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FIXED POINT THEOREMS FOR A PAIR OF MAPPINGS SATISFYING A GENERALIZED WEAKLY CONTRACTIVE CONDITION IN ORDERED METRIC SPACES

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Abstract

Fixed point theorems for a pair of mappings which satisfy the condition of being generalized weakly contractive in complete ordered metric spaces are derived. At the end of the paper, applications of the previous results to new fixed point results of integral type are also shown.

Keywords: Fixed point, Complete metric space, Altering distance function, Weakly contractive condition, Partially ordered set.

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1. Introduction and preliminaries

There are a lot of generalizations of the Banach contraction mapping principle in the literature. One of the most interesting of them is the result of Khan *et al.* [14]. They addressed a new category of fixed point problems for a single self-map with the help of a control function which they called an altering distance function.

A function $\varphi : [0, \infty) \to [0, \infty)$ is called an *altering distance function* if φ is continuous, nondecreasing and $\varphi(0) = 0$ holds.

Khan *et al.* [14] gave the following result.

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1.1. Theorem. *Let* $(\mathcal{X}, d)$ *be a complete metric space, let* φ *be an altering distance function, and let* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a self-mapping which satisfies the following inequality:*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq c\varphi(d(x, y))$$

for all $x, y \in \mathcal{X}$ *and for some* $0 < c < 1$. *Then* $\mathcal{T}$ *has a unique fixed point.* □

In fact Khan *et al.* proved a more general theorem of which the above result is a corollary. Another generalization of the contraction principle was suggested by Alber and Guerre-Delabriere [2] in Hilbert Spaces by introducing the concept of weakly contractive mappings.

A self-mapping $\mathcal{T}$ on a metric space $\mathcal{X}$ is called *weakly contractive* if for each $x, y \in \mathcal{X}$,

$$d(\mathcal{T}x, \mathcal{T}y) \leq d(x, y) - \phi(d(x, y)), \tag{1.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is positive on $(0, \infty)$ and $\phi(0) = 0$.

Rhoades [21] showed that most results of [2] are still true for any Banach space. Also Rhoades [21] proved the following very interesting fixed point theorem which contains contractions as the special case $\phi(t) = (1 - k)t$.

1.2. Theorem. *Let* $(\mathcal{X}, d)$ *be a complete metric space. If* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *is a weakly contractive mapping, and in addition* ϕ *is a continuous and nondecreasing function, then* $\mathcal{T}$ *has a unique fixed point.* □

In fact, Alber and Guerre-Delabriere [2] assumed an additional condition on ϕ which is $lim_{t\to\infty}\phi(t) = \infty$. But Rhoades [21] obtained the result noted in Theorem 1.2 without using this particular assumption. Also, the weak contractions are closely related to the maps of Boyd and Wong [9] and to the Reich type ones [20]. Namely, if ϕ is a lower semi-continuous function from the right then $\psi(t) = t - \phi(t)$ is an upper semi-continuous function from the right, and moreover, (1.1) turns into $d(\mathcal{T}x, \mathcal{T}y) \leq \psi(d(x, y))$. Therefore the weak contraction is of Boyd and Wong type. And if we define $\beta(t) = 1 - \frac{\phi(t)}{t}$ for $t > 0$ and $\beta(0) = 0$, then (1.1) is replaced by $d(\mathcal{T}x, \mathcal{T}y) \leq \beta(d(x, y))d(x, y)$. Therefore the weak contraction becomes a Reich type one.

Recently, the following generalized result has given by Dutta and Choudhoury [11], combining Theorem 1.1 and Theorem 1.2.

1.3. Theorem. *Let* $(\mathcal{X}, d)$ *be a complete metric space and let* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a self-mapping satisfying the inequality*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(d(x, y)) - \phi(d(x, y))$$

for all $x, y \in \mathcal{X}$, *where* $\varphi, \phi : [0, \infty) \to [0, \infty)$ *are both continuous and nondecreasing functions with* $\varphi(t) = 0 = \phi(t)$ *if and only if* $t = 0$. *Then* $\mathcal{T}$ *has a unique fixed point.* □

Also, Zhang and Song [22] have given the following generalized version of Theorem 1.2.

1.4. Theorem. *Let* $(\mathcal{X}, d)$ *be a complete metric space and let* $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ *be two mappings such that for each* $x, y \in \mathcal{X}$,

$$d(\mathcal{T}x, \mathcal{S}y) \leq \Phi(x, y) - \phi(\Phi(x, y)),$$

where $\phi : [0, \infty) \to [0, \infty)$ *is a lower semi-continuous function with* $\phi(t) > 0$ *for* $t > 0$ *and* $\phi(0) = 0$,

$$\Phi(x, y) = \max\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{S}y), \frac{1}{2}[d(y, \mathcal{T}x) + d(x, \mathcal{S}y)]\}.$$

Then there exists a unique point $z \in \mathcal{X}$ *such that* $z = \mathcal{T}z = \mathcal{S}z$. □

Also, Abbas and Khan [3] gave an extension of Theorem 1.3 as follows:

