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ON GENERALIZED FUZZY INTERIOR IDEALS IN Γ-SEMIGROUPS

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Abstract

In this paper, using the idea of quasi-coincidence of a fuzzy point with a fuzzy set, the concept of $(\in, \in \vee q)$- fuzzy interior ideals in a Γ-semigroup has been introduced and some of their important related properties have been investigated. Characterizations of regular Γ-semigroups, intra-regular Γ-semigroups, and semisimple Γ-semigroups in terms of $(\in, \in \vee q)$-fuzzy interior ideals have also been obtained.

Keywords: Γ-semigroup, Belonging to or quasi-coincident with, $(\in, \in \vee q)$-fuzzy ideal, $(\in, \in \vee q)$-fuzzy interior ideal, Regular Γ-semigroup, Intra-regular Γ-semigroup, Semisimple Γ-semigroup.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [36]. Since then many researchers have explored generalizations of fuzzy sets. Many papers on fuzzy sets have appeared showing the importance of the concept and its application to logic, set theory, group theory, semigroup theory, real analysis, measure theory, topology etc. It was first applied to the theory of groups by A. Rosenfeld [20]. In [34], Yuan *et al.* introduced the definition of fuzzy subgroup with thresholds, which is a generalization of Rosenfeld's fuzzy subgroup and Bhakat and Das's fuzzy subgroup. Murali [18] proposed a definition of fuzzy point belonging to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi coincidence of a fuzzy point with a fuzzy set, which is mentioned in [19], played a vital role in generating some different types of fuzzy subgroup. Bhakat and Das [2, 3]

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gave the concept of (α, β)-fuzzy subgroups by using the *belong to* relation $(\in)$ and *quasi-coincidence with* relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. In particular, the $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. We see the fuzzification of different concepts of semigroups in [15, 16, 17]. Yunqiang Yin and Dehua Xu [35] introduced the concepts of $(\in, \in \vee q)$-fuzzy subsemigroup and $(\in, \in \vee q)$-fuzzy ideals in semigroups. In [14], Y. B. Jun and S. Z. Song introduced the notion of generalized fuzzy interior ideals in semigroups. X. Huang and Y. Yin [13] introduced the notions of $(\in, \in \vee q)$-fuzzy prime ideals and $(\in, \in \vee q)$-fuzzy maximal ideals in a semigroup. Sen and Saha in [31] defined the concepts of Γ-semigroups as a generalization of semigroups and ternary semigroups.

Γ-semigroups have been analyzed by many mathematicians, for instance Chattopadhay [4, 5, 29], Dutta and Adhikari [7, 8], Hila [11, 12], Chinram [6], Saha [21, 30] and Seth [32]. Sardar and Majumder[22, 24, 28] characterized subsemigroups, bi-ideals, interior ideals (along with B. Davvaz [25]), quasi ideals, ideals, prime (along with D. Mandal [23]) and semiprime ideals, ideal extensions (along with T.K. Dutta [9, 10]) of a Γ-semigroup in terms of fuzzy subsets. They also studied their different properties directly and via operator semigroups of a Γ-semigroup. In [26, 27], S. K. Sardar, B. Davvaz, S. K. Majumder and S. Kayal introduced the concept of generalized fuzzy subsemigroup and generalized fuzzy bi-ideal, generalized fuzzy ideals in a Γ-semigroup. The purpose of this paper is as stated in the abstract.

2. Preliminaries

In this section we discuss some elementary definitions which will be used in the sequel.

A function μ from a non-empty set X to the unit interval $[0, 1]$ is called a *fuzzy subset* [36] of X.

Let $S = \{x, y, z, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two non-empty sets. Then S is called a Γ-*semigroup* [31] if there exists a mapping $S \times \Gamma \times S \to S$ (images to be denoted by $a\alpha b$) satisfying

(1) $x\gamma y \in S$,

(2) $(x\beta y)\gamma z = x\beta(y\gamma z)$, for all $x, y, z \in S$ and for all $\beta, \gamma \in \Gamma$.

2.1. Example. [25] Let $\Gamma = \{5, 7\}$. For any $x, y \in N$ and $\gamma \in \Gamma$, define $x\gamma y = x \cdot \gamma \cdot y$ where $\cdot$ is the usual multiplication on N. Then N is a Γ-semigroup.

A non-empty subset A of a Γ-semigroup S is called a *subsemigroup* [28] of S if $A\Gamma A \subseteq A$.

A non-empty subset A of a Γ-semigroup S is called a *left* (*right*) *ideal* [22] of S if $S\Gamma A \subseteq A$ $(A\Gamma S \subseteq A)$. A non-empty subset A of a Γ-semigroup S is called an *ideal* (*two sided ideal*) [22] of S if it is both a left ideal and right ideal of S.

A subsemigroup A of a Γ-semigroup S is called an *interior ideal* [25] of S if $S\Gamma A\Gamma S \subseteq A$.

A Γ-semigroup S is called *regular* [7], if for each $a \in S$, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

A Γ-semigroup S is called *intra-regular* [7], if for each $a \in S$, there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = x\alpha a\beta a\gamma y$.

A Γ-semigroup S is called *right* (*left*) *zero* [33] if $x\gamma y = y$ (resp. $x\gamma y = x$) $\forall\, x, y \in S$, $\forall\, \gamma \in \Gamma$.

A Γ-semigroup S is called *semisimple* [33] if for each $a \in S$ there exist $x, y, z \in S, \alpha, \beta, \gamma, \delta \in \Gamma$ such that $a = x\alpha a\beta y\gamma a\delta z$.

2.2. Definition. [18] A fuzzy subset μ of a set X of the form

$$\mu(y) = \begin{cases} t(\neq 0) & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point with support x and value t*, and is denoted by x_t.

2.3. Definition. [28] A non-empty fuzzy subset μ of a Γ-semigroup S is called a *fuzzy subsemigroup of S* if $\mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\} \forall x, y \in S,\ \forall \gamma \in \Gamma$.

2.4. Definition. [25] A non-empty fuzzy subset μ of a Γ-semigroup S is called a *fuzzy interior ideal of S* if

(I1) $\mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\}\ \forall x, y \in S,\ \gamma \in \Gamma$.
(I2) $\mu(x\alpha a\beta y) \geq \mu(a)\ \forall a, x, y \in S,\ \forall \alpha, \beta \in \Gamma$.

