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A NOTE ON NEIGHBOURHOODS FOR APPROACH SPACES

Gunther Jäger*

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Abstract

We characterize approach spaces by suitable systems of neighbourhoods. We further characterize the lower separation axioms T_0, T_1, T_2, and regularity and the measure of compactness using these neighbourhood systems. Also the approach space underlying an approach uniform space is described using the neighbourhood systems.

Keywords: Approach space, Neighbourhood, Approach limit, Separation axioms, Regularity, Measure of compactness, Approach uniform space.

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1. Introduction

Approach spaces were introduced by R. Lowen as a common framework for metric and topological spaces [4]. They can be defined in various ways, e.g. via an approach distance, a system of closure operators, approach systems, hull operators or limit operators. Having a system of closure operators at our disposal, Lowen [5] points out that the definition of neighbourhoods is natural. Yet this has not been published so far, although from another direction (limit towers as stacks of limit structures, see [1]), approach spaces have been characterized and also here the definition of neighbourhoods appears natural. In this note we try to close this gap by giving a set of "natural" axioms for neighbourhood systems for the points of an approach space. We characterize approach spaces by these neighbourhood systems.

In order to underline the appropriateness of this approach, we use the neighbourhood systems to characterize the lower separation axioms T_0, T_1 and T_2 [6] and regularity [2]. We further characterize the measure of compactness [5]. All these characterization have a nice similarity to characterizations of the corresponding properties for topological spaces. We further show that the approach space underlying an approach uniform space [7] can be easily described using the neighbourhoods. The latter yields a nice description of the limit operator for an approach uniform space.

*Department of Statistics, Rhodes University, 6140 Grahamstown, South Africa.
E-mail: g.jager@ru.ac.za

2. Preliminaries

For an ordered set $(L, \leq)$, we write, upon existence, $\alpha \wedge \beta$ for the infimum of $\alpha, \beta \in L$ and $\alpha \vee \beta$ for their supremum. Similarly we denote, upon existence, the infimum of a subset $A \subseteq L$ by $\bigwedge A$ and the supremum of a subset by $\bigvee A$.

Let X be a set and $\mathbf{F}(X)$ the set of all filters on X. The set $\mathbf{F}(X)$ is ordered by set inclusion, i.e. $\mathbf{F} \leq \mathbf{G} \iff \mathbf{F} \subseteq \mathbf{G}$. The fixed ultrafilter generated by $\{x\}$ is denoted by $\dot{x} = \{A \subseteq X : x \in A\}$. For a set J and a family of filters on X, $(\mathbf{F}_j)_{j\in J}$, indexed by J and a filter $\mathbf{G} \in \mathbf{F}(J)$ the *compression operator* $\kappa(\mathbf{G}, (\mathbf{F}_j)_{j\in J})$ is defined by ([3])

$$\kappa(\mathbf{G}, (\mathbf{F}_j)_{j\in J}) = \bigvee_{G\in\mathbf{G}} \bigwedge_{j\in G} \mathbf{F}_j.$$

Approach spaces can be defined in various ways [5]. We choose here the definition via the limit operator as this suits us best.

2.1. Definition. [5] A function $\lambda : \mathbf{F}(X) \longrightarrow [0,\infty]^X$ is called a *limit operator* iff

(L1) For all $x \in X$: $\lambda(\dot{x})(x) = 0$;

(L2) For all $\mathbf{F}, \mathbf{G} \in \mathbf{F}(X)$, $x \in X$: $\mathbf{F} \leq \mathbf{G}$ implies $\lambda(\mathbf{G})(x) \leq \lambda(\mathbf{F})(x)$;

(L3') For all $(\mathbf{F}_j)_{j\in J} \in \mathbf{F}(X)^J$, $x \in X$: $\lambda(\bigwedge_{j\in J} \mathbf{F}_j)(x) = \bigvee_{j\in J} \lambda(\mathbf{F}_j)(x)$;

(L4) For all $\mathbf{G} \in \mathbf{F}(X)$, $(\mathbf{F}_y)_{y\in X} \in \mathbf{F}(X)^X$, $x \in X$: $\lambda(\kappa(\mathbf{G}, (\mathbf{F}_y)_{y\in X}))(x) \leq \lambda(\mathbf{G})(x) + \bigvee_{y\in X} \lambda(\mathbf{F}_y)(y)$.

The pair (X, λ) is then called an *approach space.* A mapping $f : (X, \lambda) \longrightarrow (X', \lambda')$ between two approach spaces is called a *contraction* if $\lambda'(f(\mathbf{F}))(f(x)) \leq \lambda(\mathbf{F})(x)$ for all $\mathbf{F} \in \mathbf{F}(X), x \in X$.

The category AP with approach spaces as objects and contractions as morphisms is a topological construct that contains both TOP, the category of topological spaces (as a bireflective and bicoreflective subcategory) and $\infty pqMET$ of extended pseudo-quasimetric spaces (as a bicoreflective subcategory), see Lowen [5].

