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THE METRIC CONNECTION WITH RESPECT TO THE SYNECTIC METRIC

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Abstract

The purpose of this paper is to investigate the metric connection of the synectic metric ${}^{S}g$ and to compute the components $\widetilde{R}^{A}_{DCB}$ of the curvature tensor $\widetilde{R}$ of the metric connection of the synectic metric ${}^{S}g$ in the tangent bundle $T(M_n)$ of the Riemannian manifold (M_n).

Keywords: Tangent bundle, Synectic metric, Metric connection, Riemannian connection, Curvature tensor.

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1. Introduction

Let M_n be an n-dimensional differentiable manifold of class C^{∞} and $T_P(M_n)$ the tangent space at a point P of M_n, that is, the set of all tangent vectors of M_n at P. Then the set

$$T(M_n) = \bigcup_{P \epsilon M_n} T_P(M_n)$$

is, by definition, the tangent bundle over the manifold (M_n) [2]. We denote by $\Im^p_q(M_n)$ the set of all tensor fields of type (p, q) in M_n and by $\pi : T(M_n) \to M_n$ the naturel projection over M_n.

For $U \subset M_n$, (x^h, y^h) are local coordinates in a neighborhood $\pi^{-1}(U) \subset T(M_n)$. If $\left\{U', x^{h'}\right\}$ is another coordinate neighborhood in M_n containing the point $P = \pi\left(\widetilde{P}\right)$ ($P \epsilon U$ and $\widetilde{P} \epsilon T_P(M_n)$), then $\pi^{-1}(U')$ contains $\widetilde{P}$ and the induced coordinates of $\widetilde{P}$ with respect to $\pi^{-1}(U')$ will be given by $\left(x^{h'}, y^{h'}\right)$, where

$$x^{h'} = x^{h'}(x),$$

$$y^{h'} = \frac{\partial x^{h'}}{\partial x^h} y^h,$$

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$x^{h'}(x)$ being differentiable functions (of class C^∞). Putting $x^{h'} = y^h$, $x^{\overline{h'}} = y^{h'}$, we write $x^{P'} = x^{P'}(x)$.

The Jacobian is given by the matrix

$$\left(\frac{\partial x^{P'}}{\partial x^P}\right) = \begin{pmatrix} \frac{\partial x^{h'}}{\partial x^h} & 0 \\ \frac{\partial^2 x^{h'}}{\partial x^h \partial x^i} y^i & \frac{\partial x^{h'}}{\partial x^h} \end{pmatrix}.$$

Let M_n be a Riemannian manifold with metric g whose components in a coordinate neighborhood U are g_{ji}. In the neighborhood $\pi^{-1}(U)$ of $T(M_n)$, U being a neighborhood of M_n, we put

$$\delta y^h = dy^h + \Gamma_i^h dx^i$$

with respect to the induced coordinates $\left(x^h, y^h\right)$ in $\pi^{-1}(U) \subset T(M_n)$, where $\Gamma_i^h = y^j \Gamma_{ji}^h$.

Suppose that there is given the following Riemannian metric

$$ {}^S\widetilde{g}_{CB} dx^C dx^B = a_{ji} dx^j dx^i + 2 g_{ji} dx^j \delta y^i \tag{1}$$

in the tangent bundle in $T(M_n)$ over a Riemannian manifold M_n with metric g, where a_{ji} are components of a symmetric tensor field of type $(0,2)$ in M_n. We call this metric the *synectic metric*. The synectic metric ${}^S g = {}^C g + {}^V a$ has components [3]

$$ {}^S g = \left({}^S\widetilde{g}_{CB}\right) = \begin{pmatrix} a_{ji} + \partial g_{ji} & g_{ji} \\ g_{ji} & 0 \end{pmatrix}, \tag{2}$$

where $\partial g_{ji} = x^{\overline{s}} \partial_s g_{ji}$.

