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SHARPENING AND GENERALIZATIONS OF CARLSON'S DOUBLE INEQUALITY FOR THE ARC COSINE FUNCTION

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Abstract

In this paper, we sharpen and generalize Carlson's double inequality for the arc cosine function.

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1. Introduction and main results

In [1, p. 700, (1.14)] and [8, p. 246, 3.4.30], it was listed that

$$\frac{6(1-x)^{1/2}}{2\sqrt{2}+(1+x)^{1/2}} < \arccos x < \frac{\sqrt[3]{4}\,(1-x)^{1/2}}{(1+x)^{1/6}}, \quad 0 \le x < 1. \tag{1.1}$$

The first aim of this paper is to sharpen and generalize the right-hand side inequality in (1.1) as follows.

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1.1. Theorem. *For real numbers a and b, let*

$$f_{a,b}(x)=\frac{(1+x)^b}{(1-x)^a}\arccos x \tag{1.2}$$

on the open unit interval $(0,1)$.

(1) *If and only if*

$$(a,b)\in\left\{b\le\frac{2}{\pi}-a\right\}\cap\left\{a\le\frac12\right\}, \tag{1.3}$$

the function $f_{a,b}(x)$ is strictly decreasing;

(2) *If*

$$\begin{aligned}(a,b)\in\left\{\frac{2}{\pi}-a\le b\le a-\frac{4}{\pi^2}\right\}\cup\left\{\frac12\le a\le b+\frac13\right\}\\ \cup\left\{\frac13<a-b<\frac{4}{\pi^2},\ a+b\ge\frac{2(a-b)^{3/2}}{\sqrt{4(a-b)-1}}\right\},\end{aligned} \tag{1.4}$$

the function $f_{a,b}(x)$ is strictly increasing;

(3) *If*

$$(a,b)\in\left\{\frac13<a-b<\frac{4}{\pi^2}\right\}\cap\left\{\frac{2}{\pi}-b<a\le\frac12\right\}, \tag{1.5}$$

the function $f_{a,b}(x)$ has a unique maximum;

(4) *If*

$$(a,b)\in\left\{\frac13<a-b<\frac{4}{\pi^2}\right\}\cap\left\{\frac12<a\le\frac{2}{\pi}-b\right\}, \tag{1.6}$$

the function $f_{a,b}(x)$ has a unique minimum;

(5) *If*

$$\begin{aligned}(a,b)\in\left\{\frac13<a-b<\frac{4}{\pi^2}\right\}\cap\left\{\frac{2}{\pi}<a+b<\frac{2(a-b)^{3/2}}{\sqrt{4(a-b)-1}}\right\}\\ \cap\left\{a>\frac12\right\},\end{aligned} \tag{1.7}$$

the function $f_{a,b}(x)$ has a unique maximum and a unique minimum in sequence;

(6) *A necessary condition for the function $f_{a,b}(x)$ to be strictly increasing is*

$$(a,b)\in\left\{b\ge\frac{2}{\pi}-a\right\}\cap\left\{a\ge\frac12\right\}. \tag{1.8}$$

As direct consequences of the monotonicity of the function $f_{a,b}(x)$, the following inequalities may be deduced.

1.2. Theorem. *For $x\in(0,1)$, the double inequality*

$$\frac{\pi}{2}\cdot\frac{(1-x)^{1/2}}{(1+x)^b}<\arccos x<2^{b+1/2}\cdot\frac{(1-x)^{1/2}}{(1+x)^b} \tag{1.9}$$

holds provided that $b\ge\frac16$.

The right-hand side inequality in (1.9) *is valid if and only if $b\ge\frac16$.*

The reversed version of (1.9) *is valid provided that $b\le\frac{2}{\pi}-\frac12$.*

The reversed version of the left-hand side inequality in (1.9) *is valid if and only if $b\le\frac{2}{\pi}-\frac12$.*

If (a,b) satisfies (1.5), $16ab(b-a)+(a+b)^2>0$ *and*

$$x_1=\frac{(a+b)(2b-2a+1)-\sqrt{16ab(b-a)+(a+b)^2}}{2(a-b)^2}>0, \tag{1.10}$$

then

$$\left.\begin{aligned}&\min\left\{2^{b+1/2},\frac{\pi}{2}\right\}\frac{(1-x)^a}{(1+x)^b}, && a=\frac{1}{2}\\ &0, && a<\frac{1}{2}\end{aligned}\right\}\le \arccos x \le \frac{(1+x_1)^{b+1/2}(1-x_1)^{1/2-a}}{a+b+(a-b)x_1}\cdot\frac{(1-x)^a}{(1+x)^b}. \tag{1.11}$$

