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ON SOLUTIONS OF A GENERALIZED QUADRATIC FUNCTIONAL EQUATION OF PEXIDER TYPE

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Abstract

In this paper we study general solutions of the following Pexider functional equation

$$(4-k)f_1\Big(\sum_{i=1}^{k} x_i\Big) + \sum_{j=2}^{k} f_j\Big(\Big(\sum_{i=1, i\neq j}^{k} x_i\Big) - x_j\Big)\Big) + f_{k+1}\Big(-x_1 + \sum_{i=2}^{k} x_i\Big) = 4\sum_{j=1}^{k} f_{k+j+1}(x_j),$$

on a vector space over a field of characteristic different from 2, for $k \geq 3$.

Keywords: Quadratic functional equation, Pexiderized quadratic functional equation.

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1. Introduction

It is easy to see that the quadratic function $f(x) = x^2$ is a solution for each of the following functional equations

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

$$f(x+y+z) + f(x-y+z) + f(x+y-z) + f(-x+y+z) = 4f(x) + 4f(y) + 4f(z). \tag{1.2}$$

So, it is natural that these equations are called quadratic functional equations. In particular, every solution of the original quadratic functional equation (1.1) is said to be a quadratic function.

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It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B such that $f(x) = B(x,x)$, for all x (see [1], [2], [7], [14], [15]).

It was proved in [12] that the functional equations (1.1) and (1.2) are equivalent. Solutions and the Hyers–Ulam–Rassias stability of the following generalization of (1.2),

$$Df(x_1,\dots,x_k) := (4-k)f(\sum_{i=1}^{k} x_i) + \sum_{j=1}^{k} f((\sum_{i=1,i\neq j}^{k} x_i) - x_j)) - 4\sum_{i=1}^{k} f(x_i), \tag{1.3}$$

also were investigated in [8].

Solutions and the stability of various functional equations in several variables are also studied by many authors (see for example [3]-[6], [9]-[11], [13], [16], [17]-[21]).

Solutions of the following Pexiderized form of (1.2) were considered in [12]

$$\begin{aligned} f_1(x+y+z) + f_2(x-y+z) + f_3(x+y-z) + f_4(-x+y+z) \\ = 4f_5(x) + 4f_6(y) + 4f_7(z). \end{aligned} \tag{1.4}$$

Indeed the following theorem was proved in [12].

1.1. Theorem. *Assume that X and Y are vector spaces over fields of characteristic different 2, respectively. The functions $f_i : X \to Y$, $i = 1,\dots,7$, satisfy the functional equation (1.4) if and only if there exist a quadratic function $Q : X \to Y$, constants $c_i \in Y$, $i = 1,\dots,7$, and additive functions $a_i : X \to Y$, $i = 1,\dots,4$, such that*

$$\begin{aligned} f_1(x) &= Q(x) + 2a_1(x) + a_2(x) + a_3(x) - a_4(x) + c_1, \\ f_2(x) &= Q(x) - a_2(x) + a_3(x) + a_4(x) + c_2, \\ f_3(x) &= Q(x) + a_2(x) - a_3(x) + a_4(x) + c_3, \\ f_4(x) &= Q(x) - 2a_1(x) + a_2(x) + a_3(x) + a_4(x) + c_4, \\ f_5(x) &= Q(x) + a_1(x) + c_5, \\ f_6(x) &= Q(x) + a_2(x) + c_6, \\ f_7(x) &= Q(x) + a_3(x) + c_7, \end{aligned}$$

with

$$c_1 + c_2 + c_4 + c_4 = 4c_5 + 4c_6 + 4c_7.$$

To prove our main result, we need the proof of this theorem. In this proof with $F_i(x) = f_i(x) - f_i(0)$, if F_i^e and F_i^o denote the even part and odd part of F_i, respectively, then it is proved that there exist additive functions $a_i : X \to Y$, $i = 1,\dots,4$, such that

$$F_1^o = 2a_1 + a_2 + a_3 - a_4, \qquad F_2^o = -a_2 + a_3 + a_4, \qquad F_3^o = a_2 - a_3 + a_4, \tag{1.5}$$