3. Characterization of approach spaces by neighbourhood systems

Let $(X, \lambda) \in |AP|$. We define for $0 \leq \alpha \leq \infty$ and $x \in X$ the *α-neighbourhood filter at x* by

$$\mathbf{U}^x_\alpha = \bigwedge_{\lambda(\mathbf{F})(x)\leq\alpha} \mathbf{F}.$$

We note that $\mathbf{U}^x_\alpha \in \mathbf{F}(X)$ by (L1).

3.1. Lemma. *Let $\lambda : \mathbf{F}(X) \longrightarrow [0,\infty]^X$ be a mapping that satisfies the axioms* (L1) *and* (L2) *of Definition 2.1. The following are equivalent.*

(1) (L3')*;*

(2) *For all* $\mathbf{F} \in \mathbf{F}(X)$, $x \in X, 0 \leq \alpha \leq \infty$*:* $\lambda(\mathbf{F})(x) \leq \alpha \iff \mathbf{F} \geq \mathbf{U}^x_\alpha$.

Proof. We assume first that (L3') is true. If $\lambda(\mathbf{F})(x) \leq \alpha$, then $\mathbf{F} \geq \mathbf{U}^x_\alpha$ by the definition of $\mathbf{U}^x_\alpha$. Conversely let $\mathbf{F} \geq \mathbf{U}^x_\alpha$. Then by (L2) and (L3')

$$\lambda(\mathbf{F})(x) \leq \lambda(\mathbf{U}^x_\alpha)(x) = \lambda\Big(\bigwedge_{\lambda(\mathbf{F})(x)\leq\alpha} \mathbf{F}\Big)(x) = \bigvee_{\lambda(\mathbf{F})(x)\leq\alpha} \lambda(\mathbf{F})(x) \leq \alpha.$$

Let us now assume that condition (2) is true. By (L2) it follows that always $\bigvee_{j\in J} \lambda(\mathbf{F}_j)(x) \leq \lambda(\bigwedge_{j\in J} \mathbf{F}_j)(x)$. For the other inequality we let $\alpha = \bigvee_{j\in J} \lambda(\mathbf{F}_j)(x)$. Then $\lambda(\mathbf{F}_j)(x) \leq \alpha$

for all $j \in J$ and hence, by (2), $\mathbf{F}_j \geq \mathbf{U}^x_\alpha$ for all $j \in J$. Therefore, $\bigwedge_{j\in J} \mathbf{F}_j \geq \mathbf{U}^x_\alpha$ and, again by (2), finally $\lambda(\bigwedge_{j\in J} \mathbf{F}_j)(x) \leq \alpha$. $\square$

3.2. Lemma. *Let $(X,\lambda) \in |AP|$. The system $\mathcal{U} = (\mathbf{U}^x_\alpha)_{x\in X,\alpha\in[0,\infty]}$ has the following properties:*

(U0) $\mathbf{U}^x_\alpha \in \mathbf{F}(X)$ *for all* $x \in X,\ \alpha \in [0,\infty]$*;*
(U1) $\mathbf{U}^x_\alpha \leq \dot{x}$ *for all* $x \in X,\ \alpha \in [0,\infty]$*;*
(U2) $\mathbf{U}^x_{\alpha+\beta} \leq \kappa(\mathbf{U}^x_\beta, (\mathbf{U}^y_\alpha)_{y\in X})$ *for all* $\alpha,\ \beta \in [0,\infty],\ x \in X$*;*
(U3) $0 \leq \alpha \leq \beta$ *implies* $\mathbf{U}^x_\beta \leq \mathbf{U}^x_\alpha$*;*
(U4) *For all* $\emptyset \neq A \subset [0,\infty]$*:* $\bigvee_{\alpha\in A} \mathbf{U}^x_\alpha = \mathbf{U}^x_{\bigwedge A}$.

Proof. (U0), (U1) and (U3) are easy and are left for the reader. For (U2) we use Lemma 3.1 and show that $\lambda(\kappa(\mathbf{U}^x_\beta, (\mathbf{U}^y_\alpha)_{y\in X}))(x) \leq \alpha + \beta$. By Lemma 3.1 we have $\lambda(\mathbf{U}^x_\beta)(x) \leq \beta$ and $\bigvee_{y\in X} \lambda(\mathbf{U}^y_\alpha)(y) \leq \alpha$. Hence, by (L4)

$$\lambda(\kappa(\mathbf{U}^x_\beta, (\mathbf{U}^y_\alpha)_{y\in X}))(x) \leq \lambda(\mathbf{U}^x_\beta)(x) + \bigvee_{y\in X} \lambda(\mathbf{U}^y_\beta)(y) \leq \alpha + \beta.$$

