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ON SOME COMMON FIXED POINT THEOREMS WITH RATIONAL EXPRESSIONS ON CONE METRIC SPACES OVER A BANACH ALGEBRA

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Abstract

In the present paper we have defined a new space called a BA-cone metric space by taking a Banach algebra instead of a Banach space. Some common fixed point theorems involving rational expressions have been proved and some consequences obtained in these spaces. Also we have extended this work to four mappings with a weak commutativity property in BA-cone metric spaces.

Keywords: Fixed point , Cone metric space, Metric space, Rational expression, Weak commutativity, Banach Algebra.

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1. Introduction

Fixed point theory plays a basic role in applications of various branches of mathematics, from elementary calculus and linear algebra to topology and analysis. Fixed point theory is not restricted to mathematics and this theory has many applications in other disciplines. This theory is closely related to game theory, military, economics, statistics and medicine.

Much work has been done involving fixed points using the Banach contraction principle. This principle has been extended to other kinds of contraction principle, such as contractive conditions involving product, rational expressions and many others. The Banach contraction principle with rational expressions have been extended and some fixed

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and common fixed point theorems obtained in [4-5]. In [3], common fixed points for a pair of self mappings satisfying a rational expression have been obtained.

Quiet recently; Huang and Zhang [6] generalized the notion of metric space by replacing the real numbers by an ordered Banach space, thereby defining cone metric spaces. They have investigated convergence in cone metric spaces, introduced completeness of cone metric spaces, and proved a Banach contraction mapping theorem, and some other fixed point theorems involving contractive type mappings in cone metric spaces using the normality condition. Later, various authors have proved some common fixed point theorems with normal and non-normal cones in these spaces.

The aim of this paper is to extend the result in [3] to BA-cone metric spaces which we have defined using a Banach algebra instead of a Banach space. We get some consequences related to some special properties of mappings.

2. Basic facts and definitions

We give some facts and definitions which we need in the sequel.

Let B be a real Banach space and K a subset of B. Then K is called a *cone* if and only if

1. K is closed, nonempty and $K \neq \{0\}$,
2. $a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in K \implies ax + by \in K$,
3. $x \in K$ and $-x \in K \implies x = 0$.

If we take a Banach algebra instead of Banach space, then we say that K is a *BA-cone*.

Given a cone $K \subset B$, we define a partial ordering $\leq$ with respect to K by $x \leq y$ if and only if $y - x \in K$. We write $x < y$ if $x \leq y$ but $x \neq y$; $x \ll y$ if $y - x \in \operatorname{int} K$, where $\operatorname{int} K$ is the interior of K. The cone K is called *normal* if there is a number $M > 0$ such that for all $x, y \in B$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq M \|y\| . \tag{2.1}$$

2.1. Definition. [6] Let X be nonempty set, B a real Banach space and $K \subset B$ a cone. Suppose the mapping $d : X \times X \to B$ satisfies

d1. $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
d2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
d3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric on* X and (X, d) is called a *cone metric space*. It is obvious that the concept of a cone metric space is more general than that of a metric space.

If we replace the Banach space with a Banach algebra in Definition 2.1 [6] then we obtain a new space which is called a *BA-cone metric space*.

2.2. Example. Let $B = R^2$, $K = \{(x, y) : x, y \geq 0\}$, $X = R$ and let $d : X \times X \to B$ be defined by $d(x, y) = (|x - y|, \alpha |x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a BA-cone metric space since B is a real commutative Banach algebra.

2.3. Example. Let $C_R^2([0, 1])$ be the space of all real functions on $[0, 1]$ whose second derivative is continuous. We recall that for $a, b > 0$, the space $C_R^2([0, 1])$ with the norm

$$\|f\| = \|f\|_\infty + a \|f'\|_\infty + b \|f''\|_\infty$$

is a Banach space, where $\|f\|_\infty = \sup_{t \in [0,1]} |f(t)|$. This space is a Banach algebra if and only if $2b \leq a^2$, see [17, page 272].

If we take $X = B = C_R^2([0,1])$ with the above norm and $K = \{u \in B : u \geq 0\}$, then (X,d) becomes a cone metric space where $d(x,y) = \left(\sup_{t\in[0,1]} |x(t) - y(t)| \right) f(t)$ and $f : [0,1] \to R$, $f(t) = e^t$. But if we take $2b > a^2$ then B is not Banach Algebra, hence (X,d) is not a BA-cone metric space.

2.4. Definition. Let (X,d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$. If for every $c \in B$ with $0 \ll c$,

1. there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \ll c$ then $\{x_n\}$ is said to be *convergent*,
2. there is $N \in \mathbb{N}$ such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a *Cauchy sequence* in X.

A cone metric space X is said to be *complete* if every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

2.5. Remark. [8] Let us recall that if X is a normal cone, $x \in K$, $a \in \mathbb{R}$, $a \in [0,1)$ and $x \leq ax$, then $x = 0$.

Let $f : X \to X$ and $x_0 \in X$. The function f is continuous at x_0 if for any sequence $x_n \to x_0$ we have $f(x_n) \to f(x_0)$.