Let M_n be a Riemannian manifold with metric g, whose local components are g_{ji}. Suppose that we are given a Riemannian metric $\widetilde{g}$ in $T(M_n)$ having local expression

$$\widetilde{g}_{CB} dx^C dx^B = 2 g_{ji} dx^j \delta y^i$$

with respect to the induced coordinates $\left(X^A\right)$, i.e., $\left(x^h, y^h\right)$, where

$$\delta y^h = dy^h + \Gamma_i^h dx^i,\ \Gamma_i^h = y^k \Gamma_{ki}^h$$

and Γ_{ji}^h are the Christoffel symbols formed with g_{ji}. We call this metric the metric Π. $\widetilde{g}$ has components

$$(\widetilde{g}_{CB}) = \begin{pmatrix} \partial g_{ji} & g_{ji} \\ g_{ji} & 0 \end{pmatrix}.$$

The metric connection $\overline{\nabla}$ of the metric Π is the unique connection which satisfies

$$\overline{\nabla}_C \widetilde{g}_{BA} = 0$$

and has non-trivial torsion tensor $\overline{T}_{CB}^A$, which is skew-symmetric in the indices C and B. The connection $\overline{\nabla}$ satisfies

$$\overline{\nabla}_C \widetilde{g}_{BA} = 0 \text{ and } \overline{\Gamma}_{CB}^A - \overline{\Gamma}_{BC}^A = \overline{T}_{CB}^A.$$

Then the metric connection $\overline{\nabla}$ of the metric Π has components $\overline{\Gamma}_{AB}^N$ such that

$$\begin{cases} \overline{\Gamma}_{ji}^h = \overline{\Gamma}_{\bar{j}i}^{\bar{h}} = \overline{\Gamma}_{j\bar{i}}^{\bar{h}} = \Gamma_{ji}^h, \\ \overline{\Gamma}_{\bar{j}i}^h = \overline{\Gamma}_{j\bar{i}}^h = \overline{\Gamma}_{\bar{j}\bar{i}}^h = \overline{\Gamma}_{\bar{j}\bar{i}}^{\bar{h}} = 0, \\ \overline{\Gamma}_{ji}^{\bar{h}} = \partial \Gamma_{ji}^h - y^k R_{kjih} \end{cases}$$

with respect to the induced coordinates in $T(M_n)$, where Γ_{ji}^k are the components of ∇ in M_n [4].

2. Riemannian connection of ${}^S g$

The components of the Riemannian connection determined by the metric ${}^S g$ are given by

$$ {}^S\Gamma^K_{JI} = \frac{1}{2}\widetilde{g}^{KM}\left(\partial_J {}^S g_{MI} + \partial_I {}^S g_{JM} - \partial_M\ {}^S g_{JI}\right), \tag{3} $$

where $\widetilde{g}^{KM}$ are the contravariant components of the metric ${}^S g$ with respect to the induced coordinates in $T(M_n)$:

$$ {}^S\widetilde{g}^{CB} = \begin{pmatrix} 0 & g^{ji} \\ g^{ji} & x^{\overline{s}}\partial_s g^{ji} - a^{ji}_{\cdot\cdot} \end{pmatrix},\ a^{ti}_{\cdot\cdot} = g^{jt}a_{js}g^{si} \tag{4} $$

where g^{ji} denote the contravariant components of g in M_n [4], i.e.,

$$ {}^S g_{IM}\widetilde{g}^{MJ} = \delta^J_I = \begin{cases} 0 & I \neq J, \\ 1 & I = J. \end{cases} \tag{5} $$

Then, taking account (2) and (4), we have

$$ \begin{cases} {}^S\Gamma^k_{ji} = \Gamma^k_{ji}, \quad {}^S\Gamma^{\overline{k}}_{\overline{j}i} = \Gamma^k_{ji}, \quad {}^S\Gamma^{\overline{k}}_{j\overline{i}} = \Gamma^k_{ji}, \quad {}^S\Gamma^{\overline{k}}_{\overline{j}\overline{i}} = 0 \\ {}^S\Gamma^k_{j\overline{i}} ={}^S \Gamma^k_{\overline{j}i} ={}^S \Gamma^k_{\overline{j}\overline{i}} = 0, \quad {}^S\Gamma^{\overline{k}}_{ji} = x^{\overline{t}}\partial_t\Gamma^k_{ji} + H^k_{ji} \end{cases} \tag{6} $$

with respect to the induced coordinates in $T(M_n)$, Γ^k_{ji} being the Christoffel symbols constructed with g_{ji}, $H^k_{ji} = \frac{1}{2}g^{ks}(\nabla_j a_{si} + \nabla_i a_{js} - \nabla_s a_{ji})$ is a tensor of type $(1,2)$ and $\nabla_s a_{ji} = \partial_s a_{ji} - \Gamma^l_{kj}a_{li} - \Gamma^l_{ki}a_{jl}$.