$$F_4^o = -2a_1 + a_2 + a_3 + a_4, \qquad F_5^o = a_1, \qquad F_6^o = a_2, \qquad F_7^o = a_3 \tag{1.6}$$

and there exists a quadratic function $Q : X \to Y$ such that

$$F_1^e = F_2^e = F_3^e = F_4^e = F_5^e = F_6^e = F_7^e = Q. \tag{1.7}$$

In this paper we consider the following Pexiderized form of (1.3)

$$\begin{aligned} (4-k)f_1(\sum_{i=1}^{k} x_i) + \sum_{j=2}^{k} f_j((\sum_{i=1,i\neq j}^{k} x_i) - x_j)) + f_{k+1}(-x_1 + \sum_{i=2}^{k} x_i) \\ = 4\sum_{j=1}^{k} f_{k+j+1}(x_j), \end{aligned} \tag{1.8}$$

for $k \geq 3$, and general solutions of this functional equation will be investigated in Section 2. Note that f_1 does not exist when $k = 4$.

2. Solution of equation (1.8)

2.1. Theorem. *Assume that X and Y are vector spaces over fields of characteristic different from 2, respectively and k is a natural number with $k \geq 3$. Then functions $f_i : X \to Y$, $i = 1, \ldots, 2k+1$, satisfy the functional equation (1.8) if and only if there exist a quadratic function $Q : X \to Y$, constants $c_i \in Y$, $i = 1, \ldots, 2k+1$, and additive functions $a_i : X \to Y$, $i = 1, \ldots, k$, and $a'_i : X \to Y$, $i = 3, \ldots, k$ such that*

$$\begin{aligned} f_1(x) &= Q(x) + \frac{1}{4-k}(2a_1 + (5-k)a_2 + \sum_{i=3}^{k} a_i - \sum_{i=3}^{k} a'_i)(x) + c_1, \ k \neq 4, \\ f_2(x) &= Q(x) - a_2(x) + a_3(x) + a'_3(x) + c_2, \\ f_j(x) &= Q(x) + a_2(x) - a_j(x) + a'_j(x) + c_j, \ j = 3, 4, \ldots, k \\ f_{k+j+1}(x) &= Q(x) + a_j(x) + c_{k+j+1}, \ j = 0, 1, 2, \ldots, k \end{aligned} \tag{2.1}$$

with

$$\sum_{i=1}^{k+1} c_i = 4 \sum_{i=k+2}^{2k+1} c_i. \tag{2.2}$$

Proof. Define $c_i = f_i(0)$, $i = 1, \ldots, 2k+1$. By letting $x_i = 0$, $i \geq 1$, in (1.8) it is clear that the c_i's satisfy the relation (2.2). For $i = 1, \ldots, 2k+1$, define $F_i(x) = f_i(x) - c_i$. It then follows from (1.8) and (2.2) that the F_i's satisfy the functional equation (1.8) with $F_i(0) = 0$.

Denote by $F_i^e(x)$ and $F_i^o(x)$ the even part and the odd part of $F_i(x)$, respectively. If we replace x_i in (1.8) by $-x_i$, for any $i = 1, \ldots, k$, and if we add (subtract) the resulting equation to (from) (1.8), we can see that the F_i^o's as well as the F_i^e's also satisfy (1.8).

Let us consider (1.8) for the F_i^o's

$$\begin{aligned} (4-k)F_1^o(\sum_{i=1}^{k} x_i) + \sum_{j=2}^{k} F_j^o((\sum_{i=1, i\neq j}^{k} x_i) - x_j)) + F_{k+1}^o(-x_1 + \sum_{i=2}^{k} x_i) \\ = 4\sum_{j=1}^{k} F_{k+j+1}^o(x_j). \end{aligned} \tag{2.3}$$

Step I By letting $x_i = 0$, $i \geq 4$, in (2.3) we get

$$\begin{aligned} \big((4-k)F_1^o + F_4^o + F_5^o + \cdots + F_k^o\big)(x_1 + x_2 + x_3) + F_2^o(x_1 - x_2 + x_3) \\ + F_3^o(x_1 + x_2 - x_3) + F_{k+1}^o(-x_1 + x_2 + x_3) \\ = 4F_{k+2}^o(x_1) + 4F_{k+3}^o(x_2) + 4F_{k+4}^o(x_3). \end{aligned}$$