For (U4) we first note that by (U3) for any $\emptyset \neq A \subseteq [0,\infty]$, $\mathbf{U}^x_\alpha \leq \mathbf{U}^x_{\bigwedge A}$ for all $\alpha \in A$. This implies that $\bigvee_{\alpha\in A} \mathbf{U}^x_\alpha \in \mathbf{F}(X)$ and that $\bigvee_{\alpha\in A} \mathbf{U}^x_\alpha \leq \mathbf{U}^x_{\bigwedge A}$. Furthermore, we know that $\bigvee_{\alpha\in A} \mathbf{U}^x_\alpha \geq \mathbf{U}^x_\beta$ for all $\beta \in A$. Hence, by (L2), for all $\beta \in A$ we have that

$$\lambda(\bigvee_{\alpha\in A} \mathbf{U}^x_\alpha)(x) \leq \lambda(\mathbf{U}^x_\beta)(x) \leq \beta$$

(again by Lemma 3.1). Therefore also $\lambda(\bigvee_{\alpha\in A} \mathbf{U}^x_\alpha)(x) \leq \bigwedge A$ and, again invoking Lemma 3.1, we obtain $\bigvee_{\alpha\in A} \mathbf{U}^x_\alpha \geq \mathbf{U}^x_{\bigwedge A}$. $\square$

We remark that (U3) is a consequence of (U4). Also it can easily be shown that (U4) is equivalent to $\mathbf{U}^x_\alpha = \bigvee_{\beta>\alpha} \mathbf{U}^x_\beta$.

Given now a system of filters on X, $\mathcal{U} = (\mathbf{U}^x_\alpha)_{x\in X,\alpha\in[0,\infty]}$, we define $\lambda_{\mathcal{U}} : \mathbf{F}(X) \longrightarrow [0,\infty]^X$ by

$$\lambda_{\mathcal{U}}(\mathbf{F})(x) = \bigwedge\{\alpha \in [0,\infty]\ :\ \mathbf{U}^x_\alpha \leq \mathbf{F}\}.$$

3.3. Lemma. *Let $\mathcal{U} = (\mathbf{U}^x_\alpha)_{x\in X,\alpha\in[0,\infty]}$ satisfy* (U0) *and* (U4) *of Lemma 3.2. Let $\mathbf{F} \in \mathbf{F}(X)$, $x \in X$ and $\alpha \in [0,\infty]$. The following are equivalent.*

(1) $\lambda_{\mathcal{U}}(\mathbf{F})(x) \leq \alpha$*;*
(2) $\mathbf{F} \geq \mathbf{U}^x_\alpha$.

Proof. If $\mathbf{F} \geq \mathbf{U}^x_\alpha$, then by definition $\lambda_{\mathcal{U}}(\mathbf{F})(x) \leq \alpha$. Conversely, let $\lambda_{\mathcal{U}}(\mathbf{F})(x) = \bigwedge\{\beta \in [0,\infty] : \mathbf{U}^x_\beta \leq \mathbf{F}\} \leq \alpha$. Then

$$\mathbf{U}^x_\alpha \leq \mathbf{U}^x_{\bigwedge\{\beta:\mathbf{U}^x_\beta\leq\mathbf{F}\}} = \bigvee_{\beta:\mathbf{U}^x_\beta\leq\mathbf{F}} \mathbf{U}^x_\beta \leq \mathbf{F}. \qquad \square$$

3.4. Lemma. *Let $\mathcal{U} = (\mathbf{U}^x_\alpha)_{x\in X,\ \alpha\in[0,\infty]}$ satisfy* (U0)–(U4) *of Lemma 3.2. Then $(X,\lambda_{\mathcal{U}}) \in |AP|$.*

Proof. (L1) and (L2) are easy and are left for the reader. For (L3') we use the complete distributivity of $[0,\infty]$. We have $\bigvee_{j\in J} \lambda_{\mathcal{U}}(\mathbf{F}_j)(x) = \bigvee_{j\in J} \bigwedge A_j$ with $A_j = \{\alpha \in [0,\infty] : \mathbf{U}^x_\alpha \leq \mathbf{F}\}$. Then

$$\bigvee_{j\in J} \bigwedge A_j = \bigwedge_{(\alpha_j)\in\prod_{j\in J} A_j} \bigvee_{j\in J} \alpha_j = \bigwedge_{\alpha_j:\mathbf{U}^x_{\alpha_j}\leq\mathbf{F}_j\forall j\in J} \bigvee_{j\in J} \alpha_j.$$

If we fix for each $j \in J$, $\alpha_j \in A_j$, then with $\alpha = \bigvee_{j\in J} \alpha_j$ we obtain from (U3) for every $j \in J$, $\mathbf{U}^x_\alpha \le \mathbf{U}^x_{\alpha_j} \le \mathbf{F}_j$. Hence

$$\bigvee_{j\in J} \lambda_{\mathcal{U}}(\mathbf{F}_j)(x) \ge \bigwedge_{\alpha:\mathbf{U}^x_\alpha \le \mathbf{F}_j \forall j\in J} \alpha = \bigwedge_{\alpha:\mathbf{U}^x_\alpha \le \bigwedge_{j\in J}\mathbf{F}_j} \alpha = \lambda_{\mathcal{U}}\Big(\bigwedge_{j\in J} \mathbf{F}_j\Big)(x).$$