Hence, from (6) we have:

2.1. Remark. If $\nabla a = 0$, then ${}^S\Gamma ={}^C \Gamma$, where ${}^C\Gamma$ is the Riemannian connection of ${}^C g$ [4].

2.2. Remark. If $a_{ji} = g_{ji}$, then ${}^S\Gamma ={}^C \Gamma$.

Thus we have

2.3. Theorem. ${}^S\Gamma ={}^C \Gamma +{}^V H$, *where* ${}^V H$ *is the vertical lift of* $H \in T^1_2(M_n)$. □

3. The Metric connection with respect to the synectic metric ${}^S g$

Let $\widetilde{\nabla}$ be a connection which satisfies

$$ \widetilde{\nabla}{}^S g = 0, \tag{7} $$

and has torsion, where ${}^S g$ is the synectic metric ${}^S g ={}^C g +{}^V a$ in $T(M_n)$.

The connection $\widetilde{\nabla}$ has the non-trivial torsion tensor $\widetilde{T}^C_{AB}$, which is skew-symmetric in the indices C and B. We denote this connection by $\widetilde{\nabla}$ and its components by $\widetilde{\Gamma}^C_{AB}$. Then the connection $\widetilde{\nabla}$ satisfies

$$ \widetilde{\nabla}{}^S g = 0 \text{ and } \widetilde{\Gamma}^C_{AB} - \widetilde{\Gamma}^C_{BA} = \widetilde{T}^C_{AB}. \tag{8} $$

On solving (8) with respect to $\widetilde{\Gamma}^C_{AB}$, we find [1]

$$ {}^S\Gamma^N_{AB} + U^N_{AB} = \widetilde{\Gamma}^N_{AB}, \tag{9} $$

where ${}^S\Gamma^C_{AB}$ are the Christoffel symbols constructed with the metric ${}^S g$,

$$ U_{ABC} = \frac{1}{2}\left(\widetilde{T}_{CAB} + \widetilde{T}_{CBA} + \widetilde{T}_{ABC}\right) \tag{10} $$

and

$$ U_{ABC} = U^N_{AB}{}^S g_{NC}, \quad \widetilde{T}_{ABC} = \widetilde{T}^N_{AB}{}^S g_{NC}. \tag{11} $$

If we put

(12) $$\widetilde{T}_{ji}^{\overline{h}} = R_{jik}^{h} y^{k},$$

all other $\widetilde{T}_{CB}^{A}$ not related to $\widetilde{T}_{ji}^{\overline{h}}$ being assumed to be zero, then we get a tensor field $\widetilde{T}_{CB}^{A}$ of type (1,2) in $T(M_n)$ which is skew-symmetric in the indices A and B. We take this $\widetilde{T}_{CB}^{A}$ as the torsion tensor and determine a metric connection in $T(M_n)$ with respect to the metric ${}^{S}g$.

Since

$$\widetilde{T}_{jih} = R_{jikh}\, y^{k}, \quad R_{jikh} = R_{jik}^{n}\, g_{nh},$$

we have for $\widetilde{T}_{CAB} + \widetilde{T}_{CBA} + \widetilde{T}_{ABC}$

$$\begin{aligned}\widetilde{T}_{jih} + \widetilde{T}_{hji} + \widetilde{T}_{hij} &= \left(R_{jikh} + R_{hjki} + R_{hikj}\right) y^{k}\\ &= -2R_{kjih} y^{k},\end{aligned}$$

from which

$$U_{jih} = \frac{1}{2}\left(\widetilde{T}_{jih} + \widetilde{T}_{hji} + \widetilde{T}_{hij}\right) = -R_{kjih} y^{k},$$

that is,

(13) $$U_{ji}^{\overline{h}} = -R_{kji}^{h} y^{k},$$

all the other U_{AB}^{N} being zero. Thus, substituting (13) and (6) in (9) we have