1.5. Theorem. *Let* $(\mathcal{X}, d)$ *be a complete metric space and* $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ *a self mapping satisfying*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(d(\mathcal{S}x, \mathcal{S}y)) - \phi(d(\mathcal{S}x, \mathcal{S}y)),$$

where $\phi, \varphi : [0, \infty) \to [0, \infty)$ *are both continuous and monotone decreasing functions with* $\phi(t) = 0 = \varphi(t)$ *if and only if* $t = 0$. *Then* $\mathcal{T}$ *and* $\mathcal{S}$ *have a unique fixed point.* □

In recent years, many results have appeared in the literature related to fixed point theorems in complete metric spaces endowed with a partial ordering, $\preceq$, see [1], [4]–[8], [10], [15]–[19]. Most of them are a hybrid of two fundamental principle: the Banach contraction theorem and the weak contractive condition. Indeed, they deal with a monotone (either order-preserving or order-reversing) mapping satisfying, with some restriction, a classical contractive condition, and such that for some $x_0 \in \mathcal{X}$, either $x_0 \preceq \mathcal{T}x_0$ or $\mathcal{T}x_0 \preceq x_0$, where $\mathcal{T}$ is a self-map on a metric space. The first result in this direction was given by Ran and Reurings [19, Theorem 2.1] who presented applications to matrix equations. Subsequently, Nieto and Rodŕiguez-López [15] extended the result of Ran and Reurings [19] for nondecreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions.

Further, Harjani and Sadarangani [12] proved the ordered version of Theorem 1.2, Amini-Harandi and Emami [5] proved the ordered version of Rich type fixed point theorems and Harjani and Sadarangani [13] proved an ordered version of Theorem 1.3.

Here an attempt has been made to give a generalized ordered version of Theorem 1.4 and 1.5. We will do this using the concept of relatively non-decreasing mapping mentioned by Ciric *et al* [10].

2. Main results

To state the theorem, we need the following definition, which is given in [10].

2.1. Definition. [10] Suppose $(\mathcal{X}, \preceq)$ is a partially ordered set and $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ are mappings of $\mathcal{X}$ into itself. One says $\mathcal{T}$ is $\mathcal{S}$-*non-decreasing* if for $x, y \in \mathcal{X}$,

$$\mathcal{S}(x) \preceq \mathcal{S}(y) \text{ implies } \mathcal{T}(x) \preceq \mathcal{T}(y). \tag{2.1}$$

The following theorem is a generalized version of Harjani Sadarangani [13, Theorems 2.1 and 2.2].

2.2. Theorem. *Let* $(\mathcal{X}, \preceq)$ *be a partially ordered set and suppose that there exists a metric* d *in* $\mathcal{X}$ *such that* $(\mathcal{X}, d)$ *is a complete metric space. Let* $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ *be such that* $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, $\mathcal{T}$ *is a* $\mathcal{S}$-*non-decreasing mapping such that*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(\Theta(x, y)) - \phi(\Theta(x, y)) \text{ for } \mathcal{S}y \preceq \mathcal{S}x, \tag{2.2}$$

where

$$\Theta(x, y) = \max\left\{d(\mathcal{S}x, \mathcal{S}y), d(\mathcal{S}x, \mathcal{T}x), d(\mathcal{S}y, \mathcal{T}y), \tfrac{1}{2}[d(\mathcal{S}y, \mathcal{T}x) + d(\mathcal{S}x, \mathcal{T}y)]\right\}$$

and $\varphi, \phi : [0, \infty) \to [0, \infty)$, φ *is a continuous non-decreasing function,* ϕ *a lower semi-continuous function and* $\varphi(t) = 0 = \phi(t)$ *if and only if* $t = 0$. *Also suppose, there exists* $x_0 \in \mathcal{X}$ *with* $\mathcal{S}x_0 \preceq \mathcal{T}x_0$. *If*

$$\mathcal{T} \text{ and } \mathcal{S} \text{ are continuous,}$$

or

$$\mathcal{S}(\mathcal{X}) \text{ is a closed subspace of } \mathcal{X}, \text{ and}$$

$$\begin{cases} \text{whenever } \{\mathcal{S}x_n\} \subset \mathcal{X} \text{ is a nondecreasing sequence with } \mathcal{S}x_n \to \mathcal{S}z \text{ in } \mathcal{S}(\mathcal{X}), \\ \text{then } \mathcal{S}x_n \preceq \mathcal{S}z \text{ for all } n \text{ and } \mathcal{S}z \preceq \mathcal{S}(\mathcal{S}(z)) \end{cases} \tag{2.3}$$

hold then $\mathcal{T}$ and $\mathcal{S}$ have a coincidence point.

Further, if $\mathcal{T}$ and $\mathcal{S}$ commute at their coincidence points then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point.