If (a,b) satisfies (1.6), $16ab(b-a)+(a+b)^2>0$ *and*

$$x_2=\frac{(a+b)(2b-2a+1)+\sqrt{16ab(b-a)+(a+b)^2}}{2(a-b)^2}\in(0,1), \tag{1.12}$$

then

$$\arccos x\ge\frac{(1+x_2)^{b+1/2}(1-x_2)^{1/2-a}}{a+b+(a-b)x_2}\cdot\frac{(1-x)^a}{(1+x)^b}. \tag{1.13}$$

The second aim of this paper is to sharpen and generalize the left-hand side inequality in (1.1) as follows.

1.3. Theorem. *For $x\in(0,1)$, the function*

$$F_{1/2,1/2,2\sqrt{2}}(x)=\frac{2\sqrt{2}+(1+x)^{1/2}}{(1-x)^{1/2}}\arccos x \tag{1.14}$$

is strictly decreasing. Consequently, the double inequality

$$\frac{6(1-x)^{1/2}}{2\sqrt{2}+(1+x)^{1/2}}<\arccos x<\frac{\left(1/2+\sqrt{2}\right)\pi(1-x)^{1/2}}{2\sqrt{2}+(1+x)^{1/2}} \tag{1.15}$$

holds on $(0,1)$ and the constants 6 and $\left(\frac{1}{2}+\sqrt{2}\right)\pi$ in (1.15) are the best possible.

2. Remarks

Before proving our theorems, we list several remarks on them.

2.1. Remark. From Theorem 1.2, we obtain

$$\frac{\pi(1-x)^{1/2}}{2(1+x)^{1/6}}<\arccos x<\frac{\sqrt[3]{4}\,(1-x)^{1/2}}{(1+x)^{1/6}} \tag{2.1}$$

and

$$\frac{4^{1/\pi}(1-x)^{1/2}}{(1+x)^{(4-\pi)/2\pi}}<\arccos x<\frac{\pi(1-x)^{1/2}}{2(1+x)^{(4-\pi)/2\pi}} \tag{2.2}$$

for $x\in(0,1)$.

Except that the right-hand side inequality in (2.1) and the left-hand side inequality in (1.15) are the same as the corresponding ones in (1.1) and that the left-hand side inequality in (1.1) is better than the corresponding one in (2.2), other corresponding inequalities in (1.1), (1.15), (2.1) and (2.2) are not included in one another.

2.2. Remark. Setting $\arccos x = t$ in (2.1) and (2.2) yields

$$\cos t < \left(\frac{\sin t}{t}\right)^3 < \frac{32}{\pi^3}\cos t \tag{2.3}$$

and

$$\left(\frac{2^{\pi+2}}{\pi^\pi}\right)^{1/(4-\pi)}\cos t < \left(\frac{\sin t}{t}\right)^{\pi/(4-\pi)} < \cos t \tag{2.4}$$

for $0 < t < \frac{\pi}{4}$. They may be rearranged as

$$\left(\frac{\sin t}{t}\right)^{\pi/(4-\pi)} < \cos t < \left(\frac{\sin t}{t}\right)^3 \tag{2.5}$$

and

$$\frac{\pi^3}{32}\left(\frac{\sin t}{t}\right)^3 < \cos t < \left(\frac{\pi^\pi}{2^{\pi+2}}\right)^{1/(4-\pi)}\left(\frac{\sin t}{t}\right)^{\pi/(4-\pi)} \tag{2.6}$$

for $0 < t < \frac{\pi}{4}$. These two inequalities are not included in one another.

For more information connected with the above two inequalities, please refer to [10], [15, Sections 7.5 and 7.6], and closely related references therein.

2.3. Remark. The approach used to prove our theorems in the next section can be utilized to establish similar bounds for some inverse trigonometric functions (see [6, 11, 13, 14, 16, 17]) and is simpler than those methods used in [4, 7, 18]. In other words, although the techniques used in this paper are nothing more than calculus this may be all that is needed to get good results.