Throughout the paper, we take B to be a Banach commutative division algebra. Recall that, a division algebra is an algebra with identity e, in which every non-zero element is a unit, where the identity is a non-zero element such that $xe = ex = x$ for all x and in any algebra with identity e, an element which has an inverse is called a unit, i.e. x is a unit if and only if there exists an inverse y such that $xy = yx = e$. We write $y = x^{-1}$ and observe that x^{-1} is unique when it exists [15].

Also, throughout we will use a cone which has non empty interior (i.e. $\text{int}K \neq \emptyset$). Therefore the uniqueness of the limit for a convergence sequence will be guaranteed.

3. Main results

In the following theorem we carry [3, Theorem 1] over to BA-cone metric spaces.

3.1. Theorem. *Let (X,d) be a BA-complete cone metric space, K a BA-normal cone with normal constant M. Suppose the mappings S and T are two self maps of X such that S and T satisfy the inequality*

$$d(Sx,Ty) \leq \alpha \, \frac{d(x,Sx)\, d(x,Ty) + [d(x,y)]^2 + d(x,Sx)\, d(x,y)}{d(x,Sx) + d(x,y) + d(x,Ty)} \tag{3.1}$$

for all x, y in X with $x \neq y$, $0 < \alpha < 1$ and $d(x,Sx) + d(x,y) + d(x,Ty) \neq 0$. Then S and T have a common fixed point. Further if $d(x,Sx) + d(x,y) + d(x,Ty) = 0$ implies $d(Sx,Ty) = 0$, then S and T have a unique common fixed point.

Proof. Let an x_0 be arbitrary point of X, and define $\{x_n\}$ by

$$x_{2n+2} = Tx_{2n+1},$$
$$x_{2n+1} = Sx_{2n}, \; n = 0, 1, 2, \ldots.$$

Let $d(x,Sx)+d(x,y)+d(x,Ty)\neq 0$. Then using (3.1),

$$\begin{aligned}
&d(x_{2n+1},x_{2n+2})\\
&\quad= d(Sx_{2n},Tx_{2n+1})\\
&\quad\le \alpha\left\{\frac{d(x_{2n},Sx_{2n})\,d(x_{2n},Tx_{2n+1})+[d(x_{2n},x_{2n+1})]^2}{d(x_{2n},Sx_{2n})+d(x_{2n},x_{2n+1})+d(x_{2n},Tx_{2n+1})}\right.\\
&\qquad\left.+\frac{d(x_{2n},Sx_{2n})\,d(x_{2n},x_{2n+1})}{d(x_{2n},Sx_{2n})+d(x_{2n},x_{2n+1})+d(x_{2n},Tx_{2n+1})}\right\}\\
&\quad= \alpha\left\{\frac{d(x_{2n},x_{2n+1})\,d(x_{2n},x_{2n+2})+[d(x_{2n},x_{2n+1})]^2}{d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+2})}\right.\\
&\qquad\left.+\frac{d(x_{2n},x_{2n+1})\,d(x_{2n},x_{2n+1})}{d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+2})}\right\}\\
&\quad= d(x_{2n},x_{2n+1})\left\{\alpha\frac{d(x_{2n},x_{2n+2})+d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+1})}{d(x_{2n},x_{2n+2})+d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+1})}\right\}.
\end{aligned}$$

Hence,

$$d(x_{2n+1},x_{2n+2})\le \alpha d(x_{2n},x_{2n+1}).$$

Similarly;

$$\begin{aligned}
d(x_{2n},x_{2n+1}) &= d(Sx_{2n},Tx_{2n-1})\\
&\le \alpha\left\{\frac{d(x_{2n},Sx_{2n})\,d(x_{2n},Tx_{2n-1})+[d(x_{2n},x_{2n-1})]^2}{d(x_{2n},Sx_{2n})+d(x_{2n},x_{2n-1})+d(x_{2n},Tx_{2n-1})}\right.\\
&\quad\left.+\frac{d(x_{2n},Sx_{2n})\,d(x_{2n},x_{2n-1})}{d(x_{2n},Sx_{2n})+d(x_{2n},x_{2n-1})+d(x_{2n},Tx_{2n-1})}\right\}.
\end{aligned}$$

Hence,

$$d(x_{2n},x_{2n+1})\le \alpha\ d(x_{2n-1},x_{2n}).$$

By this way, if we continue, we get

$$d(x_{2n+1},x_{2n+2})\le \alpha d(x_{2n},x_{2n+1})\le \alpha^2 d(x_{2n-1},x_{2n})\le\cdots\le \alpha^{2n+1}d(x_0,x_1).$$

It is obvious that the following inequality holds for $m>n$.

$$\begin{aligned}
d(x_n,x_{n+m}) &\le \sum_{i=1}^{m} d(x_{n+i-1},x_{n+i})\\
&\le \sum_{i=1}^{m}\alpha^{n+i-1}d(x_0,x_1)\\
&\le \frac{\alpha^n}{1-\alpha}d(x_0,x_1).
\end{aligned}$$

By (2.1),

$$\|d(x_n,x_{n+m})\|\le M\frac{\alpha^n}{1-\alpha}\|d(x_0,x_1)\|,$$

which implies that $d(x_n,x_{n+m})\to 0$ as $n\to\infty$. Hence, $\{x_n\}$ is a Cauchy sequence, so by the completeness of X this sequence must be convergent in X. Let z be the limit of $\{x_n\}$, i.e. $x_n\to z$ as $n\to\infty$.

