Hence $\mathrm{Im}(d) \times \mathrm{Ker}(d)$ is also a subtraction algebra with the binary operation "−" defined by

$$(x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$$

for all $(x_1, y_1), (x_2, y_2) \in \mathrm{Im}(d) \times \mathrm{Ker}(d)$.

4.13. Theorem. *Let d be a derivation of a subtraction algebra X. If $\phi = (d, \mu) : X \to \mathrm{Im}(d) \times \mathrm{Ker}(d)$ is the map defined by*

$$\phi(x) = (d(x), \mu(x))$$

for all $x \in X$, then ϕ is a monomorphism.

Proof. Suppose that $\phi = (d, \mu) : X \to \mathrm{Im}(d) \times \mathrm{Ker}(d)$ is the map defined by $\phi(x) = (d(x), \mu(x))$ for all $x \in X$. Then for any $x, y \in X$ we have

$$\begin{aligned} \phi(x - y) &= (d(x - y), \mu(x - y)) \\ &= (d(x) - d(y), \mu(x) - \mu(y)) \\ &= (d(x), \mu(x)) - (d(y), \mu(y)) \\ &= \phi(x) - \phi(y). \end{aligned}$$

If $\phi(x) = \phi(y)$, then $(d(x), \mu(x)) = (d(y), \mu(y))$, and by Proposition 4.12,

$$x = d(x) \vee \hat{x} = d(x) \vee \mu(x) = d(y) \vee \mu(y) = d(y) \vee \hat{y} = y.$$

Hence ϕ is a monomorphism. □

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