Proof. If $\mathcal{T}x_0 = \mathcal{S}x_0$, then we have the coincidence point. Suppose $\mathcal{T}x_0 \neq \mathcal{S}x_0$ for the given $x_0 \in \mathcal{X}$. Now since $\mathcal{S}x_0 \preceq \mathcal{T}x_0$ and $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, we can choose $x_1 \in \mathcal{X}$ so that $\mathcal{S}(x_1) = \mathcal{T}(x_0)$. Again, from $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, we can find $x_2 \in \mathcal{X}$ so that $\mathcal{S}(x_2) = \mathcal{T}(x_1)$. Continuing this process we find a sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$\mathcal{S}x_{n+1} = \mathcal{T}x_n \text{ for all } n \geq 0. \tag{2.4}$$

Since $\mathcal{S}(x_0) \preceq \mathcal{T}(x_0)$ and $\mathcal{T}(x_0) = \mathcal{S}(x_1)$, we have $\mathcal{S}(x_0) \preceq \mathcal{S}(x_1)$. Then from (2.1),

$$\mathcal{T}(x_0) \preceq \mathcal{T}(x_1).$$

Thus, by (2.4), $\mathcal{S}(x_1) \preceq \mathcal{S}(x_2)$. Again from (2.1),

$$\mathcal{T}(x_1) \preceq \mathcal{T}(x_2),$$

that is, $\mathcal{S}(x_1) \preceq \mathcal{S}(x_2)$. Continuing, we obtain

$$\mathcal{T}(x_0) \preceq \mathcal{T}(x_1) \preceq \mathcal{T}(x_2) \preceq \cdots \preceq \mathcal{T}(x_n) \preceq \mathcal{T}(x_{n+1}) \cdots .$$

If there exists $n_0 \in \{1, 2, \cdots\}$ such that $\Theta(x_{n_0}, x_{n_0-1}) = 0$ then it is clear that $\mathcal{S}(x_{n_0-1}) = \mathcal{T}(x_{n_0}) = \mathcal{T}x_{n_0-1}$ and so $\mathcal{T}$ and $\mathcal{S}$ have a coincidence at $x = x_{n_0-1}$, and so we have finished the proof. Now we can suppose

$$\Theta(x_n, x_{n-1}) > 0 \tag{2.5}$$

for all $n \geq 1$.

First we will prove that $\lim_{n\to\infty} d(\mathcal{T}x_{n+1}, \mathcal{T}x_n) = 0$. From (2.4), we have for $n \geq 1$,

$$\begin{aligned}\Theta(x_n, x_{n-1}) &= \max\big\{d(\mathcal{S}x_n, \mathcal{S}x_{n-1}), d(\mathcal{S}x_n, \mathcal{T}x_n), d(\mathcal{S}x_{n-1}, \mathcal{T}x_{n-1}), \\ &\qquad \tfrac{1}{2}[d(\mathcal{S}x_{n-1}, \mathcal{T}x_n) + d(\mathcal{S}x_n, \mathcal{T}x_{n-1})]\big\} \\ &= \max\big\{d(\mathcal{T}x_{n-1}, \mathcal{T}x_{n-2}), d(\mathcal{T}x_{n-1}, \mathcal{T}x_n), \tfrac{1}{2}d(\mathcal{T}x_{n-2}, \mathcal{T}x_n)\big\} \\ &\leq \max\{d(\mathcal{T}x_{n-1}, \mathcal{T}x_{n-2}), d(\mathcal{T}x_{n-1}, \mathcal{T}x_n)\}.\end{aligned}$$

Now we claim that

$$d(\mathcal{T}x_{n+1}, \mathcal{T}x_n) \leq d(\mathcal{T}x_n, \mathcal{T}x_{n-1}) \tag{2.6}$$

for all $n \geq 1$. Suppose this is not true, that is, there exists $n_0 \geq 1$ such that

$$d(\mathcal{T}(x_{n_0+1}), \mathcal{T}(x_{n_0})) > d(\mathcal{T}(x_{n_0}), \mathcal{T}(x_{n_0-1})).$$

Now since $\mathcal{T}(x_{n_0}) \preceq \mathcal{T}(x_{n_0+1})$, we can use the inequality (2.2) for these elements, and then we have:

$$\begin{aligned}\varphi(d(\mathcal{T}x_{n_0}, \mathcal{T}x_{n_0-1})) &\leq \varphi(\Theta(x_{n_0}, x_{n_0-1})) - \phi(\Theta(x_{n_0}, x_{n_0-1})) \\ &\leq \varphi(\max\{d(\mathcal{T}x_{n_0-1}, \mathcal{T}x_{n_0-2}), d(\mathcal{T}x_{n_0-1}, \mathcal{T}x_{n_0})\}) \\ &\qquad - \phi(\Theta(x_{n_0}, x_{n_0-1})) \\ &= \varphi(d(\mathcal{T}x_{n_0-1}, \mathcal{T}x_{n_0})) - \phi(\Theta(x_{n_0}, x_{n_0-1})).\end{aligned}$$

