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FIXED AND RELATED FIXED POINT THEOREMS FOR THREE MAPS IN G-METRIC SPACES

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Abstract

Using the setting of G-metric spaces, unique common fixed points of three maps that satisfy a generalized (φ, ψ)-weak contractive condition are obtained. It is noted that the existence of a fixed point of any one of the mappings implies that the three mappings have a common fixed point. These results extend and generalize various well known comparable results in the existing literature.

Keywords: Common fixed point, Generalized weak contractive condition, Lower semicontinous functions, G-metric space.

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1. Introduction and preliminaries

The study of fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity. Mustafa and Sims [7] generalized the concept of a metric space. Based on the notion of generalized metric spaces, Mustafa *et al.* [7, 8, 9, 10, 11] obtained some fixed point theorems for mappings satisfying different contractive conditions. Abbas and Rhoades [1] initiated the study of a common fixed point theory in generalized metric spaces. Saadati *et al.* [13] proved some fixed point results for contractive mappings in partially ordered G-metric spaces. Abbas *et al.* [3] studied some coupled common fixed point theorems in two generalized metric spaces. Meanwhile, Shatanawi [14] obtained a coupled fixed point theorem in G-metric space. Abbas *et al* [2] and Chugh *et al.* [5] obtained some fixed point results for maps satisfying property P in G-metric spaces. Recently, Shatanawi [15] proved some fixed point results for self mapping in a complete G-metric space under some contractive conditions related

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to a nondecreasing map $\phi : R^+ \to R^+$ with $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \geq 0$. For more works in G-metric spaces see [4, 16].

The aim of this paper is to initiate the study of a common fixed point for three mappings in complete G-metric space under the various generalized (φ, ψ)-weak contractive conditions. It is worth mentioning that our results do not rely on the notion of continuity and any type of commutativity of mappings involved therein. We generalize various results of Mustafa *et al.* [9, 10] and Shatanawi [15].

Consistent with Mustafa and Sims [8], the following definitions and results will be needed in the sequel.

1.1. Definition. Let X be a nonempty set. Suppose that the mapping $G : X \times X \times X \to R^+$ satisfies:

(a) $G(x, y, z) = 0$ if $x = y = z$;
(b) $0 < G(x, x, y)$ for all $x, y \in X$. with $x \neq y$;
(c) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
(d) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables); and
(e) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a G-metric on X and (X, G) is called a $G-$ metric space.

Mustafa and Sims [8, Proposition 1] have also shown that if $G(x, y, z) = 0$ then $x = y = z$. For more properties of a G-metric we refer the reader to [8].

1.2. Definition. A sequence $\{x_n\}$ in a G-metric space X is:

(i) a *G-Cauchy* sequence if, for every $\varepsilon > 0$, there is a natural number n_0 such that for all $n, m, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$.
(ii) a *G-Convergent* sequence if, for any $\varepsilon > 0$, there is an $x \in X$ and an $n_0 \in N$ such that for all $n, m \geq n_0$, $G(x_n, x_m, x) < \varepsilon$.

A G-metric space on X is said to be *G-complete* if every G-Cauchy sequence in X is G-convergent in X. It is known that $\{x_n\}$ G-converges to $x \in X$ if and only if $G(x_m, x_n, x) \to 0$ as $n, m \to \infty$.

1.3. Proposition. [8] *Let X be a G-metric space. Then the following are equivalent:*

(1) *The sequence $\{x_n\}$ is G-convergent to x.*
(2) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$.
(3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$.
(4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to \infty$. □

1.4. Proposition. [8] *Let X be a G-metric space. Then the following are equivalent:*

(1) *The sequence $\{x_n\}$ is G-Cauchy.*
(2) *For every for every $\varepsilon > 0$ there exists $n_0 \in N$ such that for all $n, m \geq n_0$, $G(x_n, x_m, x_m) < \varepsilon$; that is, if $G(x_n, x_m, x_m) \to 0$ as $n, m \to \infty$.*

1.5. Definition. A G-metric on X is said to be *symmetric* if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

1.6. Proposition. *Every G-metric on X will define a metric d_G on X by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \ \forall\, x, y \in X. \tag{1.1}$$

For a symmetric G-metric space, one obtains

$$d_G(x, y) = 2G(x, y, y), \ \forall\, x, y \in X. \tag{1.2}$$

However, if G is not symmetric, then the following inequality holds:

$$\tfrac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y), \ \forall\, x, y \in X. \tag{1.3}$$ □

1.7. Definition. The following two classes of mappings are defined as

$$\Phi = \{\varphi \mid \varphi : [0,\infty) \to [0,\infty) \text{ is lower semi continuous, } \varphi(t) > 0 \text{ for all } t > 0,\ \varphi(0) = 0\}.$$

$$\Psi = \{\psi \mid \psi : [0,\infty) \to [0,\infty) \text{ is continuous and nondecreasing with } \psi(t) = 0 \text{ if and only if } t = 0\}.$$

2. Common fixed point theorems

In this section, we obtain common fixed point theorems for three mappings defined on a generalized metric spaces.

2.1. Theorem. *Let f. g and h be self maps on a complete G-metric space X satisfying*

$$\psi(G(fx, gy, hz)) \le \psi(M(x, y, z)) - \varphi(M(x, y, z)), \tag{2.1}$$

where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$M(x,y,z) = \max\{G(x,y,z), G(x,x,fx), G(y,y,gy), G(z,z,hz), G(x,fx,gy), G(y,gy,hz), G(z,hz,fx)\}$$

for all $x, y, z \in X$. Then f, g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h and conversely.