(14) $$\begin{cases}\widetilde{\Gamma}_{ji}^{h} = \widetilde{\Gamma}_{\overline{j}i}^{\overline{h}} = \widetilde{\Gamma}_{j\overline{i}}^{\overline{h}} = \Gamma_{ji}^{h},\\ \widetilde{\Gamma}_{ji}^{h} = \widetilde{\Gamma}_{\overline{j}i}^{h} = \widetilde{\Gamma}_{j\overline{i}}^{\overline{h}} = \Gamma_{ji}^{h},\\ \widetilde{\Gamma}_{ji}^{\overline{h}} = x^{\overline{t}}\partial_t \Gamma_{ji}^{h} + H_{ji}^{h} - y^{k} R_{kjih},\end{cases}$$

with respect to the induced coordinates, Γ_{ji}^{k} being the Christoffel symbols formed with g_{ji}, where $H_{ji}^{k} = \frac{1}{2} g^{ks}\left(\nabla_j a_{si} + \nabla_i a_{js} - \nabla_s a_{ji}\right)$. Thus we have:

3.1. Remark. If $\nabla a = 0$, then the metric connection $\widetilde{\nabla}$ in the tangent bundle $T(M_n)$ with respect to the metric ${}^{S}g$ coincides with the metric connection $\overline{\nabla}$ with the metric ${}^{C}g$. That is,

$$\widetilde{\nabla} = \overline{\nabla}.$$

3.2. Remark. If $a_{ji} = g_{ji}$, then the metric connection $\widetilde{\nabla}$ in the tangent bundle $T(M_n)$ with respect to the metric ${}^{S}g$ coincides with the metric connection $\overline{\nabla}$ with the metric ${}^{C}g$. That is,

$$\widetilde{\nabla} = \overline{\nabla}.$$

Thus we have

3.3. Theorem. $\widetilde{\nabla} = \overline{\nabla} +^{V} H$, *where* $H_{ji}^{k} = \frac{1}{2} g^{ks}\left(\nabla_j a_{si} + \nabla_i a_{js} - \nabla_s a_{ji}\right)$. □

4. The Curvature tensor of the Metric connection $\widetilde{\nabla}$

Components of the curvature tensor of the metric connection are given by

(15) $$\widetilde{R}_{KJI}^{H} = \partial_K \widetilde{\Gamma}_{JI}^{H} - \partial_J \widetilde{\Gamma}_{KI}^{H} + \widetilde{\Gamma}_{KT}^{H}\widetilde{\Gamma}_{JI}^{H} - \widetilde{\Gamma}_{JT}^{H}\widetilde{\Gamma}_{KI}^{H},$$

where $\widetilde{\Gamma}_{JI}^{H}$ are the components of the metric connection $\widetilde{\nabla}$ with respect to the metric ${}^{S}g$.

Taking into account (14)–(15), we have

(16) $$\begin{cases}\widetilde{R}_{kji}^{h} = \widetilde{R}_{\overline{k}ji}^{\overline{h}} = \widetilde{R}_{k\overline{j}i}^{\overline{h}} = \widetilde{R}_{kj\overline{i}}^{\overline{h}} = R_{kji}^{h}\\ \widetilde{R}_{kji}^{\overline{h}} = \partial R_{kji}^{h} + y^{n}\left(\nabla_j R_{nki}^{h} - \nabla_k R_{nji}^{h}\right) + \nabla_k H_{ji}^{h} - \nabla_j H_{ki}^{h},\end{cases}$$

all the others not related to these being zero, with respect to the induced coordinates.

The contracted curvature tensor of the metric connection $\widetilde{\nabla}$ has components $\widetilde{R}_{CB} = \widetilde{R}_{ECB}^{E}$ such that

$$\widetilde{R}_{Eji}^{E} = R_{ji}, \quad \widetilde{R}_{E\bar{j}i}^{E} = 0, \quad \widetilde{R}_{Ej\bar{i}}^{E} = 0, \quad \widetilde{R}_{E\bar{j}\bar{i}}^{E} = 0 \tag{17}$$

because of (16), where $R_{ji} = R_{hji}^{h}$ denote the components of the Ricci tensor of the Riemannian manifold M_n. Thus we have

4.1. Theorem. *The tangent bundle $T(M_n)$ with the metric connection $\widetilde{\nabla}$ has a vanishing contracted curvature tensor if and only if M_n has a vanishing Ricci tensor.* □

For the scalar curvature of $T(M_n)$ with the metric connection, we have

$$\widetilde{R} = \widetilde{g}^{CB}\widetilde{R}_{CB} = 0 \tag{18}$$

by means of (4) and (17), where $\widetilde{g}^{CB}$ denote the contravariant components of the metric ${}^{S}g$. Thus we have

4.2. Theorem. *The tangent bundle $T(M_n)$ with the metric connection of the synectic metric ${}^{S}g$ has vanishing scalar curvature.* □

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