Now if we assume $z \neq Tz$, then $d(z, Tz) > 0$. If we use the triangle inequality and Inequality (3.1) we have

$$\begin{aligned} &d(z, Tz) \\ &\quad \leq d(z, x_{2n+1}) + d(x_{2n+1}, Tz) \\ &\quad = d(z, x_{2n+1}) + d(Sx_{2n}, Tz) \\ &\quad \leq d(z, x_{2n+1}) \\ &\qquad + \alpha \, \frac{d(x_{2n}, Sx_{2n})\, d(x_{2n}, Tz) + [d(x_{2n}, z)]^2 + d(x_{2n}, Sx_{2n})\, d(x_{2n}, z)}{d(x_{2n}, Sx_{2n}) + d(x_{2n}, z) + d(x_{2n}, Tz)} \\ &\quad \leq d(z, x_{2n+1}) \\ &\qquad + \alpha \, \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n}, Tz) + [d(x_{2n}, z)]^2 + d(x_{2n}, x_{2n+1})\, d(x_{2n}, z)}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, z) + d(x_{2n}, Tz)}, \end{aligned}$$

so using the condition of a normal cone;

$$\begin{aligned} &\|d(z, Tz)\| \\ &\quad \leq M \Bigg\{ \|d(z, x_{2n+1})\| \\ &\qquad + \alpha \left\| \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n}, Tz) + [d(x_{2n}, z)]^2 + d(x_{2n}, x_{2n+1})\, d(x_{2n}, z)}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, z) + d(x_{2n}, Tz)} \right\| \Bigg\} \end{aligned}$$

As $n \to \infty$, we have

$$\|d(z, Tz)\| \leq 0,$$

which is a contradiction. Hence, we get $z = Tz$; i.e. z is a fixed point of T.

Similarly; let us suppose that $z \neq Sz$, and $d(z, Sz) > 0$.

$$\begin{aligned} d(z, Sz) &\leq d(z, x_{2n+2}) + d(x_{2n+2}, Sz) \\ &= d(z, x_{2n+2}) + d(Sz, Tx_{2n+1}) \\ &\leq d(z, x_{2n+2}) \\ &\qquad + \alpha \frac{d(z, Sz)\, d(z, Tx_{2n+1}) + [d(z, x_{2n+1})]^2 + d(z, Sz)\, d(z, x_{2n+1})}{d(z, Sz) + d(z, x_{2n+1}) + d(z, Tx_{2n+1})} \\ &= d(z, x_{2n+2}) \\ &\qquad + \alpha \frac{d(z, Sz)\, d(z, x_{2n+2}) + [d(z, x_{2n+1})]^2 + d(z, Sz)\, d(z, x_{2n+1})}{d(z, Sz) + d(z, x_{2n+1}) + d(z, x_{2n+2})}, \end{aligned}$$

so by (2.1),

$$\begin{aligned} &\|d(z, Sz)\| \\ &\quad \leq M \{ \|d(z, x_{2n+2})\| \\ &\qquad + \alpha \left\| \frac{d(z, Sz)\, d(z, x_{2n+2}) + [d(z, x_{2n+1})]^2 + d(z, Sz)\, d(z, x_{2n+1})}{d(z, Sz) + d(z, x_{2n+1}) + d(z, x_{2n+2})} \right\| \Bigg\}. \end{aligned}$$

Hence,

$$\|d(z, Sz)\| \leq 0,$$

a contradiction. Therefore $d(z, Sz) = 0$ and so $z = Sz$, i.e. z is a fixed point of S.

Hence we find that z is a common fixed point of S and T.

For the uniqueness of z, let us suppose that $d(x, Sx) + d(x, y) + d(x, Ty) = 0$ implies $d(Sx, Ty) = 0$ and that w is another fixed point of T in X. Then,

$$d(z, Sz) + d(z, w) + d(z, Tw) = 0 \text{ implies } d(Sz, Tw) = 0.$$

Therefore, we get

$$d(z,w)=d(Sz,Tw)=0,$$

which implies that $z=w$, and this is the desired consequence. $\square$

If S is a map which has a fixed point z, then z is a fixed point of S^n for every $n\in\mathbb{N}$ too. However, the converse need not to be true. Jeong and Rhoades [9] discussed this situation and gave examples for metric spaces, while Abbas and Rhoades [1] examined this for cone metric spaces. If a map satisfies $F(S)=F(S^n)$ for each $n\in\mathbb{N}$ then it is said to have property P. If $F(S^n)\cap F(T^n)=F(S)\cap F(T)$ then we say that S and T have property Q.