2.4. Remark. Motivated by the papers [5, 6, 16], some inequalities of Carlson type were also sharpened and improved in [2, 9].

2.5. Remark. It is noted that there are some applications in [3] of this type of inequality obtained in [7].

2.6. Remark. This paper is a slightly revised version of the preprint [5] and a sisterly article of [6] and its preprint [12].

3. Proofs of the theorems

Now we are in a position to verify our theorems.

Proof of Theorem 1.1. Straightforward differentiation yields

$$\begin{aligned} f'_{a,b}(x) &= \frac{(1+x)^{b-1}}{(1-x)^{a+1}}(\arccos x)\left[a+b+(a-b)x-\frac{\sqrt{1-x^2}}{\arccos x}\right] \\ &\triangleq \frac{(1+x)^{b-1}}{(1-x)^{a+1}}(\arccos x)g_{a,b}(x) \end{aligned} \tag{3.1}$$

with

$$g'_{a,b}(x) = a-b-\frac{1}{(\arccos x)^2}+\frac{x}{\sqrt{1-x^2}\,\arccos x},$$

$$\begin{aligned}
g''_{a,b}(x) &= \frac{(\arccos x)^2 + x\sqrt{1-x^2}\,\arccos x + 2x^2 - 2}{(1-x^2)^{3/2}(\arccos x)^3} \\
&\triangleq \frac{h(x)}{(1-x^2)^{3/2}(\arccos x)^3}, \\
h'(x) &= \frac{(1+2x^2)}{\sqrt{1-x^2}}\left[\frac{3x\sqrt{1-x^2}}{1+2x^2} - \arccos x\right] \\
&\triangleq \frac{(1+2x^2)}{\sqrt{1-x^2}}q(x), \\
q'(x) &= \frac{4(1-x^2)^{3/2}}{(1+2x^2)^2}.
\end{aligned}$$

It is clear that $q'(x)$ is positive, and so $q(x)$ is increasing on $[0,1)$. By virtue of $q(1)=0$, we obtain that $q(x)<0$ on $[0,1)$, which is equivalent to $h'(x)<0$ and $h(x)$ is decreasing on $[0,1)$. Due to $h(1)=0$, it follows that $h(x)>0$ and $g''_{a,b}(x)>0$ on $[0,1)$, and so the function $g'_{a,b}(x)$ is increasing on $[0,1)$. It is easy to obtain that $\lim_{x\to 0^+} g'_{a,b}(x) = a-b-\frac{4}{\pi^2}$ and $\lim_{x\to 1^-} g'_{a,b}(x) = a-b-\frac{1}{3}$. Hence,

(1) if $a-b \ge \frac{4}{\pi^2}$, then $g'_{a,b}(x)>0$ and $g_{a,b}(x)$ is increasing on $(0,1)$;
(2) if $a-b \le \frac{1}{3}$, then $g'_{a,b}(x)<0$ and $g_{a,b}(x)$ is decreasing on $(0,1)$;
(3) if $\frac{1}{3} < a-b < \frac{4}{\pi^2}$, then $g'_{a,b}(x)$ has a unique zero and $g_{a,b}(x)$ has a unique minimum on $(0,1)$.