The other inequality follows from (L2). In order to prove (L4), we set $\alpha = \lambda_{\mathcal{U}}(\mathbf{G})(x)$ and $\beta = \bigvee_{y\in X} \lambda_{\mathcal{U}}(\mathbf{F}_y)(y)$. By Lemma 3.3 then $\mathbf{G} \ge \mathbf{U}^x_\alpha$ and $\mathbf{F}_y \ge \mathbf{U}^y_\beta$ for all $y \in X$. Hence by (U2), $\mathbf{U}^x_{\alpha+\beta} \le \kappa\big(\mathbf{U}^x_\alpha, (\mathbf{U}^y_\beta)_{y\in X}\big) \le \kappa\big(\mathbf{G}, (\mathbf{F}_y)_{y\in X}\big)$. Again, by Lemma 3.3, we obtain finally $\lambda_{\mathcal{U}}\big(\kappa\big(\mathbf{G}, (\mathbf{F}_y)_{y\in X}\big)\big)(x) \le \alpha + \beta$. □

3.5. Lemma. *Let $(X, \lambda) \in |AP|$. If we denote the neighbourhood system by $\mathcal{U}^\lambda = \big(\mathbf{U}^x_\alpha\big)_{x\in X,\alpha\in[0,\infty]}$, then $\lambda_{(\mathcal{U}^\lambda)} = \lambda$.*

Proof. Let $\mathbf{F} \in \mathbf{F}(X)$ and $x \in X$. Then $\lambda_{(\mathcal{U}^\lambda)}(\mathbf{F})(x) = \bigwedge\{\alpha \in [0,\infty] : \mathbf{F} \ge \mathbf{U}^x_\alpha\}$. From Lemma 3.1 we know that $\mathbf{F} \ge \mathbf{U}^x_\alpha$ if and only if $\lambda(\mathbf{F})(x) \le \alpha$. Hence we obtain

$$\lambda_{(\mathcal{U}^\lambda)}(\mathbf{F})(x) = \bigwedge\{\alpha \in [0,\infty] : \lambda(\mathbf{F})(x) \le \alpha\} = \lambda(\mathbf{F})(x). \qquad \square$$

3.6. Lemma. *Let $\mathcal{U}$ satisfy* (U0)–(U4) *from Lemma 3.2. Then $\mathcal{U}^{(\lambda_{\mathcal{U}})} = \mathcal{U}$.*

Proof. Let $x \in X$ and $\alpha \in [0,\infty]$. We set

$$\mathcal{U}^{(\lambda_{\mathcal{U}})} = (\mathbf{U}^x_\alpha)_{x\in X,\alpha\in[0,\infty]} \text{ and } \mathcal{U} = (\mathbf{V}^x_\alpha)_{x\in X,\alpha\in[0,\infty]}.$$

By Lemma 3.3 we obtain that from $\lambda_{\mathcal{U}}(\mathbf{F})(x) \le \alpha$ it follows that $\mathbf{V}^x_\alpha \le \mathbf{F}$. Hence $\mathbf{U}^x_\alpha = \bigwedge_{\lambda_c u(\mathbf{F})(x)\le\alpha} \mathbf{F} \ge \mathbf{V}^x_\alpha$. In order to show $\mathbf{U}^x_\alpha \le \mathbf{V}^x_\alpha$ it is sufficient to show that $\lambda_{\mathcal{U}}(\mathbf{V}^x_\alpha)(x) \le \alpha$. But this follows straight from $\lambda_{\mathcal{U}}(\mathbf{V}^x_\alpha)(x) = \bigwedge\{\beta \in [0,\infty] : \mathbf{V}^x_\beta \le \mathbf{V}^x_\alpha\} \le \alpha$. □

We finally characterize contractions using neighbourhood systems.

3.7. Lemma. *Let $(X, \lambda), (X', \lambda') \in |AP|$ and let $f : X \longrightarrow X'$ be a mapping. The following are equivalent.*

(1) *f is a contraction;*

(2) *$\mathbf{V}^{f(x)}_\alpha \le f(\mathbf{U}^x_\alpha)$ for all $\alpha \in [0,\infty]$ and all $x \in X$, where $\mathcal{U}^\lambda = (\mathbf{U}^x_\alpha)_{x\in X,\alpha\in[0,\infty]}$ and $\mathcal{U}^{\lambda'} = (\mathbf{V}^y_\alpha)_{y\in X',\alpha\in[0,\infty]}$ are the neighbourhood systems for (X, λ), (X', λ'), respectively.*

Proof. If f is a contraction, then with $\mathbf{F} = \mathbf{U}^x_\alpha$ we obtain $\lambda'(f(\mathbf{U}^x_\alpha))(f(x)) \le \lambda(\mathbf{U}^x_\alpha)(x) \le \alpha$ by Lemma 3.1. If we use Lemma 3.1 again, we obtain from this $f(\mathbf{U}^x_\alpha) \ge \mathbf{V}^{f(x)}_\alpha$.