2.5. Definition. [22] A non-empty fuzzy subset μ of a Γ-semigroup S is called a *fuzzy left ideal of S* if $\mu(x\gamma y) \geq \mu(y)\ \forall x, y \in S,\ \forall \gamma \in \Gamma$.

2.6. Definition. [22] A non-empty fuzzy subset μ of a Γ-semigroup S is called a *fuzzy right ideal of S* if $\mu(x\gamma y) \geq \mu(x)\ \forall x, y \in S,\ \forall \gamma \in \Gamma$.

2.7. Definition. [22] A non-empty fuzzy subset μ of a Γ-semigroup S is called a *fuzzy ideal* or *fuzzy two-sided ideal of S* if it is both a fuzzy left ideal and a fuzzy right ideal of S.

2.8. Definition. [35] A fuzzy point x_t is said to *belong to* (*be quasi coincident with*) a fuzzy subset μ, written as $x_t \in \mu$(resp. $x_t q\mu$) if $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$).

$x_t \in \mu$ or $x_t q\mu$ will be denoted by $x_t \in \vee q\mu, x_t \in \mu$ and $x_t q\mu$ will be denoted by $x_t \in \wedge q\mu$. Also, $x_t \overline{\in} \mu$, $x_t \overline{\in \vee q} \mu$ and $x_t \overline{\in \wedge q} \mu$ will respectively mean $x_t \notin \mu,\ x_t \notin \vee q\mu$ and $x_t \notin \wedge q\mu$.

2.9. Definition. [13] Let X be a non-empty set and μ a fuzzy subset of X. Then for any $t \in (0, 1]$, the sets $\mu_t = \{x \in X : \mu(x) \geq t\}$ and $\text{Supp}(\mu) = \{x \in X : \mu(x) > 0\}$ are respectively called the *t-level set* and *supporting set of μ*, respectively. The t-level subset μ_t of μ is also denoted by $U(\mu; t)$.

3. Main Results

3.1. Definition. [26] A non-empty fuzzy subset μ of a Γ-semigroup S is called an $(\in, \in \vee q)$-fuzzy subsemigroup of S if $\forall x, y \in S,\ \forall \gamma \in \Gamma,\ t, r \in (0, 1],\ x_t, y_r \in \mu \implies (x\gamma y)_{\min(t,r)} \in \vee q\mu$.

3.2. Definition. An $(\in, \in \vee q)$-fuzzy subsemigroup μ of a Γ-semigroup S is called an *$(\in, \in \vee q)$-fuzzy interior ideal of S*, if $\forall a, x, y \in S,\ \forall \alpha, \beta \in \Gamma,\ \forall t \in (0, 1],\ a_t \in \mu \implies (x\alpha a\beta y)_t \in \vee q\mu$.

3.3. Theorem. *Let μ be a fuzzy subset of a Γ-semigroup S. Then μ_t is an interior ideal of S for all $t \in (0.5, 1]$, provided μ_t is non-empty, if and only if μ satisfies the following conditions:*

(1) $\max\{\mu(x\gamma y), 0.5\} \geq \min\{\mu(x), \mu(y)\},\ \forall x, y \in S,\ \forall \gamma \in \Gamma,$
(2) $\max\{\mu(x\alpha a\beta y), 0.5\} \geq \mu(a),\ \forall a, x, y \in S,\ \forall \alpha,\ \beta \in \Gamma.$

Proof. Let us suppose that μ_t is an interior ideal of S for all $t \in (0.5, 1]$. Then by definition, μ_t is a subsemigroup of S. Let $x, y \in S, \gamma \in \Gamma$. If possible, let $\max\{\mu(x\gamma y), 0.5\} < \min\{\mu(x), \mu(y)\}$. Let $\min\{\mu(x), \mu(y)\} = s$. Then $\mu(x) \geq s,\ \mu(y) \geq s,\ s > 0.5$ and $\mu(x\gamma y) < s$. Thus $x, y \in \mu_s$ but $x\gamma y \notin \mu_s$- a contradiction. Hence the condition (1) is valid. Now if possible, let there exist $a, x, y \in S, \alpha, \beta \in \Gamma$ such that $\max\{\mu(x\alpha a\beta y), 0.5\} <$

$\mu(a)$. Let $\mu(a) = r$. Then $a \in \mu_r$. Also $r > 0.5$ and $\mu(x\alpha a\beta y) < r$. Since $a \in \mu_r$ and μ_r is an interior ideal of S, $x\alpha a\beta y \in \mu_r$, which implies $\mu(x\alpha a\beta y) \geq r$, a contradiction. Hence the condition (2) is valid.

Conversely, let us suppose that the fuzzy subset μ satisfies the conditions (1) and (2) such that μ_t is non-empty for $t \in (0.5, 1]$. Let $x, y \in \mu_t,\ \gamma \in \Gamma$. Then $\mu(x) \geq t,\ \mu(y) \geq t$. Then by condition (1), $\max\{\mu(x\gamma y), 0.5\} \geq \min\{\mu(x), \mu(y)\} \implies \max\{\mu(x\gamma y), 0.5\} \geq t$ and $t > 0.5 \implies \mu(x\gamma y) \geq t$. Consequently, $x\gamma y \in \mu_t$. Hence μ_t is a subsemigroup of S.

Now let $x, y \in S, \alpha, \beta \in \Gamma$ and $a \in \mu_t$. Then $\mu(a) \geq t$. Then by condition (2), $\max\{\mu(x\alpha a\beta y), 0.5\} \geq \mu(a) \geq t$ and $t > 0.5 \implies \mu(x\alpha a\beta y) \geq t$. Consequently, $x\alpha a\beta y \in \mu_t$. Hence μ_t is an interior ideal of S. □

The following theorem is a characterization of $(\in, \in \vee q)$-fuzzy interior ideals.