We examine the property Q for those mappings which satisfy Inequality (3.1).

3.2. Theorem. *Let (X,d) be a BA-complete cone metric space and K a BA-normal cone with normal constant M. Suppose the self mappings S and T in X satisfy* (3.1). *Then S and T have property Q.*

Proof. By the above theorem, we know that S and T have a common fixed point in X. Let $z\in F(S^n)\cap F(T^n)$. Then;

$$\begin{aligned}
d(z,Tz)&=d\left(S^nz,T^{n+1}z\right)=d\left(S\left(S^{n-1}z\right),T\left(T^nz\right)\right)\\
&\le\alpha\left\{\frac{d\left(S^{n-1}z,S^nz\right)d\left(S^{n-1}z,T^{n+1}z\right)+\left[d\left(S^{n-1}z,T^nz\right)\right]^2}{d\left(S^{n-1}z,S^nz\right)+d\left(S^{n-1}z,T^nz\right)+d\left(S^{n-1}z,T^{n+1}z\right)}\right.\\
&\qquad\left.+\frac{d\left(S^{n-1}z,S^nz\right)d\left(S^{n-1}z,T^nz\right)}{d\left(S^{n-1}z,S^nz\right)+d\left(S^{n-1}z,T^nz\right)+d\left(S^{n-1}z,T^{n+1}z\right)}\right\}\\
&\le\alpha\left\{\frac{d\left(S^{n-1}z,z\right)d\left(S^{n-1}z,Tz\right)+\left[d\left(S^{n-1}z,z\right)\right]^2}{d\left(S^{n-1}z,z\right)+d\left(S^{n-1}z,z\right)+d\left(S^{n-1}z,Tz\right)}\right.\\
&\qquad\left.+\frac{d\left(S^{n-1}z,z\right)d\left(S^{n-1}z,z\right)}{d\left(S^{n-1}z,z\right)+d\left(S^{n-1}z,z\right)+d\left(S^{n-1}z,Tz\right)}\right\},
\end{aligned}$$

and,

$$\begin{aligned}
d\left(S^nz,T^{n+1}z\right)&\le\alpha\frac{d\left(S^{n-1}z,z\right)\left[d\left(S^{n-1}z,Tz\right)+2d\left(S^{n-1}z,z\right)\right]}{\left[d\left(S^{n-1}z,Tz\right)+2d\left(S^{n-1}z,z\right)\right]}\\
&=\alpha d\left(S^{n-1}z,T^nz\right)
\end{aligned}$$

Similarly;

$$\begin{aligned}
d\left(S^nz,T^{n+1}z\right)&\le\alpha d\left(S^{n-1}z,T^nz\right)=d\left(S\left(S^{n-2}z\right),T\left(T^{n-1}z\right)\right)\\
&\le\alpha\left\{\frac{d\left(S^{n-2}z,S^{n-1}z\right)d\left(S^{n-2}z,T^nz\right)+\left[d\left(S^{n-2}z,T^{n-1}z\right)\right]^2}{d\left(S^{n-2}z,S^{n-1}z\right)+d\left(S^{n-2}z,T^{n-1}z\right)+d\left(S^{n-2}z,T^nz\right)}\right.\\
&\qquad\left.+\frac{d\left(S^{n-2}z,S^{n-1}z\right)d\left(S^{n-2}z,T^{n-1}z\right)}{d\left(S^{n-2}z,S^{n-1}z\right)+d\left(S^{n-2}z,T^{n-1}z\right)+d\left(S^{n-2}z,T^nz\right)}\right\}\\
&=\alpha\left\{\frac{d\left(S^{n-2}z,S^{n-1}z\right)d\left(S^{n-2}z,T^nz\right)+\left[d\left(S^{n-2}z,S^{n-1}z\right)\right]^2}{d\left(S^{n-2}z,S^{n-1}z\right)+d\left(S^{n-2}z,S^{n-1}z\right)+d\left(S^{n-2}z,T^nz\right)}\right.\\
&\qquad\left.+\frac{d\left(S^{n-2}z,S^{n-1}z\right)d\left(S^{n-2}z,S^{n-1}z\right)}{d\left(S^{n-2}z,S^{n-1}z\right)+d\left(S^{n-2}z,S^{n-1}z\right)+d\left(S^{n-2}z,T^nz\right)}\right\},
\end{aligned}$$

and,

$$d\left(S^{n-1}z,T^{n}z\right)\leq\alpha\frac{d\left(S^{n-2}z,S^{n-1}z\right)\left[d\left(S^{n-2}z,T^{n}z\right)+2d\left(S^{n-2}z,S^{n-1}z\right)\right]}{\left[d\left(S^{n-2}z,T^{n}z\right)+2d\left(S^{n-2}z,S^{n-1}z\right)\right]}$$
$$=\alpha d\left(S^{n-2}z,T^{n-1}z\right).$$