Direct calculation gives

(3.2) $$g_{a,b}(0) = a+b-\frac{2}{\pi}$$

and

(3.3) $$\lim_{x\to 1^-} g_{a,b}(x) = 2a-1.$$

Therefore,

(1) if $a-b \ge \frac{4}{\pi^2}$ and $a+b \ge \frac{2}{\pi}$, then $g_{a,b}(x)$ and $f'_{a,b}(x)$ are positive, and so the function $f_{a,b}(x)$ is increasing on $(0,1)$;
(2) if $a-b \ge \frac{4}{\pi^2}$ and $2a \le 1$, then $g_{a,b}(x)$ and $f'_{a,b}(x)$ are negative, and so the function $f_{a,b}(x)$ is decreasing on $(0,1)$;
(3) if $a-b \le \frac{1}{3}$ and $a+b \le \frac{2}{\pi}$, then $g_{a,b}(x)$ and $f'_{a,b}(x)$ are negative, and so the function $f_{a,b}(x)$ is decreasing on $(0,1)$;
(4) if $a-b \le \frac{1}{3}$ and $2a \ge 1$, then $g_{a,b}(x)$ and $f'_{a,b}(x)$ are positive, and so the function $f_{a,b}(x)$ is increasing on $(0,1)$;
(5) if $\frac{1}{3} < a-b < \frac{4}{\pi^2}$, $a+b \le \frac{2}{\pi}$ and $a \le \frac{1}{2}$, then $g_{a,b}(x)$ and $f'_{a,b}(x)$ are negative, and so the function $f_{a,b}(x)$ is decreasing on $(0,1)$;
(6) if $\frac{1}{3} < a-b < \frac{4}{\pi^2}$, $a+b > \frac{2}{\pi}$ and $a \le \frac{1}{2}$, then $g_{a,b}(x)$ and $f'_{a,b}(x)$ have a unique zero on $(0,1)$, which is a unique maximum point of $f_{a,b}(x)$ on $(0,1)$;
(7) if $\frac{1}{3} < a-b < \frac{4}{\pi^2}$, $a+b \le \frac{2}{\pi}$ and $a > \frac{1}{2}$, then $g_{a,b}(x)$ and $f'_{a,b}(x)$ have a unique zero on $(0,1)$, which is a unique minimum point of $f_{a,b}(x)$ on $(0,1)$;
(8) if $\frac{1}{3} < a-b < \frac{4}{\pi^2}$, the minimum point $x_0 \in (0,1)$ of $g_{a,b}(x)$ satisfies

$$\frac{1}{\arccos x_0} = \frac{x_0 + \sqrt{x_0^2 + 4(a-b)(1-x_0^2)}}{2\sqrt{1-x_0^2}}$$

and the minimum of $g_{a,b}(x)$ equals

$$\begin{aligned} g_{a,b}(x_0) &= a+b+\Big(a-b-\frac{1}{2}\Big)x_0-\frac{1}{2}\sqrt{x_0^2+4(a-b)(1-x_0^2)} \\ &\ge a+b-\frac{2(a-b)^{3/2}}{\sqrt{4(a-b)-1}}, \end{aligned}$$

which means that

(a) when $\frac{1}{3}<a-b<\frac{4}{\pi^2}$ and $a+b\ge\frac{2(a-b)^{3/2}}{\sqrt{4(a-b)-1}}$, the functions $g_{a,b}(x)$ and $f'_{a,b}(x)$ are non-negative, and so the function $f_{a,b}(x)$ is strictly increasing on $(0,1)$;

(b) when $\frac{1}{3}<a-b<\frac{4}{\pi^2}$, $a+b>\frac{2}{\pi}$, $a>\frac{1}{2}$ and $a+b<\frac{2(a-b)^{3/2}}{\sqrt{4(a-b)-1}}$, the functions $g_{a,b}(x)$ and $f'_{a,b}(x)$ have two zeros which are in sequence the maximum and minimum of the function $f_{a,b}(x)$ on $(0,1)$.

As a result, the sufficiency for the function $f_{a,b}(x)$ to be monotonic on $(0,1)$ is proved.

Conversely, if the function $f_{a,b}(x)$ is strictly decreasing, then the function $g_{a,b}(x)$ must be negative on $(0,1)$, so the quantities in (3.2) and (3.3) are non-positive. Hence, the condition in (1.3) is also necessary.

By similar arguments to the above, the necessary condition (1.8) follows immediately. The proof of Theorem 1.1 is thus proved. □

Proof of Theorem 1.2. It is easy to see that $\lim_{x\to0^+}f_{a,b}(x)=\frac{\pi}{2}$ and

$$\lim_{x\to1^-}f_{a,b}(x)=2^b\lim_{x\to1^-}\frac{\arccos x}{(1-x)^a}=\begin{cases}2^{b+1/2}, & a=\frac{1}{2};\\ 0, & a<\frac{1}{2};\\ \infty, & a>\frac{1}{2}.\end{cases}$$

From Theorem 1.1, it follows that the function $f_{1/2,b}(x)$ is strictly increasing (or strictly decreasing respectively) on $(0,1)$ if $b\ge\frac{1}{6}$ (or if and only if $b\le\frac{2}{\pi}-\frac{1}{2}$ respectively). Consequently, if $b\ge\frac{1}{6}$, then

$$\frac{\pi}{2}=\lim_{x\to0^+}f_{1/2,b}(x)<f_{1/2,b}(x)<\lim_{x\to1^-}f_{1/2,b}(x)=2^{b+1/2} \tag{3.4}$$

on $(0,1)$, which can be rearranged as the inequality (1.9); if $b\le\frac{2}{\pi}-\frac{1}{2}$, the inequality (3.4), and so the inequality (1.9), reverses.