3.4. Theorem. *Let μ be any $(\in, \in \vee q)$-fuzzy subsemigroup of a Γ-semigroup S. Then the following statements are equivalent:*

(1) *μ is an $(\in, \in \vee q)$-fuzzy interior ideal of S,*
(2) *for any $x, y \in S, a \in Supp(\mu), \alpha, \beta \in \Gamma$, $\mu(x\alpha a\beta y) \geq \min\{\mu(a), 0.5\}$,*
(3) *$\forall r \in (0, 0.5]$, if μ_r is non-empty, then μ_r is an interior ideal of S.*

Proof. (1) $\Longrightarrow$ (2) Suppose μ is an $(\in, \in \vee q)$-fuzzy interior ideal of S. Let $x, y \in S, a \in \mathrm{Supp}(\mu)$ and $\alpha, \beta \in \Gamma$. Then $\mu(a) > 0$. If possible, suppose $\mu(x\alpha a\beta y) < \min\{\mu(a), 0.5\}$. Let $\min\{\mu(a), 0.5\} = t$. Then $\mu(a) \geq t, t \leq 0.5$ and $\mu(x\alpha a\beta y) < t$. Hence $a_t \in \mu$ and $\mu(x\alpha a\beta y) + t < 1$. But by (1), $a_t \in \mu \implies \mu(x\alpha a\beta y) \geq t$ or $\mu(x\alpha a\beta y) + t > 1$. Thus we get a contradiction. Hence (2) follows.

(2) $\Longrightarrow$ (3) Let us suppose that (2) holds. Let $r \in (0, 0.5]$ be such that μ_r is non-empty. Since μ is a $(\in, \in \vee q)$-fuzzy subsemigroup of S, then μ_r is a subsemigroup of S [26]. Let $a \in \mu_r,\ x, y \in S$ and $\alpha, \beta \in \Gamma$. Then $\mu(a) \geq r$, whence $a \in \mathrm{Supp}(\mu)$. Then by (2) we obtain $\mu(x\alpha a\beta y) \geq \min\{\mu(a), 0.5\} \geq \min\{r, 0.5\} = r$. Hence $x\alpha a\beta y \in \mu_r$. Consequently, μ_r is an interior ideal of S. Hence (3) follows.

(3) $\Longrightarrow$ (1) Suppose (3) holds. Let $a, x, y \in S,\ \alpha, \beta \in \Gamma,\ t \in (0, 1]$ be such that $a_t \in \mu$. Then $\mu(a) \geq t$, whence $a \in \mu_t$.

Case (i) Suppose $t < 0.5$. Then by (3), $x\alpha a\beta y \in \mu_t$. Hence $\mu(x\alpha a\beta y) \geq t \implies (x\alpha a\beta y)_t \in \vee q\mu$.

Case (ii) Suppose $t \geq 0.5$. If possible, suppose $(x\alpha a\beta y)_t \overline{\in \vee q}\mu$. Then $\mu(x\alpha a\beta y) < t$ and $\mu(x\alpha a\beta y) + t \leq 1$. Since $t \geq 0.5$, $\mu(x\alpha a\beta y) < 0.5$. Let s be such that $\mu(x\alpha a\beta y) < s < 0.5$. Then $\mu(a) \geq s$, whence $a \in \mu_s$. Hence by (3), $x\alpha a\beta y \in \mu_s$ whence $\mu(x\alpha a\beta y) \geq s$, a contradiction. Consequently, (1) follows. □

3.5. Proposition. *If μ is an $(\in, \in \vee q)$-fuzzy interior ideal of a Γ-semigroup S, then* $\mathrm{Supp}(\mu)$ *is an interior ideal of S.*

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy interior ideal of a Γ-semigroup S. Then $\mathrm{Supp}(\mu)$ is an fuzzy subsemigroup [26]. Let $x, y \in S,\ a \in \mathrm{Supp}(\mu),\ \alpha, \beta \in \Gamma$. Then $\mu(a) > 0$. Since μ is an $(\in, \in \vee q)$-fuzzy interior ideal of S, by Theorem 3.4 $\mu(x\alpha a\beta y) \geq \min\{\mu(a), 0.5\} > 0$, whence $x\alpha a\beta y \in \mathrm{Supp}(\mu)$. Consequently, $S\Gamma(\mathrm{Supp}(\mu))\Gamma S \subseteq \mathrm{Supp}(\mu)$. Hence $\mathrm{Supp}(\mu)$ is an interior ideal of S. □

3.6. Proposition. *If μ is a fuzzy interior ideal of a Γ-semigroup S then μ is an $(\in, \in \vee q)$-fuzzy interior ideal of S.*

Proof. Let μ be a fuzzy interior ideal of S. Then μ is a fuzzy subsemigroup of S. Hence μ is an $(\in, \in \vee q)$-fuzzy subsemigroup of S [26]. Let $a, x, y \in S$, $\alpha, \beta \in \Gamma$ and $r \in (0, 1]$ be such that $a_r \in \mu \implies \mu(a) \geq r$. Now by hypothesis, $\mu(x\alpha a\beta y) \geq \mu(a)$. Hence $\mu(x\alpha a\beta y) \geq r$. Therefore $(x\alpha a\beta y)_r \in \vee q\mu$. Consequently, μ is an $(\in, \in \vee q)$-fuzzy interior ideal of S. □

The converse of the above proposition is not necessarily true, as is clear from the following example.

3.7. Example. Let $S = \{x, y, z, r, t\}$ and $\Gamma = \{\gamma\}$, where γ is defined on S with the following Cayley table:

γ	x	y	z	r	t
x	x	r	x	r	r
y	x	y	x	r	r
z	x	r	z	r	t
r	x	r	x	r	r
t	x	r	z	r	t

Then S is a Γ-semigroup. We define a fuzzy subset $\mu : S \to [0, 1]$ as $\mu(x) = 0.8,\ \mu(y) = 0.7,\ \mu(z) = 0.6,\ \mu(r) = 0.5,\ \mu(t) = 0.3$.

Then it is easy to verify that μ is an $(\in, \in \vee q)$-fuzzy interior ideal of S, but it is not a fuzzy interior ideal of S.