Let now the condition (2) be true and let $\mathbf{F} \in \mathbf{F}(X)$ and $x \in X$. With $\alpha = \lambda(\mathbf{F})(x)$ we obtain $\mathbf{F} \ge \mathbf{U}^x_\alpha$ and hence $f(\mathbf{F}) \ge f(\mathbf{U}^x_\alpha) \ge \mathbf{V}^{f(x)}_\alpha$. With (L2) and Lemma 3.1 we obtain from this

$$\lambda'(f(\mathbf{F}))(f(x)) \le \lambda'(\mathbf{V}^{f(x)}_\alpha)(f(x)) \le \alpha,$$

and f is a contraction. □

We could define the category of *neighbourhood approach spaces* with spaces with neighbourhood systems that satisfy (U0)–(U4) as objects and mappings which satisfy (2) of Lemma 3.7 as morphisms . The Lemmas 3.2, 3.4, 3.5, 3.6 and 3.7, however, show that the category AP of approach spaces and contractions is isomorphic to this new category. So it makes not much sense introducing a new notation. We rather consider our neighbourhood systems as an alternative way of describing approach spaces. In the following sections we will show that this description is very nice and leads to characterizations of certain properties of approach spaces which are similar to characterizations of corresponding properties in *TOP*.

4. Characterization of approach properties by neighbourhood systems

Separation axioms. In [6] Lowen and Sioen defined several separation axioms in AP. Given an approach space (X, λ) its topological coreflection is denoted by $(X, \mathcal{T}_\lambda) \in |TOP|$. It is shown in [5] that convergence in $(X, \mathcal{T}_\lambda)$ is characterized by $\mathbf{F} \to x \iff \lambda(\mathbf{F})(x) = 0$ for $x \in X$ and $\mathbf{F} \in \mathbf{F}(X)$. An approach space (X, λ) is called a T_0-*space* if $(X, \mathcal{T}_\lambda)$ is a topological T_0-space. The T_0-axiom in *TOP* can be characterized by convergence via the following axiom:

(T_0) $\quad x \neq y$ implies $\dot{y} \not\to x$ or $\dot{x} \not\to y$.

Translating this, we find that an approach space is a T_0-space if and only if $\lambda(\dot{x})(y) = 0 = \lambda(\dot{y})(x)$ implies $x = y$.

4.1. Lemma. *Let $(X, \lambda) \in |AP|$. Then (X, λ) is a T_0-space if and only if $\mathbf{U}_0^x \leq \dot{y}$ and $\mathbf{U}_0^y \leq \dot{x}$ implies $x = y$.*

Proof. We note that $\mathbf{U}_0^x \leq \dot{y}$ if and only if $\lambda(\dot{y})(x) = 0$ by Lemma 3.1. □

T_1-spaces can be characterized similarly. An approach space (X, λ) is a T_1-*space* [6] if and only if $(X, \mathcal{T}_\lambda)$ is a T_1-space. Again a characterization via convergence yields that (X, λ) is a T_1-space if and only if $\lambda(\dot{x})(y) = 0$ implies $x = y$.

4.2. Lemma. *Let $(X, \lambda) \in |AP|$. Then (X, λ) is a T_1-space if and only if $\mathbf{U}_0^y \leq \dot{x}$ implies $x = y$.* □

Also T_2-spaces in AP are defined via the topological bicoreflection: (X, λ) is a T_2-space if and only if $(X, \mathcal{T}_\lambda)$ is a T_2-space in *TOP*, see [6]. The T_2-axiom can again be characterized using the limit operator as: $\lambda(\mathbf{F})(x) = 0 = \lambda(\mathbf{F})(y)$ implies $x = y$.

4.3. Lemma. *Let $(X, \lambda) \in |AP|$. Then (X, λ) is a T_2-space if and only if $\mathbf{U}_0^x \vee \mathbf{U}_0^y \in \mathbf{F}(X)$ implies $x = y$.*

Proof. If (X, λ) is a T_2-space and $\mathbf{U}_0^x \vee \mathbf{U}_0^y$ exists, then by (L2) and Lemma 3.1 $\lambda(\mathbf{U}_0^x \vee \mathbf{U}_0^y) \leq \lambda(\mathbf{U}_0^x) = 0$ and $\lambda(\mathbf{U}_0^x \vee \mathbf{U}_0^y) \leq \lambda(\mathbf{U}_0^y) = 0$ and hence by (T2), $x = y$. Conversely, if $\lambda(\mathbf{F})(x) = 0 = \lambda(\mathbf{F})(y)$, then, again by Lemma 3.1, $\mathbf{F} \geq \mathbf{U}_0^x$ and $\mathbf{F} \geq \mathbf{U}_0^y$. Consequently, $\mathbf{U}_0^x \vee \mathbf{U}_0^y$ exists and hence $x = y$. □