This implies $\phi(\Theta(x_{n_0}, x_{n_0-1})) = 0$, by the property of ϕ we have $\Theta(x_{n_0}, x_{n_0-1}) = 0$, which contradict (2.5). Therefore, (2.6) is true and so the sequence $\{d(\mathcal{T}(x_{n+1}), \mathcal{T}(x_n))\}$ is non-increasing and bounded below. Thus there exists $\rho \geq 0$ such that $\lim_{n\to\infty} d(\mathcal{T}(x_{n+1}), \mathcal{T}(x_n)) = \rho$. Therefore, from (2.4),

$$\begin{aligned}\lim_{n\to\infty} d(\mathcal{T}(x_n), \mathcal{T}(x_{n-1})) &\leq \lim_{n\to\infty} \Theta(x_n, x_{n-1})\\ &= \lim_{n\to\infty} \max\big\{d(\mathcal{S}x_n, \mathcal{S}x_{n-1}), d(\mathcal{S}x_n, \mathcal{T}x_n),\\ &\qquad d(\mathcal{S}x_{n-1}, \mathcal{T}x_{n-1}), \tfrac{1}{2}[d(\mathcal{S}x_{n-1}, \mathcal{T}x_n) + d(\mathcal{S}x_n, \mathcal{T}x_{n-1})]\big\}\\ &= \lim_{n\to\infty} \max\{d(\mathcal{T}x_{n-1}, \mathcal{T}x_{n-2}), d(\mathcal{T}x_{n-1}, \mathcal{T}x_n),\\ &\qquad \tfrac{1}{2}d(\mathcal{T}x_{n-2}, \mathcal{T}x_n)\}.\end{aligned}$$

This implies $\rho \leq \lim_{n\to\infty} \Theta(x_n, x_{n-1}) \leq \rho$ and so $\lim_{n\to\infty} \Theta(x_n, x_{n-1}) = \rho$. By the lower semi-continuity of ϕ we have

$$\phi(\rho) \leq \liminf_{n\to\infty} \phi(\Theta(x_n, x_{n+1})).$$

Now we claim that $\rho = 0$. From (2.2), we have

$$\varphi(d(\mathcal{T}x_n, \mathcal{T}x_{n-1})) \leq \varphi(\Theta(x_n, x_{n-1})) - \phi(\Theta(x_n, x_{n-1})),$$

and taking the upper limit as $n \to \infty$, we have

$$\begin{aligned}\varphi(\rho) &\leq \varphi(\rho) - \liminf_{n\to\infty} \phi(\Theta(x_n, x_{n+1}))\\ &\leq \varphi(\rho) - \phi(\rho),\end{aligned}$$

that is, $\phi(\rho) = 0$. Thus by the property of ϕ, we have $\rho = 0$.

Next we show that $\{x_n\}$ is Cauchy. Suppose this is not true. Then there is an $\varepsilon > 0$ such that for any integer k there exist integers $m(k) > n(k) > k$ such that

$$d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)}) \geq \varepsilon. \tag{2.7}$$

For every integer k, let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying (2.7) and such that

$$d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1}) < \varepsilon. \tag{2.8}$$

Now

$$\begin{aligned}\varepsilon &\leq d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)})\\ &\leq d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1}) + d(\mathcal{T}x_{m(k)-1}, \mathcal{T}x_{m(k)}).\end{aligned}$$

Then by $\rho = 0$ and (2.8) it follows that

$$\lim_{k\to\infty} d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)}) = \varepsilon. \tag{2.9}$$

Also, by the triangle inequality, we have

$$\left|d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1}) - d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)})\right| < d(\mathcal{T}x_{m(k)-1}, \mathcal{T}x_{m(k)}).$$

By using (2.9) we get

$$\lim_{k\to\infty} d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1}) = \varepsilon. \tag{2.10}$$

Now by (2.4) we get

$$\begin{aligned}
d(\mathcal{T}x_{n(k)},\mathcal{T}x_{m(k)-1}) &\le \Theta(x_{n(k)},x_{m(k)-1})\\
&= \max\big\{d(\mathcal{S}x_{n(k)},\mathcal{S}x_{m(k)-1}),d(\mathcal{S}x_{n(k)},\mathcal{T}x_{n(k)}),\\
&\quad d(\mathcal{S}x_{m(k)-1},\mathcal{T}x_{m(k)-1}),\\
&\qquad \tfrac{1}{2}[d(\mathcal{S}x_{m(k)-1},\mathcal{T}x_{n(k)})+d(\mathcal{S}x_{n(k)},\mathcal{T}x_{m(k)-1})]\big\}\\
&\le \max\big\{d(\mathcal{T}x_{n(k)-1},\mathcal{T}x_{m(k)-2}),d(\mathcal{T}x_{n(k)-1},\mathcal{T}x_{n(k)}),\\
&\quad d(\mathcal{T}x_{m(k)-2},\mathcal{T}x_{m(k)-1})\\
&\qquad \tfrac{1}{2}[d(\mathcal{T}x_{m(k)-2},\mathcal{T}x_{n(k)})+d(\mathcal{T}x_{n(k)-1},\mathcal{T}x_{m(k)-1})]\big\}\\
&\le \max\big\{d(\mathcal{T}x_{n(k)-1},\mathcal{T}x_{m(k)-2}),d(\mathcal{T}x_{n(k)-1},\mathcal{T}x_{n(k)}),\\
&\quad d(\mathcal{T}x_{m(k)-2},\mathcal{T}x_{m(k)-1})\frac{1}{2}[d(\mathcal{T}x_{m(k)-2},\mathcal{T}x_{n(k)-1})\\
&\qquad +d(\mathcal{T}x_{n(k)-1},\mathcal{T}x_{n(k)})+d(\mathcal{T}x_{n(k)-1},\mathcal{T}x_{m(k)-1})]\big\},
\end{aligned}$$