3.8. Theorem. *Let $\{\mu_i : i \in I\}$ be any family of $(\in, \in \vee q)$-fuzzy interior ideals of S. Then $\bigcap_{i\in I} \mu_i$ is an $(\in, \in \vee q)$-fuzzy interior ideal of S.*

Proof. Clearly, $\bigcap_{i\in I} \mu_i$ is an $(\in, \in \vee q)$-fuzzy subsemigroup of S [26]. Let $a, x, y \in S,\ \alpha, \beta \in \Gamma$ be such that $a \in \text{Supp}(\bigcap_{i\in I} \mu_i)$. Then $a \in \text{Supp}(\mu_i)\ \forall\, i \in I$, so

$$\begin{aligned}\Big(\bigcap_{i\in I} \mu_i\Big)(x\alpha a\beta y) &= \inf_{i\in I}(\mu_i(x\alpha a\beta y))\\ &\geq \inf_{i\in I}(min\{\mu_i(a), 0.5\}) \text{ (cf. Theorem 3.4)}\\ &= \min\{\inf_{i\in I}(\mu_i(a)), 0.5\}\\ &= \min\Big\{\Big(\bigcap_{i\in I} \mu_i\Big)(a), 0.5\Big\}.\end{aligned}$$

Hence, by Theorem 3.4, $\bigcap_{i\in I} \mu_i$ is an $(\in, \in \vee q)$-fuzzy interior ideal of S. □

The following theorem shows that the image and inverse image of $(\in, \in \vee q)$-fuzzy interior ideals of a Γ-semigroup are also $(\in, \in \vee q)$-fuzzy interior ideals.

3.9. Theorem. *Let S and S' be Γ-semigroups, μ and μ' respectively $(\in, \in \vee q)$-fuzzy interior ideals of S and S', and f be a homomorphism from S onto S'. Then*

(1) *$f(\mu)$ is an $(\in, \in \vee q)$-fuzzy interior ideal of S',*
(2) *$f^{-1}(\mu')$ is an $(\in, \in \vee q)$-fuzzy interior ideal of S.*

Proof. (1) Let μ be an $(\in, \in \vee q)$-fuzzy interior ideal of S. Then μ is an $(\in, \in \vee q)$-fuzzy subsemigroup of S. Then $f(\mu)$ is an $(\in, \in \vee q)$ fuzzy subsemigroup of S' [26].

Let $x', y' \in S'$, $a' \in \text{Supp}(f(\mu))$, $\alpha, \beta \in \Gamma$. Since f is onto, there exist $a, x, y \in S$ such that $f(a) = a'$, $f(x) = x'$, $f(y) = y'$ and $\mu(a) > 0$. Then $f(x\alpha a\beta y) = x'\alpha a'\beta y'$. Then

$$\begin{aligned} f(\mu)(x'\alpha a'\beta y') &= \sup\{\mu(z) : z \in f^{-1}(x'\alpha a'\beta y')\} \\ &\geq \sup\{\mu(x\alpha a\beta y) : x \in f^{-1}(x'),\ a \in f^{-1}(a'),\ y \in f^{-1}(y')\} \\ &\geq \sup\{\min\{\mu(a), 0.5\} : a \in f^{-1}(a') \\ &= \min\{\sup_{a \in f^{-1}(a')} \mu(a), 0.5\} \\ &= \min\{f(\mu)(a'), 0.5\}. \end{aligned}$$

Hence $f(\mu)$ is an $(\in, \in\vee q)$-fuzzy interior ideal of S'.

(2) Let μ' be an $(\in, \in\vee q)$-fuzzy interior ideal of S'. Then μ' is an $(\in, \in\vee q)$-fuzzy subsemigroup of S'. Then $f^{-1}(\mu')$ is an $(\in, \in\vee q)$-fuzzy subsemigroup of S [26].

Let $x, y \in S$, $a \in \text{Supp}(f^{-1}(\mu'))$, $\alpha, \beta \in \Gamma$. Then $f^{-1}(\mu')(a) > 0 \implies \mu'(f(a)) > 0 \implies f(a) \in \text{Supp}(\mu')$. Then we have

$$\begin{aligned} f^{-1}(\mu')(x\alpha a\beta y) &= \mu'(f(x\alpha a\beta y)) \\ &= \mu'(f(x)\alpha f(a)\beta f(y)) \\ &\geq \min\{\mu'(f(a)), 0.5\} \\ &= \min\{f^{-1}(\mu')(a), 0.5\}. \end{aligned}$$

Hence $f^{-1}(\mu')$ is an $(\in, \in\vee q)$-fuzzy interior ideal of S. □

4. (α, β)-fuzzy interior ideals

In what follows, unless otherwise mentioned, α, β denote any one of $\in$, $\in\vee q$ or $\in\wedge q$.

4.1. Definition. A non-empty fuzzy subset μ of a Γ-semigroup S is called an (α, β)-*fuzzy interior ideal of* S, where $\alpha \neq\in\wedge q$, if it satisfies

(1) $\forall\, x, y \in S,\ \forall\, \gamma \in \Gamma,\ \forall\, t, r \in (0, 1],\ x_t\alpha\mu,\ y_r\alpha\mu \implies (x\gamma y)_{\min(t,r)}\beta\mu.$
(2) $\forall\, a, x, y \in S,\ \forall\, \eta, \delta \in \Gamma,\ \forall\, t \in (0, 1],\ a_t\alpha\mu \implies (x\eta a\delta y)_t\beta\mu.$

In the following theorem it is shown that every fuzzy interior ideal of a Γ-semigroup S is an $(\in, \in)$-fuzzy interior ideal of S, and conversely.

4.2. Theorem. *For any fuzzy subset μ of a Γ-semigroup S, the conditions* (I1) *and* (I2) *in Definition 2.3 are equivalent, respectively, to the conditions* (I3) *and* (I4) *stated below:*

(I3) $\forall\, x, y \in S,\ \forall\, \gamma \in \Gamma,\ \forall\, t, r \in (0, 1],\ x_t \in \mu, y_r \in \mu \implies (x\gamma y)_{\min(t,r)} \in \mu.$
(I4) $\forall\, a, x, y \in S,\ \forall\, \beta, \delta \in \Gamma,\ \forall\, t \in (0, 1],\ a_t \in \mu \implies (x\beta a\delta y)_t \in \mu.$

Proof. The equivalence of (I1) and (I3) is shown in [26, Theorem 5.3].