Regularity in AP is defined differently. In [2] a diagonal condition is used. There, $(X, \lambda) \in |AP|$ is called *regular* if for all sets J, all mappings $\psi : J \longrightarrow X$, all selections of filters $(\mathbf{F}_j)_{j \in J} \in \mathbf{F}(X)^J$, all filters $\mathbf{G} \in \mathbf{F}(J)$ and all $x \in X$ we have

$$\lambda(\psi(\mathbf{G}))(x) \leq \lambda(\kappa(\mathbf{G}, (\mathbf{F}_j)_{j \in J}))(x) + \bigvee_{j \in J} \lambda(\mathbf{F}_j)(\psi(j)).$$

This condition can be characterized as follows [2]. For $F \subseteq X$ and $\epsilon \geq 0$ we define

$$F^{(\epsilon)} = \{y \in X : \bigwedge_{\mathbf{U} \in \mathbf{F}(X) \text{ ultra}: A \in \mathbf{U}} \lambda(\mathbf{U})(y) \leq \epsilon\}.$$

It was noted in [2] that $F^{(\epsilon)} = \{y \in X : \delta(y, F) \leq \epsilon\}$, where δ is the approach distance (see [5]). For $\mathbf{F} \in \mathbf{F}(X)$ we define $\mathbf{F}^{(\epsilon)}$ to be the filter generated by the filter base $\{F^{(\epsilon)} : F \in \mathbf{F}\}$. Then (X, λ) is regular if and only if for all $\mathbf{F} \in \mathbf{F}(X)$, for all $\epsilon \geq 0$ and for all $x \in X$ we have $\lambda(\mathbf{F}^{(\epsilon)})(x) \leq \lambda(\mathbf{F})(x) + \epsilon$.

4.4. Lemma. *Let $(X,\lambda) \in |AP|$. Then (X,λ) is regular if and only if $(\mathbf{U}_\alpha^x)^{(\epsilon)} \geq \mathbf{U}_{\alpha+\epsilon}^x$ for all $x \in X$ and all $\alpha, \epsilon \geq 0$.*

Proof. Let (X,λ) be regular and let $\alpha, \epsilon \geq 0$. Then we know that $\lambda((\mathbf{U}_\alpha^x)^{(\epsilon)})(x) \leq \lambda(\mathbf{U}_\alpha^x)(x) + \epsilon \leq \alpha + \epsilon$. By Lemma 3.1 this implies $(\mathbf{U}_\alpha^x)^{(\epsilon)} \geq \mathbf{U}_{\alpha+\epsilon}^x$.

Conversely, let $\mathbf{F} \in \mathbf{F}(X)$ and set $\alpha = \lambda(\mathbf{F})(x)$. Then $\mathbf{F} \geq \mathbf{U}_\alpha^x$ and hence $\mathbf{F}^{(\epsilon)} \geq (\mathbf{U}_\alpha^x)^{(\epsilon)} \geq \mathbf{U}_{\alpha+\epsilon}^x$. With (L2) and Lemma 3.1 this yields $\lambda(\mathbf{F}^{(\epsilon)})(x) \leq \lambda((\mathbf{U}_\alpha^x)^{(\epsilon)})(x) \leq \alpha + \epsilon$. □

Reading $F^{(\epsilon)}$ as the ϵ-closure of F, we can rephrase Lemma 4.4 in the following way: Each $(\alpha - \epsilon)$-neighbourhood of a point contains the ϵ-closure of an α-neighbourhood of the point. This is perfectly analogous to the definition of regularity in TOP.

Measure of compactness. In [5], for an approach space (X,λ), a measure of compactness is defined as

$$\mu_C(X) = \bigvee_{\mathbf{U} \in \mathbf{F}(X) \text{ ultra}} \bigwedge_{x \in X} \lambda(\mathbf{U})(x).$$

It is shown in [5] that $\mu_C(X)$ generalizes the well-known Hausdorff measure of non-compactness for pseudo-metric spaces. We have a nice characterization of $\mu_C(X)$ using neighbourhood systems.