and letting $k\to\infty$ and using (2.9) and (2.10), we have

$$\varepsilon \le \lim_{k\to\infty}\Theta(x_{n(k)},x_{m(k)-1}) \le \varepsilon,$$

and so

$$\lim_{k\to\infty}\Theta(x_{n(k)},x_{m(k)-1}) = \varepsilon.$$

By the lower semi-continuity of ϕ we have

$$\phi(\varepsilon) \le \liminf_{k\to\infty}\phi(\Theta(x_{n(k)},x_{m(k)-1})).$$

Now by (2.2) we get

$$\begin{aligned}
\varphi(\varepsilon) &= \limsup_{k\to\infty}\varphi(d(\mathcal{T}x_{n(k)},\mathcal{T}x_{m(k)}))\\
&\le \limsup_{k\to\infty}\varphi(d(\mathcal{T}x_{n(k)},\mathcal{T}x_{n(k)+1})+d(\mathcal{T}x_{n(k)},\mathcal{T}x_{m(k)-1}))\\
&= \limsup_{k\to\infty}\varphi(d(\mathcal{T}x_{n(k)},\mathcal{T}x_{m(k)-1}))\\
&\le \limsup_{k\to\infty}[\varphi(\Theta(x_{n(k)},x_{m(k)-1}))-\phi(\Theta(x_{n(k)},x_{m(k)-1}))]\\
&= \varphi(\varepsilon)-\liminf_{k\to\infty}\phi(\Theta(x_{n(k)},x_{m(k)-1}))\\
&\le \varphi(\varepsilon)-\phi(\varepsilon),
\end{aligned}$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. From the completeness of $\mathcal{X}$ there exists $z\in\mathcal{X}$ such that $x_n\to z$ as $n\to\infty$. If $\mathcal{T}$ and $\mathcal{S}$ are continuous then it is clear that $\mathcal{T}z=\mathcal{S}z$. If (2.3) holds, then we have $\mathcal{S}x_n\preceq\mathcal{S}z$ for all n. Since by (2.4) we have $\{\mathcal{T}(x_n)\}=\{\mathcal{S}(x_{n+1})\}\subseteq\mathcal{S}(\mathcal{X})$ and $\mathcal{S}(\mathcal{X})$ is closed, there exists $z\in\mathcal{X}$ such that $\lim_{n\to\infty}\mathcal{S}(x_n)=\mathcal{S}(z)$.

Therefore, for all n, we can use the inequality (2.2) for x_n and z. Since

$$\begin{aligned}
\Theta(z,x_n) &= \max\big\{d(\mathcal{S}z,\mathcal{S}x_n),d(\mathcal{S}z,\mathcal{T}z),d(\mathcal{S}x_n,\mathcal{T}x_n),\tfrac{1}{2}[d(\mathcal{S}x_n,\mathcal{T}z)+d(\mathcal{S}z,\mathcal{T}x_n)]\big\}\\
&= \max\big\{d(\mathcal{S}z,\mathcal{S}x_n),d(\mathcal{S}z,\mathcal{T}z),d(\mathcal{S}x_n,\mathcal{S}x_{n+1}),\\
&\qquad \tfrac{1}{2}[d(\mathcal{S}x_n,\mathcal{T}z)+d(\mathcal{S}z,\mathcal{S}x_{n+1})]\big\},
\end{aligned}$$

and so $\lim_{n\to\infty}\Theta(z,x_n)=d(\mathcal{S}z,\mathcal{T}z)$. Hence we have

$$\begin{aligned}\varphi(d(\mathcal{T}z,\mathcal{S}z)) &= \limsup_{n\to\infty}\varphi(d(\mathcal{T}z,\mathcal{S}x_{n+1}))\\ &= \limsup_{n\to\infty}\varphi(d(\mathcal{T}z,\mathcal{T}x_n))\\ &\le \limsup_{n\to\infty}[\varphi(\Theta(z,x_n))-\phi(\Theta(z,x_n))]\\ &\le \varphi(d(\mathcal{T}z,\mathcal{S}z))-\phi(d(\mathcal{T}z,\mathcal{S}z)).\end{aligned}$$

By the property of ϕ we have $\mathcal{T}z=\mathcal{S}z$. Thus we have proved that $\mathcal{T}$ and $\mathcal{S}$ have a coincidence.