The right-hand side inequality in (1.9) may be rewritten as

$$\begin{aligned} b &> \frac{\ln\arccos x-[\ln(1-x)+(\ln2)]/2}{\ln2-\ln(1+x)} \\ &\to (1+x)\bigg[\frac{1}{\sqrt{1-x^2}\arccos x}-\frac{1}{2(1-x)}\bigg] \\ &\to \frac{2(x-1)+\sqrt{1-x^2}\arccos x}{(x-1)\sqrt{1-x^2}\arccos x} \\ &\to \frac{x\arccos x/\sqrt{1-x^2}-1}{(x-1)\big[1+(1+2x)\arccos x/\sqrt{1-x^2}\,\big]} \\ &\to \frac{x\arccos x/\sqrt{1-x^2}-1}{4(x-1)} \\ &\to \frac{1}{6} \end{aligned}$$

as $x \to 1^-$. Therefore, the condition $b \geq \frac{1}{6}$ is also a necessary condition such that the right-hand side inequality in (1.9) is valid.

The reversed version of the left-hand side inequality in (1.9) may be rearranged as

$$b < \frac{\ln \pi - \ln 2 + [\ln(1-x)]/2 - \ln \arccos x}{\ln(1+x)} \to \frac{2}{\pi} - \frac{1}{2}$$

as $x \to 0^+$. Hence, the necessity of $\frac{2}{\pi} - \frac{1}{2}$ is proved.

By the equation (3.1) in the proof of Theorem 1.1, it follows that the extreme points $\xi \in (0,1)$ of the function $f_{a,b}(x)$ satisfy $g_{a,b}(\xi) = 0$, that is,

$$\arccos \xi = \frac{\sqrt{1-\xi^2}}{a+b+(a-b)\xi},$$

so the extremes of $f_{a,b}(x)$ equal

$$f_{a,b}(\xi) = \frac{(1+\xi)^{b+1/2}}{(1-\xi)^{a-1/2}[a+b+(a-b)\xi]} \triangleq g(\xi)$$

and

$$g'(x) \triangleq \frac{(x+1)^{b-1/2} h(x)}{[a+b+(a-b)x]^2 (1-x)^{a+1/2}},$$

where

$$h(x) = (a-b)^2 x^2 + (a+b)(2a-2b-1)x + (a+b)^2 - a + b$$

has two zero points x_1 and x_2 which are also the zeros of the function $g'(x)$ and the extreme points of $g(x)$ for $x \in (0,1)$.

When $16ab(b-a) + (a+b)^2 > 0$ and $x_{1,2} \in (0,1)$, the point x_1 is the maximum point and x_2 is the minimum point of $g(x)$, so we have the inequality

$$\frac{(1+x_2)^{b+1/2}(1-x_2)^{1/2-a}}{a+b+(a-b)x_2} \leq f_{a,b}(\xi) \leq \frac{(1+x_1)^{b+1/2}(1-x_1)^{1/2-a}}{a+b+(a-b)x_1}. \tag{3.5}$$

When $16ab(b-a) + (a+b)^2 > 0$ such that $x_1 \leq 0$ and $x_2 \in (0,1)$, the function $g(x)$ has only one minimum and the left-hand side inequality in (3.5) is valid.

When $16ab(b-a) + (a+b)^2 > 0$ such that $x_1 \in (0,1)$ and $x_2 \geq 1$, the function $g(x)$ has only one maximum and the right-hand side inequality in (3.5) is valid.