So, by relations (1.5) and (1.6), there exist additive functions $a_i : X \to Y$, $i = 1, 2, 3$, and an additive function $a'_3 : X \to Y$ such that

$$\begin{aligned} F_{k+2}^o &= a_1 \\ F_{k+3}^o &= a_2 \\ F_{k+4}^o &= a_3 \\ F_3^o &= a_2 - a_3 + a'_3 \\ F_2^o &= -a_2 + a_3 + a'_3 \end{aligned}$$

$$(2.4)\qquad (4-k)F_1^o + F_4^o + F_5^o + \cdots + F_k^o = 2a_1 + a_2 + a_3 - a_3'$$

and also

$$F_{k+1}^o = -2a_1 + a_2 + a_3 + a_3'.$$

Step II By putting $x_i = 0$, for $i \geq 3, i \neq 4$, in (2.3) we have

$$\begin{aligned}\big((4-k)F_1^o + F_3^o + F_5^o + F_6^o + \cdots + F_k^o\big)(x_1+x_2+x_4) + F_2^o(x_1-x_2+x_4)\\ + F_4^o(x_1+x_2-x_4) + F_{k+1}^o(-x_1+x_2+x_4)\\ = 4F_{k+2}^o(x_1) + 4F_{k+3}^o(x_2) + 4F_{k+5}^o(x_4).\end{aligned}$$

Then by relations (1.5) and (1.6), there exist additive functions $a_4 : X \to Y$ and $a_i' : X \to Y$, $i = 3, 4$, such that

$$\begin{aligned}&F_{k+2}^o = a_1\\ &F_{k+3}^o = a_2\\ &F_{k+5}^o = a_4\\ &F_4^o = a_2 - a_4 + a_4'\\ &F_2^o = -a_2 + a_3 + a_3'\\ &(4-k)F_1^o + F_3^o + F_5^o + F_6^o + \cdots + F_k^o = 2a_1 + a_2 + a_4 - a_4'.\end{aligned}$$

Step III If we put $x_i = 0$ for $i \geq 3, i \neq j$, in (2.3) then we get

$$\begin{aligned}\big((4-k)F_1^o + F_3^o + \cdots + F_{j-1}^o + F_{j+1}^o + F_{j+2}^o + \cdots + F_k^o\big)(x_1+x_2+x_j)\\ + F_2^o(x_1-x_2+x_j) + F_j^o(x_1+x_2-x_j) + F_{k+1}^o(-x_1+x_2+x_j)\\ = 4F_{k+2}^o(x_1) + 4F_{k+3}^o(x_2) + 4F_{k+j+1}^o(x_j).\end{aligned}$$

So, by relations (1.5) and (1.6), there exist additive functions $a_j : X \to Y$, and $a_j' : X \to Y$ such that

$$\begin{aligned}&F_{k+2}^o = a_1\\ &F_{k+3}^o = a_2\\ &F_{k+j+1}^o = a_j\\ &F_j^o = a_2 - a_j + a_j'\\ &F_2^o = -a_2 + a_3 + a_3'\\ &(4-k)F_1^o + F_3^o + \cdots + F_{j-1}^o + F_{j+1}^o + F_{j+2}^o + \cdots + F_k^o = 2a_1 + a_2 + a_j - a_j'.\end{aligned}$$

Step IV If we put $x_i = 0$ for $i \geq 3, i \neq k$, in (2.3), then we have

$$\begin{aligned}\big((4-k)F_1^o + F_3^o + F_4^o + \cdots + F_{k-1}^o\big)(x_1+x_2+x_k) + F_2^o(x_1-x_2+x_k)\\ + F_k^o(x_1+x_2-x_k) + F_{k+1}^o(-x_1+x_2+x_k)\\ = 4F_{k+2}^o(x_1) + 4F_{k+3}^o(x_2) + 4F_{2k+1}^o(x_k).\end{aligned}$$