(I2) $\implies$ (I4) Let $a, x, y \in S$; $\beta, \delta \in \Gamma$ and $t \in (0, 1]$ be such that $a_t \in \mu$. Then $\mu(a) \geq t$. So, by (I2) we have $\mu(x\beta a\delta y) \geq \mu(a) \geq t$. Hence $(x\beta a\delta y)_t \in \mu$.

(I4) $\implies$ (I2) Let $a, x, y \in S$, $\beta, \delta \in \Gamma$. Since $a_{\mu(a)} \in \mu$, so by (I4), we obtain $(x\beta a\delta y)_{\mu(a)} \in \mu$ and consequently, $\mu(x\beta a\delta y) \geq \mu(a)$. □

4.3. Definition. For any fuzzy subset μ of a Γ-semigroup S and $t \in (0, 1]$, we have $Q(\mu; t) := \{x \in S : x_t q\mu\}$ and $[\mu]_t := \{x \in S : x_t \in\vee q\mu\}$. It is clear that $[\mu]_t = U(\mu; t) \cup Q(\mu; t)$.

4.4. Theorem. *Let A be a non-empty subset of a Γ-semigroup S. Then A is an interior ideal of S if and only if μ_A is an $(\in, \in\vee q)$-fuzzy interior ideal S, where μ_A is the characteristic function of A.*

Proof. Let A be an interior ideal of S. Then the characteristic function μ_A of A is a fuzzy interior ideal of S (cf. [25, Corollary 3.4]). Hence by Proposition 3.6, μ_A is an $(\in, \in \vee q)$-fuzzy interior ideal of S.

Conversely, let μ_A be an $(\in, \in \vee q)$-fuzzy interior ideal of S. Then μ_A is an $(\in, \in \vee q)$-fuzzy subsemigroup of S. Then A is an subsemigroup of S [26]. Let $x, y \in S, a \in A, \beta, \delta \in \Gamma$. Then $\mu_A(a) = 1$. Hence by the hypothesis and Theorem 3.4, $\mu_A(x\beta a\delta y) \geq \min\{\mu_A(a), 0.5\} = 0.5 \implies \mu_A(x\beta a\delta y) = 1$. Consequently, $x\beta a\delta y \in A$. Hence A is an interior ideal of S. □

4.5. Theorem. *Let S be a Γ-semigroup and μ be a non-empty fuzzy subset of S. Then μ is an $(\in, \in \vee q)$-fuzzy interior ideal of S if and only if $[\mu]_t$ is an interior ideal of S.*

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy interior ideal of S. Then μ is an $(\in, \in \vee q)$-fuzzy subsemigroup of S. Hence by [26, Theorem 5.21], $[\mu]_t$ is a subsemigroup of S. Let $x, y \in S, a \in [\mu]_t, \alpha, \beta \in \Gamma$ and $t \in (0, 1]$. Then $a_t \in \vee q\mu$, which implies, $\mu(a) \geq t$ or $\mu(a) + t > 1$.

Case-(i) Let $\mu(a) \geq t \implies a_t \in \mu$. Then μ being an $(\in, \in \vee q)$-fuzzy interior ideal of S we obtain $(x\alpha a\beta y)_t \in \vee q\mu$, whence $x\alpha a\beta y \in [\mu]_t$.

Case-(ii) Let $\mu(a) + t > 1$. Then $\mu(x\alpha a\beta y) \geq \min\{\mu(a), 0.5\} \geq \min\{1 - t, 0.5\}$. If $t > 0.5$, $\mu(x\alpha a\beta y) > 1 - t \implies \mu(x\alpha a\beta y) + t > 1$ and consequently, $(x\alpha a\beta y)_t q\mu$. If $t \leq 0.5$, then $\mu(x\alpha a\beta y) \geq t$ and so $(x\alpha a\beta y)_t \in \mu$. Hence $(x\alpha a\beta y)_t \in \vee q\mu$, whence $x\alpha a\beta y \in [\mu]_t$. Consequently $[\mu]_t$ is an interior ideal of S.

Conversely, let $[\mu]_t$ be an interior ideal of S. Then $[\mu]_t$ is a subsemigroup of S. Then μ is an $(\in, \in \vee q)$-fuzzy subsemigroup of S [26]. Let $x, y \in S, a \in \text{Supp}(\mu), \alpha, \beta \in \Gamma$ be such that $\mu(x\alpha a\beta y) < t < \min\{\mu(a), 0.5\}$ for some $t \in (0, 0.5]$. Then $a \in U(\mu; t) \subseteq [\mu]_t$, which implies that $x\alpha a\beta y \in [\mu]_t$ (since $[\mu]_t$ is an interior ideal of S). Consequently, $\mu(x\alpha a\beta y) \geq t$ or $\mu(x\alpha a\beta y) + t > 1$, a contradiction. Thus $\mu(x\alpha a\beta y) \geq \min\{\mu(a), 0.5\}$, $\forall\, x, y \in S$, $a \in \text{Supp}(\mu)$, $\forall\ \alpha, \beta \in \Gamma$. Hence μ is an $(\in, \in \vee q)$-fuzzy interior ideal of S. □

4.6. Theorem. *Let μ be a non-zero (α, β)-fuzzy interior ideal of a Γ-semigroup S. Then the set $\mu_0 := \{x \in S : \mu(x) > 0\}$ is an interior ideal of S.*

Proof. Since μ is non-zero (α, β)-fuzzy interior ideal, μ_0 is non-empty. Then μ_0 is a subsemigroup of S, as μ is (α, β)-fuzzy subsemigroup of S [26]. Let $x, y \in S, \gamma, \delta \in \Gamma$. Let $a \in \mu_0$. Then $\mu(a) > 0$. Let us assume that $\mu(x\gamma a\delta y) = 0$. If $\alpha \in \{\in, \in \vee q\}$ then $a_{\mu(a)}\alpha\mu$ but $(x\gamma a\delta y)_{\mu(a)}\overline{\beta}\mu$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Thus $\mu(x\gamma a\delta y) > 0$. Consequently, $x\gamma a\delta y \in \mu_0$. Hence μ_0 is an interior ideal of S. □

4.7. Definition. [27]

(i) A non-empty fuzzy subset μ of a Γ-semigroup S is said to be an $(\in, \in \vee q)$-*fuzzy left ideal of* S, if $\forall\, x, y \in S,\ \forall\, \gamma \in \Gamma,\ r \in (0, 1], y_r \in \mu \implies (x\gamma y)_r \in \vee q\mu$.