When $16ab(b-a) + (a+b)^2 > 0$ such that $x_2 \leq 0$ or $x_1 \geq 1$, the function $g(x)$ is strictly increasing on $(0,1)$; since $\lim_{x\to 0^+} g(x) = \frac{1}{a+b}$ and

$$\lim_{x \to 1^-} g(x) = \begin{cases} 2^{b+1/2}, & a = \frac{1}{2}, \\ 0, & a < \frac{1}{2}, \\ \infty, & a > \frac{1}{2}, \end{cases}$$

we find

$$\frac{1}{a+b} \leq f_{a,b}(\xi) \leq \begin{cases} 2^{b+1/2}, & a = \frac{1}{2}, \\ 0, & a < \frac{1}{2}, \\ \infty, & a > \frac{1}{2}. \end{cases} \tag{3.6}$$

When $16ab(b-a) + (a+b)^2 > 0$ such that $x_1 \leq 0$ and $x_2 \geq 1$, the function $g(x)$ is strictly decreasing on $(0,1)$, and so the inequality (3.6) reverses.

When $16ab(b-a) + (a+b)^2 \leq 0$, the function $g(x)$ is strictly increasing on $(0,1)$, and so the inequality (3.6) holds.

Under the condition (1.5),

(1) If $16ab(b-a)+(a+b)^2>0$ and $x_1>0$, then

$$\left.\begin{matrix} \min\left\{2^{b+1/2},\dfrac{\pi}{2}\right\}, & a=\dfrac12 \\ 0, & a<\dfrac12 \end{matrix}\right\} \le \frac{(1+x)^b}{(1-x)^a}\arccos x \le f_{a,b}(\xi) \le \frac{(1+x_1)^{b+1/2}(1-x_1)^{1/2-a}}{a+b+(a-b)x_1};$$

(2) If either $16ab(b-a)+(a+b)^2>0$ such that $x_2\le 0$ or $x_1\ge 1$ or $16ab(b-a)+(a+b)^2\le 0$, then

$$\left.\begin{matrix} \min\left\{2^{b+1/2},\dfrac{\pi}{2}\right\}, & a=\dfrac12 \\ 0, & a<\dfrac12 \end{matrix}\right\} \le \frac{(1+x)^b}{(1-x)^a}\arccos x \le f_{a,b}(\xi) \le \begin{cases} 2^{b+1/2}, & a=\frac12, \\ 0, & a<\frac12. \end{cases}$$

Under the condition (1.6), if $16ab(b-a)+(a+b)^2>0$ and $x_2\in(0,1)$, then

$$\frac{(1+x)^b}{(1-x)^a}\arccos x \ge f_{a,b}(\xi) \ge \frac{(1+x_2)^{b+1/2}(1-x_2)^{1/2-a}}{a+b+(a-b)x_2}.$$

Straightforward simplification completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Direct computation yields

$$\begin{aligned}\frac{\mathrm{d}F_{1/2,1/2,2\sqrt2}(x)}{\mathrm{d}x} &= \frac{\bigl[1+\sqrt{2(x+1)}\,\bigr]\sqrt{1-x^2}}{(1+x)(x-1)^2} \\ &\quad\times\left[\arccos x - \frac{\bigl(\sqrt{1+x}+2\sqrt2\,\bigr)\sqrt{1-x}}{1+\sqrt{2(x+1)}}\right] \\ &\triangleq \frac{\bigl[1+\sqrt{2(x+1)}\,\bigr]\sqrt{1-x^2}}{(1+x)(x-1)^2}G(x),\end{aligned}$$

and

$$G'(x)=\frac{(x-1)\sqrt{2(1+x)}\,\bigl[\sqrt{1+x}-\sqrt2\,\bigr]}{2\sqrt{(1+x)(1-x^2)}\,\bigl[1+\sqrt{2(1+x)}\,\bigr]^2}>0.$$

Thus, the function $G(x)$ is strictly increasing on $(0,1)$. Since $\lim_{x\to1^-}G(x)=0$, the function $G(x)$ is negative on $(0,1)$, which means that the derivative $F'_{1/2,1/2,2\sqrt2}(x)$ is negative and that the function $F_{1/2,1/2,2\sqrt2}(x)$ is strictly decreasing on $(0,1)$. Further, from

$$\lim_{x\to0^+}F_{1/2,1/2,2\sqrt2}(x)=\left(\frac12+\sqrt2\right)\pi$$

and $\lim_{x\to1^-}F_{1/2,1/2,2\sqrt2}(x)=6$, the double inequality (1.15) and its best possibility follow. Theorem 1.3 is thus proved. □

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