Suppose now that $\mathcal{T}$ and $\mathcal{S}$ commute at z. Set $w=\mathcal{S}z=\mathcal{T}z$. Then,

$$\mathcal{T}(w)=\mathcal{T}(\mathcal{S}(z))=\mathcal{S}(\mathcal{T}(z))=\mathcal{S}(w). \tag{2.11}$$

Since from (2.3), we have $\mathcal{S}z\preceq\mathcal{S}(\mathcal{S}(z))=\mathcal{S}(w)$ and as $\mathcal{S}z=\mathcal{T}z$ and $\mathcal{S}w=\mathcal{T}w$, from (2.4) we get

$$\varphi(d(\mathcal{T}z,\mathcal{T}w))\le\varphi((\Theta(z,w))-\phi(\Theta(z,w))$$

where

$$\begin{aligned}\Theta(z,w) &= \max\left\{d(\mathcal{S}z,\mathcal{S}w),d(\mathcal{S}z,\mathcal{T}z),d(\mathcal{S}w,\mathcal{T}w),\tfrac{1}{2}[d(\mathcal{S}w,\mathcal{T}z)+d(\mathcal{S}z,\mathcal{T}w)]\right\}\\ &= d(w,\mathcal{T}z).\end{aligned}$$

Hence

$$\varphi(d(w,\mathcal{T}w))=\varphi(d(\mathcal{T}z,\mathcal{T}w))\le\varphi(d(w,\mathcal{T}w))-\phi(d(w,\mathcal{T}w))),$$

which implies

$$\phi(d(w,\mathcal{T}w)))\le 0,$$

which is possible only when $w=\mathcal{T}w$. Thus $w=\mathcal{T}w=\mathcal{S}w$. Hence w is a common fixed point of $\mathcal{T}$ and $\mathcal{S}$. Thus, the proof is complete. □

The following corollary is a generalized version of Harjani Sadarangani [12, Theorems 2 and 3].

2.3. Corollary. *Let $(\mathcal{X},\preceq)$ be a partially ordered set and suppose that there exists a metric d in $\mathcal{X}$ such that $(\mathcal{X},d)$ is a complete metric space. Let $\mathcal{T},\mathcal{S}:\mathcal{X}\to\mathcal{X}$ be such that $\mathcal{T}(\mathcal{X})\subset\mathcal{S}(\mathcal{X})$, $\mathcal{T}$ is a $\mathcal{S}$-non-decreasing mapping such that*

$$d(\mathcal{T}x,\mathcal{T}y))\le\Theta(x,y)-\phi(\Theta(x,y))\ \textit{for}\ \mathcal{S}y\preceq\mathcal{S}x,$$

where

$$\Theta(x,y)=\max\left\{d(\mathcal{S}x,\mathcal{S}y),d(\mathcal{S}x,\mathcal{T}x),d(\mathcal{S}y,\mathcal{T}y),\tfrac{1}{2}[d(\mathcal{S}y,\mathcal{T}x)+d(\mathcal{S}x,\mathcal{T}y)]\right\},$$

$\phi:[0,\infty)\to[0,\infty)$ is a lower semi-continuous function and $\varphi(t)=0=\phi(t)$ if and only if $t=0$. Also suppose, there exists $x_0\in\mathcal{X}$ with $\mathcal{S}x_0\preceq\mathcal{T}x_0$. If

$$\mathcal{T},\mathcal{S}\ \textit{are continuous}$$

or

$$\mathcal{S}(\mathcal{X})\ \textit{is closed subspace of}\ \mathcal{X}\ \textit{and}$$

$$\begin{cases}\textit{whenever}\ \{\mathcal{S}x_n\}\subset\mathcal{X}\ \textit{is a nondecreasing sequence with}\ \mathcal{S}x_n\to\mathcal{S}z\ \textit{in}\ \mathcal{S}(\mathcal{X}),\\ \textit{then}\ \mathcal{S}x_n\preceq\mathcal{S}z\ \textit{for all}\ n\ \textit{and}\ \mathcal{S}z\preceq\mathcal{S}(\mathcal{S}(z))\end{cases}$$

holds. Then $\mathcal{T}$ and $\mathcal{S}$ have a coincidence point.

Further, if $\mathcal{T}$ and $\mathcal{S}$ commute at their coincidence points then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point.

If we tale $\mathcal{S} = I$, the identity mapping, in Theorem 2.2 and Corollary 2.3, then we have following corollaries as generalized versions of Harjani Sadarangani [13] and Harjani Sadarangani [12], respectively.

2.4. Corollary. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and suppose that there exists a metric d in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$, where $\mathcal{T}$ is a non-decreasing mapping such that*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(\Theta(x, y)) - \phi(\Theta(x, y)) \text{ for } y \preceq x, \tag{2.12}$$

where

$$\Theta(x, y) = \max\left\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \tfrac{1}{2}[d(y, \mathcal{T}x) + d(x, \mathcal{T}y)]\right\}$$

and $\varphi, \phi : [0, \infty) \to [0, \infty)$, φ is continuous and non-decreasing, ϕ is lower semi-continuous and $\varphi(t) = 0 = \phi(t)$ if and only if $t = 0$. Also suppose there exists $x_0 \in \mathcal{X}$ with $x_0 \preceq \mathcal{T}x_0$. If

$$\mathcal{T} \text{ is continuous}$$

or

$$\begin{cases} \text{whenever } \{x_n\} \subset \mathcal{X} \text{ is a nondecreasing sequence with } x_n \to z \text{ in } \mathcal{X}, \\ \text{then } x_n \preceq z \text{ for all } n \end{cases}$$

holds then $\mathcal{T}$ has a fixed point.