(ii) A non-empty fuzzy subset μ of a Γ-semigroup S is said to be an $(\in, \in \vee q)$- *fuzzy right ideal of* S, if $\forall\, x, y \in S,\ \forall\, \gamma \in \Gamma,\ r \in (0, 1], x_r \in \mu \implies (x\gamma y)_r \in \vee q\mu$.

(iii) A non-empty fuzzy subset μ of a Γ-semigroup S is said to be an $(\in, \in \vee q)$-*fuzzy ideal of* S, if it is an $(\in, \in \vee q)$-fuzzy left ideal and an $(\in, \in \vee q)$-fuzzy right ideal of S.

4.8. Definition. [26] Let S be a Γ-semigroup and μ be a non-empty fuzzy subset of S. Then μ is called an $(\in, \in \vee q)$-*fuzzy left ideal* (*fuzzy right ideal*) *of* S if $\forall\, x, y \in S,\ \forall\, \gamma \in \Gamma,\ \mu(x\gamma y) \geq \min\{\mu(y), 0.5\}$ (resp. $\mu(x\gamma y) \geq \min\{\mu(x), 0.5\}$).

4.9. Definition. [27] Let S be a Γ-semigroup. A non-empty fuzzy subset μ of S is called an $(\in, \in \vee q)$-*fuzzy ideal of* S if $\forall x, y \in S,\ \forall\, \gamma \in \Gamma,\ \mu(x\gamma y) \geq \min[max\{\mu(x), \mu(y)\}, 0.5]$.

4.10. Theorem. *In a regular* Γ*-semigroup every* $(\in, \in \vee q)$*-fuzzy ideal is an* $(\in, \in \vee q)$*-fuzzy interior ideal and conversely.*

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy ideal of S. Let $x, y \in S$, $a \in \text{Supp}(\mu)$, $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned}\mu(x\alpha a\beta y) &= \mu(x\alpha(a\beta y)) \\ &\geq \min\{\mu(a\beta y), 0.5\}(\text{since } \mu \text{ is an } (\in, \in \vee q)\text{-fuzzy left ideal of } S, \\ &\qquad \text{by 5.8)} \\ &\geq \min[\min\{\mu(a), 0.5\}, 0.5](\text{since } \mu \text{ is an } (\in, \in \vee q)\text{-fuzzy right} \\ &\qquad \text{ideal of } S, \text{ by 5.8)} \\ &= \min\{\mu(a), 0.5\}.\end{aligned}$$

Hence by Theorem 3.4, μ is an $(\in, \in \vee q)$-fuzzy interior ideal of S.

Conversely, let us suppose that μ is an $(\in, \in \vee q)$-fuzzy interior ideal of S. Let $x, y \in S, \gamma \in \Gamma$. If $\mu(x)$=0, then $\mu(x\gamma y) \geq \min\{\mu(x), 0.5\}$. If $\mu(x) > 0$ then $x \in \text{Supp}(\mu)$. Since S is regular, there exist $a \in S$, $\alpha, \beta \in \Gamma$ such that $x = x\alpha a\beta x$. Then $x\gamma y = (x\alpha a\beta x)\gamma y = (x\alpha a)\beta x\gamma y$. Thus by hypothesis, $\mu(x\gamma y) = \mu((x\alpha a)\beta x\gamma y) \geq \min\{\mu(x), 0.5\}$ (cf.Theorem 3.4). Hence μ is an $(\in, \in \vee q)$-fuzzy right ideal of S. Similarly we can prove that μ is an $(\in, \in \vee q)$-fuzzy left ideal of S. Hence μ is an $(\in, \in \vee q)$-fuzzy ideal of S. □

4.11. Theorem. *In an intra-regular* Γ*-semigroup every* $(\in, \in \vee q)$*-fuzzy ideal is an* $(\in, \in \vee q)$*-fuzzy interior ideal and conversely.*

Proof. The first part of the theorem is similar to Theorem 5.10. Conversely, let us suppose that μ is an $(\in, \in \vee q)$-fuzzy interior ideal of S. Let $x, y \in S$, $\gamma \in \Gamma$. If $\mu(x) = 0$, then $\mu(x\gamma y) \geq \min\{\mu(x), 0.5\}$. If $\mu(x) > 0$, then $x \in \text{Supp}(\mu)$. Since S is intra-regular, there exist $a, b \in S$, $\alpha, \beta, \delta \in \Gamma$ such that $x = a\alpha x\beta x\delta b$. Then $x\gamma y = (a\alpha x\beta x\delta b)\gamma y = (a\alpha x)\beta x\delta(b\gamma y)$. Then because of the hypothesis we apply Theorem 3.4 and obtain $\mu(x\gamma y) = \mu((a\alpha x)\beta x\delta(b\gamma y)) \geq \min\{\mu(x), 0.5\}$. Hence μ is an $(\in, \in \vee q)$-fuzzy right ideal of S.

Similarly we can prove that μ is an $(\in, \in \vee q)$-fuzzy left ideal of S. Hence μ is an $(\in, \in \vee q)$-fuzzy ideal of S. □

4.12. Theorem. *In a semisimlple* Γ*-semigroup every* $(\in, \in \vee q)$*-fuzzy ideal is an* $(\in, \in \vee q)$*-fuzzy interior ideal and conversely.*

Proof. First part of the theorem is similar with Theorem 5.10. To prove the converse let us suppose that μ is an $(\in, \in \vee q)$-fuzzy interior ideal of S. Let $x, y \in S, \gamma \in \Gamma$. If $\mu(x) = 0$, then $\mu(x\gamma y) \geq \min\{\mu(x), 0.5\}$. If $\mu(x) > 0$, then $x \in \text{Supp}(\mu)$. Since S is semisimple, there exist $a, b, c \in S$, $\alpha, \beta, \delta, \eta \in \Gamma$ such that $x = a\alpha x\beta b\delta x\eta c$. Then $x\gamma y = (a\alpha x\beta b\delta x\eta c)\gamma y = a\alpha x\beta(b\delta x\eta c\gamma y)$. Thus, by hypothesis, $\mu(x\gamma y) = \mu(a\alpha x\beta(b\delta x\eta c\gamma y)) \geq \min\{\mu(x), 0.5\}$. Hence μ is an $(\in, \in \vee q)$-fuzzy right ideal of S.