Using relations (1.5) and (1.6) we may find additive functions $a_k : X \to Y$, and $a'_k : X \to Y$ such that

$$F^o_{k+2} = a_1$$
$$F^o_{k+3} = a_2$$
$$F^o_{2k+1} = a_k$$
$$F^o_k = a_2 - a_k + a'_k$$
$$F^o_2 = -a_2 + a_3 + a'_3$$
$$(4-k)F^o_1 + F^o_3 + F^o_4 + \cdots + F^o_{k-1} = 2a_1 + a_2 + a_k - a'_k.$$

By these steps we get all F^o_i's, $i = 2, \ldots, 2k+1$. If $k = 4$, then there is no need to consider F^o_1. For $k \neq 4$, using (2.4) we may find F^o_1 as follows

$$F^o_1(x) = \frac{1}{4-k}(2a_1 + (5-k)a_2 + (a_3 + a_4 + a_5 + \cdots + a_k) - (a'_3 + a'_4 + a'_5 + \cdots + a'_k)).$$

We will now deal with (1.8) associated with the F^e_i's;

$$\begin{aligned}(4-k)F^e_1(\sum_{i=1}^k x_i) + \sum_{j=2}^k F^e_j\Big(\Big(\sum_{i=1, i\neq j}^k x_i\Big) - x_j\Big) + F^e_{k+1}(-x_1 + x_2 + \cdots + x_k) \\ = 4F^e_{k+2}(x_1) + F^e_{k+3}(x_2) + F^e_{k+4}(x_3) + \cdots + F^e_{2k+1}(x_k).\end{aligned} \tag{2.5}$$

By putting $x_i = 0$, for $i \geq 4$, in (2.5) we get

$$\begin{aligned}\big((4-k)F^e_1 + F^e_4 + \cdots + F^e_k\big)(x_1 + x_2 + x_3) + F^e_2(x_1 - x_2 + x_3) \\ + F^e_3(x_1 + x_2 - x_3) + F^e_{k+1}(-x_1 + x_2 + x_3) \\ = 4F^e_{k+2}(x_1) + 4F^e_{k+3}(x_2) + 4F^e_{k+4}(x_3).\end{aligned}$$

Hence, by (1.7), there exists a quadratic function $Q : X \to Y$ such that

$$(4-k)F^e_1 + F^e_4 + \cdots + F^e_k = F^e_2 = F^e_3 = F^e_{k+1} = F^e_{k+2} = F^e_{k+3} = F^e_{k+4} = Q. \tag{2.6}$$

Also for any $j = 4, 5, \ldots, k$, by letting $x_i = 0$, for $3 \leq i \leq k, i \neq j$, in (2.5) we get

$$\begin{aligned}\big((4-k)F^e_1 + F^e_3 + \cdots + F^e_{j-1} + F^e_{j+1} + \cdots + F^e_k\big)(x_1 + x_2 + x_j) \\ + F^e_2(x_1 - x_2 + x_j) + F^e_j(x_1 + x_2 - x_j) + F^e_{k+1}(-x_1 + x_2 + x_j) \\ = 4F^e_{k+2}(x_1) + 4F^e_{k+3}(x_2) + 4F^e_{k+j+1}(x_j).\end{aligned}$$

Thus using (1.7), we get

$$\begin{aligned}(4-k)F^e_1 + F^e_3 + \cdots + F^e_{j-1} + F^e_{j+1} + \cdots + F^e_k \\ = F^e_2 = F^e_j = F^e_{k+1} = F^e_{k+2} = F^e_{k+3} = F^e_{k+j+1} = Q.\end{aligned} \tag{2.7}$$

Therefore, for $i \geq 2$, F^e_i is concluded by (2.6) and (2.7). For getting F^e_1, when $k \neq 4$, we note that

$$(4-k)F^e_1 + F^e_3 + \cdots + F^e_{k-1} = Q.$$

Also from relations (2.6) and (2.7), we have

$$F^e_3 = \cdots = F^e_{k-1} = Q,$$

and so we get

$$F^e_1 = Q.$$

Conversely, if there exist a quadratic function $Q : X \to Y$, constants $c_i \in Y$, $i = 1, \ldots, 2k+1$, satisfying (2.2), and if there exist additive functions $a_i : X \to Y$, $i = 1, \ldots, k$, and $a'_i : X \to Y$, $i = 3, \ldots, k$, such that each of the equations in (2.1) holds true, then it is obvious that the f_i's satisfy the functional equation (1.8). □

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