2.5. Corollary. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and suppose that there exists a metric d in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$, $\mathcal{T}$ a non-decreasing mapping such that*

$$d(\mathcal{T}x, \mathcal{T}y)) \leq \Theta(x, y) - \phi(\Theta(x, y)) \text{ for } \mathcal{S}y \preceq \mathcal{S}x,$$

where

$$\Theta(x, y) = \max\left\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \tfrac{1}{2}[d(y, \mathcal{T}x) + d(x, \mathcal{T}y)]\right\}$$

and $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function satidfying $\phi(t) = 0$ if and only if $t = 0$. Also suppose, there exists $x_0 \in \mathcal{X}$ with $x_0 \preceq \mathcal{T}x_0$. If

$$\mathcal{T} \text{ is continuous}$$

or

$$\begin{cases} \text{whenever } \{x_n\} \subset \mathcal{X} \text{ is a nondecreasing sequence with } x_n \to z \text{ in } \mathcal{X}, \\ \text{then } x_n \preceq z \text{ for all } n \end{cases}$$

holds, then $\mathcal{T}$ has a fixed point.

2.6. Remark. In [19, Theorem 1.1] it is proved that if

(2.13) every pair of elements has a lower bound and upper bound,

then for every $x \in \mathcal{X}$,

$$\lim_{n\to\infty} \mathcal{T}^n x = y,$$

where y is the fixed point of $\mathcal{T}$ such that

$$y = \lim_{n\to\infty} \mathcal{T}^n x_0$$

and hence $\mathcal{T}$ has a unique fixed point. If condition (2.13) fails, it is possible to find examples of functions $\mathcal{T}$ with more than one fixed point. There are some examples to illustrate this fact in [15].

2.7. Example. Let $\mathcal{X} = \mathbb{R}$ and consider a relation on $\mathcal{X}$ as follows:

$$x \preceq y \Longleftrightarrow \{(x = y) \text{ or } (x, y \in [0,1] \text{ with } x \le y).$$

It is easy to see that $\preceq$ is a partial order on $\mathcal{X}$. Let d be Euclidean metric on $\mathcal{X}$. Now define a self map of $\mathcal{X}$ as follows:

$$\mathcal{T}x = \begin{cases} 2x - \dfrac{3}{2} & \text{if } x > 1, \\ \dfrac{x}{2} & \text{if } 0 \le x \le 1, \\ 0 & \text{if } x < 0. \end{cases}$$

Now we claim that the condition (2.12) of Corollary 2.4 is satisfied with $\varphi(t) = t$, $\phi(t) = \frac{t}{2}$. Indeed, if $x, y \notin [0,1]$, then $x \preceq y \iff x = y$. Therefore, since $d(\mathcal{T}x, \mathcal{T}y) = 0$, then the condition (2.12) is satisfied.

Again if $x \in [0,1]$ and $y \notin [0,1]$, then x and y are not comparative. Now if $x, y \in [0,1]$, then $x \preceq y \iff x \le y$ and

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) &= d(\frac{x}{2}, \frac{y}{2}) \\ &= \tfrac{1}{2} d(x,y) \\ &\le \tfrac{1}{2}\Theta(x,y) \\ &= \Theta(x,y) - \tfrac{1}{2}\Theta(x,y) \\ &= \Theta(x,y) - \phi(\Theta(x,y)). \end{aligned}$$

Also it is easy to see that the other conditions of Corollary 2.4 are satisfied and so $\mathcal{T}$ has a fixed point in $\mathcal{X}$. Also note that the the weak contractive condition of Theorem 1.3 of this paper and of [22, Corollary 2.2] is not satisfied.

3. Application to integral type problems

We present here applications of the previous section. We obtain some fixed point theorems for pairs of mappings satisfying a general contractive condition of integral type in complete partially ordered metric spaces.