4.5. Lemma. *Let $(X,\lambda) \in |AP|$. Then $\mu_C(X) \leq \alpha$ if and only if for each $\epsilon > 0$ and for each collection $(V_x)_{x \in X}$ with $V_x \in \mathbf{U}_{\alpha+\epsilon}^x$ for every $x \in X$ there is a finite subcollection $V_{x_1}, V_{x_2}, \ldots, V_{x_n}$ such that $X = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_n}$.*

Proof. Let $\mu_C(X) \leq \alpha$ and assume that for an $\epsilon > 0$ and for a collection $(V_x)_{x \in X}$ with $V_x \in \mathbf{U}_{\alpha+\epsilon}^x$ for all $x \in X$ we have that for all finite subcollections $V_{x_1} \cup \cdots \cup V_{x_n} \neq X$. Then

$$\mathbf{B} = \{X \setminus (V_{x_1} \cup \cdots \cup V_{x_n}) : \{V_{x_1}, \ldots, V_{x_n}\} \subseteq \{V_x : x \in X\} \text{ finite subcollection}\}$$

is the basis of a filter. Let $\mathbf{U} \in \mathbf{F}(X)$ be a finer ultrafilter. If $\lambda(\mathbf{U})(x) \leq \alpha + \epsilon$, then $\mathbf{U} \geq \mathbf{U}_{\alpha+\epsilon}^x$ and hence $V_x \in \mathbf{U}$, a contradiction to $X \setminus V_x \in \mathbf{U}$. Therefore $\lambda(\mathbf{U})(x) > \alpha + \epsilon$ for all $x \in X$ and hence $\bigwedge_{x \in X} \lambda(\mathbf{U})(x) \geq \alpha + \epsilon$. It follows $\mu_C(X) \geq \alpha + \epsilon$, a contradiction.

To show the converse, we assume that $\mu_C(X) > \alpha$. Then also $\mu_C(X) > \alpha + \epsilon$ for some $\epsilon > 0$. Hence there is an ultrafilter $\mathbf{U} \in \mathbf{F}(X)$ such that $\bigwedge_{x \in X} \lambda(\mathbf{U})(x) > \alpha + \epsilon$. Therefore $\lambda(\mathbf{U})(x) > \alpha + \epsilon$ for all $x \in X$ and hence $\mathbf{U} \not\geq \mathbf{U}_{\alpha+\epsilon}^x$ for all $x \in X$. Hence for every $x \in X$ there is $V_x \in \mathbf{U}_{\alpha+\epsilon}^x$ such that $V_x \notin \mathbf{U}$. We choose now $x_1, x_2, \ldots, x_n$ such that $V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_n} = X$. As $\mathbf{U}$ is an ultrafilter, it follows that $V_{x_k} \in \mathbf{U}$ for some k, a contradiction. Hence $\mu_C(X) \leq \alpha$ and the proof is complete. □

Lemma 4.5 generalizes the well-known characterization of compactness in TOP by means of neighbourhoods: A space X is compact if and only if for each selection of neighbourhoods V_x for every point $x \in X$, there exist finitely many $x_1, \ldots, x_n$ such that $X = V_{x_1} \cup \cdots \cup V_{x_n}$.

5. Characterization of the approach space underlying an approach uniform spaces using neighbourhood systems

There are various ways for defining an approach uniform space [7, 8]. We will need two of them. For the first way, we can consider an ideal $\Gamma \subseteq [0,\infty]^{X \times}$ (i.e. $\gamma, \nu \in \Gamma$ implies $\gamma \vee \nu \in \Gamma$ and $\gamma \in \Gamma$, and $\nu \leq \gamma$ implies $\nu \in \Gamma$, where the functions in Γ are ordered pointwise). If Γ satisfies the following axioms

(AU1) $\forall \gamma \in \Gamma,\ \forall x \in X$: $\gamma(x,x) = 0$;
(AU2) $\forall \xi \in [0,\infty]^{X\times X}$: $(\forall \epsilon > 0,\ \forall N > \infty : \exists \gamma_\epsilon^N \in \Gamma$ s.t. $\xi \wedge N \le \gamma_\epsilon^N + \epsilon) \implies \xi \in \Gamma$;
(AU3) $\forall \gamma \in \Gamma\ \forall N < \infty \exists \gamma^N \in \Gamma$ s.t. $\forall x,y,z \in X : \gamma(x,z) \wedge N \le \gamma^N(x,y) + \gamma^N(y,z)$;
(AU4) $\forall \gamma \in \Gamma$: $\gamma^s \in \Gamma$ (where $\gamma^s(x,y) = \gamma(y,x)$ for all $x,y \in X$).

then the pair (X,Γ) is called an *approach uniform space*.

For an approach uniform space (X,Γ) we have an underlying approach space (X,λ_Γ) with

$$\lambda_\Gamma(\mathbf{F})(x) = \bigvee_{\gamma\in\Gamma} \bigwedge_{F\in\mathbf{F}} \bigvee_{y\in F} \gamma(x,y),$$

for $\mathbf{F} \in \mathbf{F}(X)$ and $x \in X$ [7].