In this way we get that

$$d\left(S^{n}z,T^{n+1}z\right)\leq\alpha d\left(S^{n-1}z,T^{n}z\right)\leq\alpha^{2}d\left(S^{n-2}z,T^{n-1}z\right)\leq\cdots\leq\alpha^{n}d\left(z,Tz\right),$$
$$d\left(z,Tz\right)\leq\alpha^{n}\ d\left(z,Tz\right),$$

and using (2.1) this inequality implies that

$$\left\|d\left(z,Tz\right)\right\|\leq M\alpha^{n}\left\|d\left(z,Tz\right)\right\|$$

as $n\to\infty$, $\|d(z,Tz)\|=0$; so $z=Tz$. By using Theorem 3.1 we get $z=Sz$, and consequently S and T have the property Q. □

3.3. Theorem. *Let T be a self mapping of a BA-complete cone metric space (X,d) with BA-normal cone K having normal constant M, which satisfies the inequality*

$$d\left(Tx,Ty\right)\leq\alpha\ \frac{d\left(x,Tx\right)d\left(x,Ty\right)+\left[d\left(x,y\right)\right]^{2}+d\left(x,Tx\right)d\left(x,y\right)}{d\left(x,Tx\right)+d\left(x,y\right)+d\left(x,Ty\right)}\tag{3.2}$$

for all x,y in X with $x\neq y$, $0<\alpha<1$ and $d(x,Tx)+d(x,y)+d(x,Ty)\neq 0$. Then T has a fixed point. Further, if $d(x,Tx)+d(x,y)+d(x,Ty)=0$ implies $d(Tx,Ty)=0$, then T has a unique fixed point.

Proof. If we take $S=T$ in Theorem 3.1 we obtain the proof. □

3.4. Theorem. *Let (X,d) be a BA-complete cone metric space and K a BA-normal cone with normal constant M. Suppose the self mapping T in X satisfies* (3.2)*. Then T has the property P.*

Proof. Let $z\in F(T^{n})$. Then

$$\begin{aligned}d\left(z,Tz\right)&=d\left(T^{n}z,T^{n+1}z\right)\\&\leq\alpha\left[\frac{d\left(T^{n-1}z,T^{n}z\right)d\left(T^{n-1}z,T^{n+1}z\right)+\left[d\left(T^{n-1}z,T^{n}z\right)\right]^{2}}{d\left(T^{n-1}z,T^{n}z\right)+d\left(T^{n-1}z,T^{n}z\right)+d\left(T^{n-1}z,T^{n+1}z\right)}\right.\\&\qquad\left.+\frac{d\left(T^{n-1}z,T^{n}z\right)d\left(T^{n-1}z,T^{n}z\right)}{d\left(T^{n-1}z,T^{n}z\right)+d\left(T^{n-1}z,T^{n}z\right)+d\left(T^{n-1}z,T^{n+1}z\right)}\right]\\&\leq\alpha\left[\frac{d\left(T^{n-1}z,z\right)d\left(T^{n-1}z,Tz\right)+\left[d\left(T^{n-1}z,z\right)\right]^{2}}{d\left(T^{n-1}z,z\right)+d\left(T^{n-1}z,z\right)+d\left(T^{n-1}z,Tz\right)}\right.\\&\qquad\left.+\frac{d\left(T^{n-1}z,z\right)d\left(T^{n-1}z,z\right)}{d\left(T^{n-1}z,z\right)+d\left(T^{n-1}z,z\right)+d\left(T^{n-1}z,Tz\right)}\right]\\&\leq\alpha d\left(T^{n-1}z,z\right)\left[\frac{d\left(T^{n-1}z,Tz\right)+d\left(T^{n-1}z,z\right)+d\left(T^{n-1}z,z\right)}{d\left(T^{n-1}z,z\right)+d\left(T^{n-1}z,z\right)+d\left(T^{n-1}z,Tz\right)}\right],\end{aligned}$$

and

$$d\left(T^{n}z,T^{n+1}z\right)\leq\alpha d\left(T^{n-1}z,T^{n}z\right).$$

Similarly,

$$\begin{aligned} d\left(T^{n-1}z,T^{n}z\right) &= d\left(T\left(T^{n-2}z\right),T\left(T^{n-1}z\right)\right) \\ &\leq \alpha\left[\frac{d\left(T^{n-2}z,T^{n-1}z\right)d\left(T^{n-2}z,T^{n}z\right)+\left[d\left(T^{n-2}z,T^{n-1}z\right)\right]^{2}}{d\left(T^{n-2}z,T^{n-1}z\right)+d\left(T^{n-2}z,T^{n-1}z\right)+d\left(T^{n-2}z,T^{n}z\right)}\right. \\ &\qquad\left.+\frac{d\left(T^{n-2}z,T^{n-1}z\right)d\left(T^{n-2}z,T^{n-1}z\right)}{d\left(T^{n-2}z,T^{n-1}z\right)+d\left(T^{n-2}z,T^{n-1}z\right)+d\left(T^{n-2}z,T^{n}z\right)}\right] \\ &\leq \alpha d\left(T^{n-2}z,T^{n-1}z\right), \end{aligned}$$

so

$$d\left(T^{n}z,T^{n+1}z\right)\leq \alpha d\left(T^{n-1}z,T^{n}z\right)\leq \alpha^{2}d\left(T^{n-2}z,T^{n-1}z\right)\leq\cdots\leq\alpha^{n}d\left(z,Tz\right),$$

and

$$d\left(z,Tz\right)\leq\alpha^{n}d\left(z,Tz\right).$$

If we use Inequality (2.1),

$$\left\|d\left(z,Tz\right)\right\|\leq M\alpha^{n}\left\|d\left(z,Tz\right)\right\|.$$