Similarly we can prove that μ is an $(\in, \in \vee q)$-fuzzy left ideal of S. Hence μ is an $(\in, \in \vee q)$-fuzzy ideal of S. □

4.13. Definition. [26] Let S be a Γ-semigroup and $x \in S$. Then we define a subset of S, denoted by I_x, defined by $I_x := \{y \in S : \mu(y) \geq \min\{\mu(x), 0.5\}\}$.

4.14. Theorem. *Let* S *be a* Γ*-semigroup and* μ *an* $(\in, \in \vee q)$*-fuzzy interior ideal of* S*. Then for every* $x \in S$*,* I_x *is an interior ideal of* S*.*

Proof. By definition, $x \in I_x$ for every $x \in S$, so I_x is non-empty. Let $a, b \in I_x$, $\gamma \in \Gamma$. Then $\mu(a) \geq \min\{\mu(x), 0.5\}$ and $\mu(b) \geq \min\{\mu(x), 0.5\}$. Now μ is an $(\in, \in \vee q)$-fuzzy subsemigroup of S. Since μ is an $(\in, \in \vee q)$-fuzzy subsemigroup, $\mu(a\gamma b) \geq \min\{\mu(a), \mu(b), 0.5\} \geq \min\{\mu(x), 0.5\}$. Hence $a\gamma b \in I_x$. Hence I_x is a subsemigroup of S.

Let $c \in I_x, \alpha, \beta \in \Gamma$ and $y, z \in S$. Then $\mu(c) \geq \min\{\mu(x), 0.5\}$. Since μ is an $(\in, \in \vee q)$-fuzzy interior ideal of S, then by Theorem 3.4, $\mu(y\alpha c\beta z) \geq \min\{\mu(c), 0.5\} \geq \min\{\min\{\mu(x), 0.5\}, 0.5\} = \min\{\mu(x), 0.5\}$. Hence $y\alpha c\beta z \in I_x$. Consequently, $S\Gamma I_x \Gamma S \subseteq I_x$. Hence I_x is an interior ideal of S. □

4.15. Definition. [26] Let S be a Γ-semigroup. For any two fuzzy subsets μ_1 and μ_2 of S we define the 0.5-*product of* μ_1 *and* μ_2 *by*

$$(\mu_1 \circ_{0.5} \mu_2)(a) := \begin{cases} \sup\limits_{(y,z)\in A_a} \min\{\mu_1(y), \mu_2(z), 0.5\} & \text{if } A_a \neq \emptyset, \\ 0 & \text{if } A_a = \emptyset \end{cases}$$

To conclude the paper we obtain the following characterization of $(\in, \in \vee q)$-fuzzy interior ideals of a Γ-semigroup S.

4.16. Theorem. *Let μ be an $(\in, \in \vee q)$-fuzzy subsemigroup of a Γ semigroup S. Then μ is an $(\in, \in \vee q)$-fuzzy interior ideal of S if and only if $\chi \circ_{0.5} \mu \circ_{0.5} \chi \subseteq \mu$, where χ is the characteristic function on S.*

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy subsemigroup of a Γ semigroup S.

Let μ be a $(\in, \in \vee q)$-fuzzy interior ideal of S. Let $a \in S$. Now if $(\chi \circ_{0.5} \mu \circ_{0.5} \chi)(a) = 0$, then $\mu(a) \geq 0$. If $(\chi \circ_{0.5} \mu \circ_{0.5} \chi)(a) \neq 0$, then there exists $x, y, z, w \in S$, $\alpha, \beta \in \Gamma$ such that $a = x\alpha y$, $y = w\beta z$. Then

$$\begin{aligned}
(\chi \circ_{0.5} \mu \circ_{0.5} \chi)(a) &= \sup_{a = x\alpha y} \min\{\chi(x), (\mu \circ_{0.5} \chi)(y), 0.5\} \\
&= \sup_{a = x\alpha y} \min\{1, \sup_{y = w\beta z} \min\{\mu(w), \chi(z), 0.5\}, 0.5\} \\
&= \sup_{a = x\alpha y} \min\{ \sup_{y = w\beta z} \min\{\mu(w), 1, 0.5\}, 0.5\} \\
&= \sup_{a = x\alpha y} \min\{ \sup_{y = w\beta z} \min\{\mu(w), 0.5\}, 0.5\} \\
&\leq \sup_{a = x\alpha w\beta z} \min\{\mu(w), 0.5\} \\
&\leq \sup_{a = x\alpha w\beta z} \mu(x\alpha w\beta z) \text{ (as } \mu \text{ is an } (\in, \in \vee q)\text{-fuzzy interior} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{ideal of } S) \\
&= \mu(a).
\end{aligned}$$

Hence $\chi \circ_{0.5} \mu \circ_{0.5} \chi \subseteq \mu$.

Conversely let $\chi \circ_{0.5} \mu \circ_{0.5} \chi \subseteq \mu$. Let $x, y \in S$, $a \in \text{Supp}(\mu)$, $\alpha, \beta \in \Gamma$. Then $\mu(x\alpha a\beta y) \geq (\chi \circ_{0.5} \mu \circ_{0.5} \chi)(x\alpha a\beta y) \geq \min\{\chi(x), \mu(a), \chi(y), 0.5\} = \min\{1, \mu(a), 1, 0.5\} = \min\{\mu(a), 0.5\}$. Hence μ is an $(\in, \in \vee q)$-fuzzy interior ideal of S. □

References

[1] Adhikari, N. C. *Study of some problems associated with Γ-Semigroups* (Ph.D. Dissertation, University of Calcutta, 1998).