Another, equivalent definition of an approach uniform space uses the concept of a *uniform tower* on $X \times X$ ([7]). For $\epsilon > 0$ we define $\mathcal{U}_\epsilon^\Gamma$ as the filter generated by the filter base $\{[\gamma < \alpha] : \gamma \in \Gamma, \alpha > \epsilon\}$. Here we have $[\gamma < \alpha] = \{(x,y) \in X \times X : \gamma(x,y) < \alpha\}$. A uniform tower then satisfies the following "natural" axioms. For all $\epsilon, \epsilon' > 0$ we have

(UT0) $\mathcal{U}_\epsilon \in \mathbf{F}(X \times X)$;
(UT1) $\mathcal{U}_\epsilon \le \dot{\Delta}$ (with $\dot{\Delta} = \{A \subseteq X \times X : (x,x) \in A \forall x \in X\}$);
(UT2) $\mathcal{U}_\epsilon \le \mathcal{U}_\epsilon^{-1}$;
(UT3) $\mathcal{U}_{\epsilon+\epsilon'} \le \mathcal{U}_\epsilon \circ \mathcal{U}_{\epsilon'}$;
(UT4) $\mathcal{U}_\epsilon = \bigvee_{\alpha>\epsilon} \mathcal{U}_\alpha$.

We note that in [7] the filter $\mathcal{U}_0$ was not defined. We will define it via (UT4) as $\mathcal{U}_0 = \bigvee_{\alpha>0} \mathcal{U}_\alpha$. The reader can easily verify that this causes no contradictions.

We define now for $U \subseteq X \times X$ and $x \in X$ the set $U(x) = \{y \in X : (x,y) \in U\}$ and with this for $\mathcal{F} \in \mathbf{F}(X \times X)$ the filter $\mathcal{F}(x) \in \mathbf{F}(X)$ generated by the filter basis $\{U(x) : U \in \mathcal{F}\}$.

5.1. Lemma. *For a uniform approach space (X,Γ) with uniform tower $(\mathcal{U}_\epsilon^\Gamma)_{\epsilon>0}$ we have*

$$\mathcal{U}_\epsilon^\Gamma(x) = \bigwedge_{\lambda_\Gamma(\mathbf{F})(x)\le\epsilon} \mathbf{F}.$$

Proof. If $\lambda_\Gamma(\mathbf{F})(x) \le \epsilon$ then for all $\gamma \in \Gamma$ we have $\bigwedge_{F\in\mathbf{F}} \bigvee_{y\in F} \gamma(x,y) \le \epsilon$. Hence, if $\alpha > \epsilon$ there is $F \in \mathbf{F}$ such that $\gamma(x,y) < \alpha$ for all $y \in F$. Therefore $F \subseteq [\gamma < \alpha](x)$ and we obtain $[\gamma < \alpha](x) \in \mathbf{F}$. Therefore it follows that $\mathcal{U}_\epsilon^\Gamma(x) \le \mathbf{F}$. Therefore $\mathcal{U}_\epsilon^\Gamma(x) \le \bigwedge_{\lambda_\Gamma(\mathbf{F})(x)\le\epsilon} \mathbf{F}$.

For the converse inequality, we show that $\lambda_\Gamma(\mathcal{U}_\epsilon^\Gamma(x))(x) \le \epsilon$. We have

$$\begin{aligned}
\lambda_\Gamma(\mathcal{U}_\epsilon^\Gamma(x))(x) &= \bigvee_{\gamma\in\Gamma} \bigwedge_{V\in\mathcal{U}_\epsilon^\Gamma(x)} \bigvee_{y\in V} \gamma(x,y) \\
&= \bigvee_{\gamma\in\Gamma} \bigwedge_{\eta\in\Gamma,\alpha>\epsilon} \bigvee_{y\in[\eta<\alpha](x)} \gamma(x,y) \\
&\le \bigvee_{\gamma\in\Gamma} \bigwedge_{\alpha>\epsilon} \bigvee_{y\in[\gamma<\alpha](x)} \gamma(x,y) \\
&= \bigvee_{\gamma\in\Gamma} \bigwedge_{\alpha>\epsilon} \bigvee_{y:\gamma(x,y)<\alpha} \gamma(x,y) \\
&\le \bigvee_{\gamma\in\Gamma} \bigwedge_{\alpha>\epsilon} \alpha \;=\; \epsilon.
\end{aligned}$$

Hence $\bigwedge_{\lambda_\Gamma(\mathbf{F})(x)\le\epsilon} \mathbf{F} \le \mathcal{U}_\epsilon^\Gamma(x)$ and the proof is complete. $\square$

Note that this implies that the neighbourhood system at $x \in X$ for the approach space underlying an approach uniform space can be obtained from the uniform tower $(\mathcal{U}_\epsilon^\Gamma)_{\epsilon \geq 0}$ as $(\mathcal{U}_\epsilon^\Gamma(x))_{\epsilon \geq 0}$. This is again in perfect analogy to the situation in the category $UNIF$ of uniform spaces. We deduce a convenient description of the approach limit for an approach uniform tower.

5.2. Corollary. *For an approach uniform space* (X, Γ) *we have* $\lambda_\Gamma(\mathbf{F})(x) = \bigwedge\{\alpha \in [0,\infty] : \mathcal{U}_\alpha^\Gamma(x) \leq \mathbf{F}\}$.

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