The right hand side of the above inequality tends to zero as $n\to\infty$, $\|d(z,Tz)\|=0$, i.e. $z=Tz$. We conclude that a mapping which satisfies the (3.2) has property P. □

Finally, we give a new theorem for four mappings. To prove the theorem we need definitions which were given for metric spaces in [11] and [18].

3.5. Definition. Two self mappings S and T of a cone metric space (X,d) are said to be *weakly commuting* if the following is satisfied for all $x\in X$;

$$d\left(STx,TSx\right)\leq d\left(Sx,Tx\right).$$

3.6. Definition. Let S and T be self mappings of a cone metric space (X,d) with a normal cone K. Then $\{S,T\}$ are said to be compatible if

$$\lim_{n\to\infty}d\left(STx_n,TSx_n\right)=0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Tx_n=\lim_{n\to\infty}Sx_n=w$ for some w in X.

3.7. Theorem. *Let (X,d) be a BA-complete cone metric space and K its BA-normal cone with normal constant M, and let $\{S,I\}$ and $\{T,J\}$ be weakly commuting pairs of self mappings satisfying the following:*

(1) *$T(X)\subset I(X)$, $S(X)\subset J(X)$.*
(2) *For all x,y in X; either*

$$\begin{aligned} &d\left(Sx,Ty\right) \\ &\leq\alpha\left\{\frac{d\left(Ix,Sx\right)d\left(Ix,Ty\right)+\left[d\left(Ix,Jy\right)\right]^{2}+d\left(Ix,Sx\right)d\left(Ix,Jy\right)}{d\left(Ix,Sx\right)+d\left(Ix,Jy\right)+d\left(Ix,Ty\right)}\right\} \\ &\qquad+\beta d\left(Ix,Jy\right) \end{aligned} \tag{3.3}$$

if $d(Ix,Sx)+d(Ix,Jy)+d(Ix,Ty)\neq0$, where $\alpha+\beta<1$ and $\beta<1$, or

$$d\left(Sx,Ty\right)=0$$

if $d(Ix,Sx)+d(Ix,Jy)+d(Ix,Ty)=0$.

If any of S,T,I or J is continuous then S,T,I and J have a unique common fixed point z. Furthermore, z is the unique common fixed point of S and I as well as of T and J.

Proof. Take x_0 as an arbitrary point of X. Since $S(X) \subset J(X)$ we can find a point x_1 in X such that $Sx_0 = Jx_1$. Also, since $T(X) \subset I(X)$ we can choose a point x_2 with $Tx_1 = Ix_2$. In general; for the point x_{2n} we can pick up a point x_{2n+1} such that $Sx_{2n} = Jx_{2n+1}$, and then a point x_{2n+2} with $Tx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, \ldots$.

Let us form $D_{2n} = d(Sx_{2n}, Tx_{2n+1})$ and $D_{2n+1} = d(Sx_{2n+2}, Tx_{2n+1})$.

Suppose $D_{2n} = d(Sx_{2n}, Tx_{2n+1}) \neq 0$ and $D_{2n+1} = d(Sx_{2n+2}, Tx_{2n+1}) \neq 0$ for $n = 1, \ldots$. Then if we employ Inequality (3.3) we have

$$\begin{aligned}
D_{2n+1} &= d(Sx_{2n+2}, Tx_{2n+1}) \\
&\leq \alpha \left\{ \frac{d(Ix_{2n+2}, Sx_{2n+2})\, d(Ix_{2n+2}, Tx_{2n+1}) + [d(Ix_{2n+2}, Jx_{2n+1})]^2}{d(Ix_{2n+2}, Sx_{2n+2}) + d(Ix_{2n+2}, Jx_{2n+1}) + d(Ix_{2n+2}, Tx_{2n+1})} \right. \\
&\qquad \left. + \frac{d(Ix_{2n+2}, Sx_{2n+2})\, d(Ix_{2n+2}, Jx_{2n+1})}{d(Ix_{2n+2}, Sx_{2n+2}) + d(Ix_{2n+2}, Jx_{2n+1}) + d(Ix_{2n+2}, Tx_{2n+1})} \right\} \\
&\qquad + \beta d(Ix_{2n+2}, Jx_{2n+1}) \\
&= \alpha \left\{ \frac{d(Tx_{2n+1}, Sx_{2n+2})\, d(Tx_{2n+1}, Tx_{2n+1}) + [d(Tx_{2n+1}, Sx_{2n})]^2}{d(Tx_{2n+1}, Sx_{2n+2}) + d(Tx_{2n+1}, Sx_{2n}) + d(Tx_{2n+1}, Tx_{2n+1})} \right. \\
&\qquad \left. + \frac{d(Tx_{2n+1}, Sx_{2n+2})\, d(Tx_{2n+1}, Sx_{2n})}{d(Tx_{2n+1}, Sx_{2n+2}) + d(Tx_{2n+1}, Sx_{2n}) + d(Tx_{2n+1}, Tx_{2n+1})} \right\} \\
&\qquad + \beta\, d(Tx_{2n+1}, Sx_{2n}) \\
&= \alpha \frac{d(Tx_{2n+1}, Sx_{2n})\, [d(Tx_{2n+1}, Sx_{2n}) + d(Tx_{2n+1}, Sx_{2n+2})]}{[d(Tx_{2n+1}, Sx_{2n}) + d(Tx_{2n+1}, Sx_{2n+2})]} \\
&\leq (\alpha + \beta)\, d(Tx_{2n+1}, Sx_{2n}),
\end{aligned}$$