[2] Bhakat, S. K. and Das, P. *On the definition of a fuzzy subgroup*, Fuzzy Sets and Systems **51**, 235–241, 1992.

[3] Bhakat, S. K. and Das, P. $(\in, \in \vee q)$-*fuzzy subgroup*, Fuzzy Sets and Systems **80** (1996) 359–368, 1996.

[4] Chattopadhyay, S. *Right inverse* Γ-*semigroup*, Bull. Cal. Math. Soc. **93**, 435–442, 2001.

[5] Chattopadhyay, S. *Right orthodox* Γ-*semigroup*, Southeast Asian Bull. Math. **29**, 23–30, 2005.

[6] Chinram, R. *On quasi*-Γ-*ideals in* Γ-*semigroups*, Science Asia **32**, 351–353, 2006.

[7] Dutta, T. K. and Adhikari, N. C. *On* Γ-*semigroup with the right and left unities*, Soochow J. Math. **19** (4), 461–474, 1993.

[8] Dutta, T. K. and Adhikari, N. C. *On prime radical of* Γ-*semigroup*, Bull. Cal. Math. Soc. **86** (5), 437–444, 1994.

[9] Dutta, T. K., Sardar, S. K. and Majumder, S. K. *Fuzzy ideal extensions of* Γ-*semigroups*, International Mathematical Forum **4** (41), 2093–2100, 2009.

[10] Dutta, T. K., Sardar, S. K. and Majumder, S. K. *Fuzzy ideal extensions of* Γ-*semigroups via its operator semigroups*, Int. J. Contemp. Math. Sciences **4** (30), 1455–1463, 2009.

[11] Hila, K. *On regular, semiprime and quasi-reflexive* Γ-*semigroup and minimal quasi-ideals*, Lobachevski J. Math. **29**, 141–152, 2008.

[12] Hila, K. *On some classes of le*-Γ-*semigroup and minimal quasi-ideals*, Algebras Groups Geom. **24**, 485–495, 2007.

[13] Huang, X. and Yin, Y. *The* $(\in, \in \vee q)$-*Fuzzy prime ideals and maximal ideals of a semigroup*, International Mathematical Forum **3** (39), 1915–1932, 2008.

[14] Jun, Y. B. and Song, S. Z. *Generalized fuzzy interior ideals in semigroups*, Inform Sci. **176**, 3079–3093, 2006.

[15] Kuroki, N. *On fuzzy semigroups*, Inform Sci. **53**, 203–236, 1991.

[16] Kuroki, N. *On fuzzy ideals and fuzzy bi-ideals in semigroups*, Fuzzy Sets and Systems **5**, 203–215, 1981.

[17] Kuroki, N. *Fuzzy semiprime ideals in semigroups*, Fuzzy Sets and Systems **158**, 277–288, 2004.

[18] Murali, V. *Fuzzy points of equivalent fuzzy subsets*, Inform. Sci. **158**, 277–288, 2004.

[19] Pu, P. M. and Liu, Y. M. *Fuzzy topology I, neighbourhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl. **76**, 571–599, 1980.

[20] Rosenfeld, A. *Fuzzy groups*, J. Math. Anal. Appl. **35**, 512–517, 1971.

[21] Saha, N. K. *On* Γ-*semigroup II*, Bull. Cal. Math. Soc. **79**, 331–335, 1987.

[22] Sardar, S. K. and Majumder, S. K. *On fuzzy ideals in* Γ-*semigroups*, International Journal of Algebra **3** (16), 775–784, 2009.

[23] Sardar, S. K., Majumder, S. K. and Mandal, D. *A note on characterization of prime ideals of* Γ-*semigroups in terms of fuzzy subsets*, Int. J. Contemp. Math. Sciences **4** (30), 1465–1472, 2009.

[24] Sardar, S. K. and Majumder, S. K. *A note on characterization of semiprime ideals of* Γ-*semigroups in terms of fuzzy subsets*, International Journal of Pure and Applied Mathematics **3** (56), 451–457, 2009.

[25] Sardar, S. K., Davvaz, B. and Majumder, S. K. *A study on fuzzy interior ideals of* Γ-*semigroups*, Computers and Mathematics with Applications **60**, 90–94, 2010.

[26] Sardar, S. K., Davvaz, B., Majumder, S. K. and Kayal, S. *On generalised fuzzy subsemigroup and generalized fuzzy bi-ideal in* Γ-*semigroups*, communicated.

[27] Sardar, S. K., Davvaz, B., Majumder, S. K. and Kayal, S. *A note on generalised fuzzy ideals in* Γ-*semigroups*, preprint.

[28] Sardar, S. K., Majumder, S. K. and Kayal, S. *On fuzzy bi-ideals and fuzzy quasi ideals in* Γ-*semigroups*, communicated.

[29] Sen, M. K. and Chattopadhyay, S. *Semidirect product of a monoid and a* Γ-*semigroup*, East-West J. Math. **6**, 131–138, 2004.

[30] Sen, M. K. and Saha, N. K. *Orthodox* Γ-*semigroups*, Internat. J. Math. Math. Sci. **13**, 527–534, 1990.

[31] Sen, M. K. and Saha, N. K. *On* Γ-*semigroup I*, Bull. Cal. Math. Soc. **78**, 180–186, 1986.

[32] Seth, A. Γ-*group congruences on regular* Γ-*semigroups*, Internat. J. Math. Math. Sci. **15**, 103–106, 1992.

[33] Uckun, M., Öztürk, M. A. and Jun, Y. B. *Intuitionistic fuzzy sets in* Γ*-semigroups*, Bull. Korean Math. Soc. **44** (2), 359–367, 2007.

[34] Yuan, X., Zhang, C. and Ren, Y. *Generalized fuzzy groups and many-valued implications*, Fuzzy Sets and Systsems **138**, 205–211, 2003.

[35] Yunqiang, Y., Xu, D. and Li, H. *The* $(\in, \in \vee q)$*-fuzzy subsemigroups and ideals of an* $(\in, \in \vee q)$*-fuzzy semigroup*, Southeast Asian Bull. Math. **33**, 391–400, 2009.

[36] Zadeh, L. A. *Fuzzy Sets*, Information and Control **8**, 338–353, 1965.