Before we start the theorem we establish the following terminology:

$$\Upsilon = \Big\{\Psi : \mathbb{R}^+ \to \mathbb{R}^+ : \Psi \text{ is a Lebesgue integrable mapping which is summable and non-negative and satisfies } \int_0^{\epsilon} \Psi(t)\,dt > 0 \text{ for each } \epsilon > 0\Big\}.$$

3.1. Theorem. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and suppose that there exists a metric d in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Let $\mathcal{S}$, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be such that $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, $\mathcal{T}$ is a $\mathcal{S}$-non-decreasing mapping and satisfying*

$$\text{(3.1)} \qquad \int_0^{\varphi(d(\mathcal{T}x,\mathcal{T}y))} \Psi(t)\,dt \le \int_0^{\varphi(\Theta(x,y))} \Psi(t)\,dt - \int_0^{\phi(\Theta(x,y))} \Psi(t)\,dt \text{ for } \mathcal{S}y \preceq \mathcal{S}x,$$

where

$$\Phi(x,y) = \max\left\{d(\mathcal{S}x, \mathcal{S}y), d(\mathcal{S}x, \mathcal{T}x), d(\mathcal{S}y, \mathcal{S}y), \tfrac{1}{2}[d(\mathcal{S}y, \mathcal{T}x) + d(\mathcal{S}x, \mathcal{S}y)]\right\}$$

and $\varphi, \phi : [0,\infty) \to [0,\infty)$, φ is a continuous, nondecreasing function, ϕ a lower semi-continuous function and $\varphi(t) = 0 = \phi(t)$ if and only if $t = 0$. If

$\mathcal{T}$ and $\mathcal{S}$ are continuous

or

$$\mathcal{S}(\mathcal{X}) \text{ is closed subspace of } \mathcal{X} \text{ and}$$

$$\begin{cases} \text{whenever } \{\mathcal{S}x_n\} \subset \mathcal{X} \text{ is a nondecreasing sequence with } \mathcal{S}x_n \to \mathcal{S}z \text{ in } \mathcal{S}(\mathcal{X}), \\ \text{then } \mathcal{S}x_n \preceq \mathcal{S}z \text{ for all } n \text{ and } \mathcal{S}z \preceq \mathcal{S}(\mathcal{S}(z)) \end{cases}$$

holds, then $\mathcal{T}$ and $\mathcal{S}$ have a coincidence point.

Further, if $\mathcal{T}$ and $\mathcal{S}$ commute at their coincidence points then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point.

Proof. Define $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ by $\Lambda(x) = \int_0^x \Psi(t)dt$. Then Λ is continuous and non-decreasing with $\Lambda(0) = 0$, and equation (3.1) becomes

$$\Lambda\left(\varphi(d(\mathcal{T}x, \mathcal{T}y))\right) \leq \Lambda\left(\varphi(\Theta(x, y))\right) - \Lambda\left(\phi(\Theta(x, y))\right),$$

which further can be written as

$$\varphi_1(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi_1(\Theta(x, y) - \phi_1(\Theta(x, y)),$$

where $\phi_1 = \Lambda \circ \phi$ and $\varphi_1 = \Lambda \circ \varphi$. Clearly, ϕ_1, φ_1 are continuous and non-decreasing and satisfy $\phi_1(t) = 0 = \varphi_1(t)$ if and only if $t = 0$. Hence, by Theorem 2.2, the proof is complete. □

3.2. Theorem. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and suppose that there exists a metric d in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Let $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ be such that $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, where $\mathcal{T}$ is an $\mathcal{S}$-non-decreasing mapping.*

Moreover, if there exists $h \in [0, 1)$ such that

$$\int_0^{d(\mathcal{T}x, \mathcal{T}y)} \Psi(t)\, dt \leq h \cdot \int_0^{\Theta(x,y)} \Psi(t)\, dt \text{ for } \mathcal{S}y \preceq \mathcal{S}x,$$

where

$$\Phi(x, y) = \max\left\{d(\mathcal{S}x, \mathcal{S}y), d(\mathcal{S}x, \mathcal{T}x), d(\mathcal{S}y, \mathcal{S}y), \tfrac{1}{2}[d(\mathcal{S}y, \mathcal{T}x) + d(\mathcal{S}x, \mathcal{S}y)]\right\}$$

and φ, $\phi : [0, \infty) \to [0, \infty)$, where φ is a continuous, non-decreasing function, ϕ a lower semi-continuous function and $\varphi(t) = 0 = \phi(t)$ if and only if $t = 0$. If

$$\mathcal{T}, \mathcal{S} \text{ are continuous}$$

or

$$\mathcal{S}(\mathcal{X}) \text{ is closed subspace of } \mathcal{X} \text{ and}$$

$$\begin{cases} \text{whenever } \{\mathcal{S}x_n\} \subset \mathcal{X} \text{ is a nondecreasing sequence with } \mathcal{S}x_n \to \mathcal{S}z \text{ in } \mathcal{S}(\mathcal{X}), \\ \text{then } \mathcal{S}x_n \preceq \mathcal{S}z \text{ for all } n \text{ and } \mathcal{S}z \preceq \mathcal{S}(\mathcal{S}(z)) \end{cases}$$

holds, then $\mathcal{T}$ and $\mathcal{S}$ have a coincidence point.

Further, if $\mathcal{T}$ and $\mathcal{S}$ commute at their coincidence points then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point.

Proof. Following a similar argument, we can define $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ by $\Lambda(x) = \int_0^x \Psi(t)\, dt$ and show the necessary properties. If we fix $\phi = (1 - h)\varphi$, then by Theorem 2.2, the proof is complete. □

3.3. Remark. We can also establish similar types of integral results as applications of Corollary 2.3 – Corollary 2.5.

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