which implies that

$$D_{2n+1} \leq \lambda D_{2n} \leq \lambda^2 D_{2n-1} \leq \cdots \leq \lambda^{2n+1} D_0,$$

where $\lambda = \alpha + \beta < 1$. Using (2.1),

$$\|D_{2n+1}\| \leq M \lambda^{2n+1} \|D_0\|.$$

In this inequality, $\|D_{2n+1}\| \to 0$ as $n \to \infty$, so $d(Sx_{2n+2}, Tx_{2n+1}) \to 0$ as $n \to \infty$. We get the following sequence

$$\{Sx_0,\ Tx_1,\ Sx_2,\ Tx_3,\ \ldots,\ Sx_{2n},\ Tx_{2n+1},\ \ldots\} \tag{3.4}$$

which is a Cauchy sequence in the complete cone metric space (X, d), and therefore converges a limit point $z \in X$. Therefore the sequences

$$\begin{aligned}
\{Sx_{2n}\} &= \{Jx_{2n+1}\} \\
\{Tx_{2n-1}\} &= \{Ix_{2n}\}
\end{aligned}$$

which are subsequences of (3.4) and hence also converge to the same point $z \in X$.

Let assume that I is continuous so that the sequences $\{I^2 x_{2n}\}$ and $\{ISx_{2n}\}$ converge to the same point Iz. We know that S and I are weakly commuting so we have;

$$d(SIx_{2n}, ISx_{2n}) \leq d(Ix_{2n}, Sx_{2n}),$$

and using (2.1)

$$\|d(SIx_{2n}, ISx_{2n})\| \leq M \|d(Ix_{2n}, Sx_{2n})\|$$

as $n \to \infty$. Hence the sequence $\{SIx_{2n}\}$ converges to the point Iz.

If we employ the triangle property and Inequality (3.3), we get

$$\begin{aligned} d(Iz,z) &\leq d(Iz,SIx_{2n}) + d(SIx_{2n},Tx_{2n+1}) + d(Tx_{2n+1},z) \\ &\leq d(Iz,SIx_{2n}) \\ &\quad + \alpha\left\{\frac{d\left(I^2x_{2n},SIx_{2n}\right)d\left(I^2x_{2n},Tx_{2n+1}\right)+\left[d\left(I^2x_{2n},Jx_{2n+1}\right)\right]^2}{d\left(I^2x_{2n},SIx_{2n}\right)+d\left(I^2x_{2n},Jx_{2n+1}\right)+d\left(I^2x_{2n},Tx_{2n+1}\right)}\right. \\ &\quad\quad \left.+\frac{d\left(I^2x_{2n},SIx_{2n}\right)d\left(I^2x_{2n},Jx_{2n+1}\right)}{d\left(I^2x_{2n},SIx_{2n}\right)+d\left(I^2x_{2n},Jx_{2n+1}\right)+d\left(I^2x_{2n},Tx_{2n+1}\right)}\right\} \\ &\quad\quad + \beta d\left(I^2x_{2n},Jx_{2n+1}\right) + d(Tx_{2n+1},z), \end{aligned}$$

which with Inequality (2.1) gives

$$\begin{aligned} \|d(Iz,z)\| &\leq M\left\{\|d(Iz,SIx_{2n})\|\right. \\ &\quad + \left\|\alpha\frac{d\left(I^2x_{2n},SIx_{2n}\right)d\left(I^2x_{2n},Tx_{2n+1}\right)+\left[d\left(I^2x_{2n},Jx_{2n+1}\right)\right]^2}{d\left(I^2x_{2n},SIx_{2n}\right)+d\left(I^2x_{2n},Jx_{2n+1}\right)+d\left(I^2x_{2n},Tx_{2n+1}\right)}\right\| \\ &\quad\quad + \left\|\frac{d\left(I^2x_{2n},SIx_{2n}\right)d\left(I^2x_{2n},Jx_{2n+1}\right)}{d\left(I^2x_{2n},SIx_{2n}\right)+d\left(I^2x_{2n},Jx_{2n+1}\right)+d\left(I^2x_{2n},Tx_{2n+1}\right)}\right\| \\ &\quad\quad + \beta\ \left\|d\left(I^2x_{2n},Jx_{2n+1}\right)\right\| + \left.\|d(Tx_{2n+1},z)\|\right\} \\ &= M\left\{\alpha\frac{\|d(Iz,z)\,d(Iz,z)\|}{2\,\|d(Iz,z)\|}\right\}, \end{aligned}$$

so

$$\|d(Iz,z)\| \leq M\left(\frac{\alpha}{2}+\beta\right)\|d(Iz,z)\|.$$

Hence $\|d(Iz,z)\| = 0$, and $Iz = z$. We want to show that $Sz = z$, too. Using the same inequalities, we have

$$\begin{aligned} & d(Sz,z) \\ &\leq d(Sz,Tx_{2n+1}) + d(Tx_{2n+1},z) \\ &\leq \alpha\left\{\frac{d(Iz,Sz)\,d(Iz,Tx_{2n+1}) + [d(Iz,Jx_{2n+1})]^2 + d(Iz,Sz)\,d(Iz,Jx_{2n+1})}{d(Iz,Sz)+d(Iz,Jx_{2n+1})+d(Iz,Tx_{2n+1})}\right\} \\ &\qquad\qquad + \beta d(Iz,Jx_{2n+1}) + d(Tx_{2n+1},z), \end{aligned}$$

and again if (2.1) is used;

$$\begin{aligned} & \|d(Sz,z)\| \\ &\leq M\left\{\left\|\alpha\left\{\frac{d(Iz,Sz)\,d(Iz,Tx_{2n+1}) + [d(Iz,Jx_{2n+1})]^2 + d(Iz,Sz)\,d(Iz,Jx_{2n+1})}{d(Iz,Sz)+d(Iz,Jx_{2n+1})+d(Iz,Tx_{2n+1})}\right\}\right\|\right. \\ &\qquad\qquad \left. + \beta\,\|d(Iz,Jx_{2n+1})\| + \|d(Tx_{2n+1},z)\|\right\} \end{aligned}$$

and, as n tends to infinity,

$$= M\left\{\alpha\frac{d(z,Sz)\,d(z,z) + [d(z,z)]^2 + d(z,Sz)\,d(z,z)}{d(z,Sz)+d(z,z)+d(z,z)} + \beta d(z,z) + d(z,z)\right\}.$$

Then, $\|d(Sz,z)\| = 0$ and hence $Sz = z$.

We have seen that $Sz = z$, and we know that $S(X) \subset J(X)$ so we can always find a point w such that $Jw = z$. Thus,

$$\begin{aligned} d(z, Tw) &= d(Sz, Tw) \\ &\leq \alpha \left\{ \frac{d(Iz, Sz)\, d(Iz, Tw) + [d(Iz, Jw)]^2 + d(Iz, Sz)\, d(Iz, Jw)}{d(Iz, Sz) + d(Iz, Jw) + d(Iz, Tw)} \right\} \\ &\qquad + \beta d(Iz, Jw), \end{aligned}$$

so that $d(z, Tw) = 0$, $Tw = z$.

Since T and J weakly commute

$$d(Tz, Jz) = d(TJw, JTw) \leq d(Jw, Tw) = d(z, z) = 0,$$

which gives $Tz = Jz$, and so

$$\begin{aligned} d(z, Tz) &= d(Sz, Tz) \\ &\leq \alpha \left\{ \frac{d(Iz, Sz)\, d(Iz, Tz) + [d(Iz, Jz)]^2 + d(Iz, Sz)\, d(Iz, Jz)}{d(Iz, Sz) + d(Iz, Jz) + d(Iz, Tz)} \right\} \\ &\qquad + \beta d(Iz, Jz), \end{aligned}$$

which gives

$$d(z, Tz) \leq \left(\frac{\alpha}{2} + \beta\right) d(z, Tz).$$

By using Remark 2.5 we get that $z = Tz$, consequently this yields $Tz = Jz = z$.

Thereby we have proved that the mappings S, T, I and J have a common fixed point.

The proof is the same if one of the mappings S, T, J is continuous instead of I.

To show that z is unique, let u be another common fixed point of S and I. Then

$$\begin{aligned} d(u, z) &= d(Su, Tz) \\ &\leq \alpha \left\{ \frac{d(Iu, Su)\, d(Iu, Tz) + [d(Iu, Jz)]^2 + d(Iu, Su)\, d(Iu, Jz)}{d(Iu, Su) + d(Iu, Jz) + d(Iu, Tz)} \right\} \\ &\qquad + \beta\, d(Iu, Jz), \end{aligned}$$

so

$$d(u, z) \leq \left(\frac{\alpha}{2} + \beta\right) d(u, z).$$

Using Remark 2.5 again we get $u = z$. In the same way it can be show that z is the unique fixed point for the mappings T and J. □

3.8. Remark. Weakly commuting mappings are obviously compatible, but the converse need not to be true. So, the condition weak commutativity can be replaced with compatibility with the same assumptions in the theorem.

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