Proof. Suppose that x_0 is an arbitrary point in X. Define a sequence $\{x_n\}$ by $x_{3n+1} = fx_{3n}$, $x_{3n+2} = gx_{3n+1}$, $x_{3n+3} = hx_{3n+2}$. We may assume that $G(x_{3n}, x_{3n+1}, x_{3n+2}) > 0$. for every n. If not, then $x_{3n} = x_{3n+1} = x_{3n+2}$ for some n. For all those n. using (2.1), we obtain

$$\begin{aligned} &\psi\left(G(x_{3n+1}, x_{3n+2}, x_{3n+3})\right) \\ &\qquad = \psi\left(G(fx_{3n}, gx_{3n+1}, hx_{3n+2})\right) \\ &\qquad \le \psi\left(M(x_{3n}, x_{3n+1}, x_{3n+2})\right) - \varphi(M(x_{3n}, x_{3n+1}, x_{3n+2})), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} &M(x_{3n}, x_{3n+1}, x_{3n+2}) \\ &\quad = \max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n}, fx_{3n}), \\ &\qquad G(x_{3n+1}, x_{3n+1}, gx_{3n+1}), G(x_{3n+2}, x_{3n+2}, hx_{3n+2}), \\ &\qquad G(x_{3n}, fx_{3n}, gx_{3n+1}), G(x_{3n+1}, gx_{3n+1}, hx_{3n+2}), \\ &\qquad G(x_{3n+2}, hx_{3n+2}, fx_{3n})\} \\ &\quad = \max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n}, x_{3n+1}), \\ &\qquad G(x_{3n+1}, x_{3n+1}, x_{3n+2}), G(x_{3n+2}, x_{3n+2}, x_{3n+3}), \\ &\qquad G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3}), \\ &\qquad G(x_{3n+2}, x_{3n+3}, x_{3n+1})\}. \end{aligned}$$

On using the fact $G(x,x,y) \le G(x,y,z)$ for all $x,y,z \in X$. with $y \ne z$. it follows that

$$\begin{aligned} M(x_{3n}, x_{3n+1}, x_{3n+2}) &= \max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3})\} \\ &= \max\{0, G(x_{3n+1}, x_{3n+2}, x_{3n+3})\} \\ &= G(x_{3n+1}, x_{3n+2}, x_{3n+3}). \end{aligned}$$

Hence

$$\psi(G(x_{3n+1}, x_{3n+2}, x_{3n+3})) \le \psi\left(G(x_{3n+1}, x_{3n+2}, x_{3n+3})\right) - \varphi(G(x_{3n+1}, x_{3n+2}, x_{3n+3})),$$

implies that $\varphi(G(x_{3n+1}, x_{3n+2}, x_{3n+3})) = 0$ and $x_{3n+1} = x_{3n+2} = x_{3n+3}$. Following similar arguments, we obtain $x_{3n+2} = x_{3n+3} = x_{3n+4}$ and hence x_{3n} becomes a common fixed point of f, g and h.

Now, by taking $G(x_{3n}, x_{3n+1}, x_{3n+2}) > 0$ for $n = 0, 1, 2, 3, \ldots$ consider

$$\begin{aligned} &\psi\left(G(x_{3n+1}, x_{3n+2}, x_{3n+3})\right) \\ &= \psi(G(fx_{3n}, gx_{3n+1}, hx_{3n+2})) \\ &\leq \psi(M(x_{3n}, x_{3n+1}, x_{3n+2})) - \varphi(M(x_{3n}, x_{3n+1}, x_{3n+2})), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} &M(x_{3n}, x_{3n+1}, x_{3n+2}) \\ &= \max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n}, fx_{3n}), \\ &\quad G(x_{3n+1}, x_{3n+1}, gx_{3n+1}), G(x_{3n+2}, x_{3n+2}, hx_{3n+2}), \\ &\quad G(x_{3n}, fx_{3n}, gx_{3n+1}), G(x_{3n+1}, gx_{3n+1}, hx_{3n+2}), \\ &\quad G(x_{3n+2}, hx_{3n+2}, fx_{3n})\} \\ &= \max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n}, x_{3n+1}), \\ &\quad G(x_{3n+1}, x_{3n+1}, x_{3n+2}), G(x_{3n+2}, x_{3n+2}, x_{3n+3}), \\ &\quad G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3}), \\ &\quad G(x_{3n+2}, x_{3n+3}, x_{3n+1})\} \\ &= \max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3})\}. \end{aligned}$$

Suppose that for infinitely many values of n

$$\max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3})\} = G(x_{3n+1}, x_{3n+2}, x_{3n+3}),$$

then we obtain that

$$\begin{aligned} \psi\left(G(x_{3n+1}, x_{3n+2}, x_{3n+3})\right) &\leq \psi\left(G(x_{3n+1}, x_{3n+2}, x_{3n+3})\right) \\ &\quad - \varphi(G(x_{3n+1}, x_{3n+2}, x_{3n+3})) \\ &< \psi(G(x_{3n+1}, x_{3n+2}, x_{3n+3})), \end{aligned}$$

a contradiction. Thus, for infinitely many values of n we have

$$\begin{aligned} \psi(G(x_{3n+1}, x_{3n+2}, x_{3n+3})) &\leq \psi(M(x_{3n}, x_{3n+1}, x_{3n+2})) \\ &\quad - \varphi(M(x_{3n}, x_{3n+1}, x_{3n+2})) \\ &< \psi(M(x_{3n}, x_{3n+1}, x_{3n+2})) \\ &= \psi(G(x_{3n}, x_{3n+1}, x_{3n+2})). \end{aligned}$$

Since the control function ψ is nondecreasing, it follows that

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq M(x_{3n}, x_{3n+1}, x_{3n+2}) = G(x_{3n}, x_{3n+1}, x_{3n+2}).$$

Similarly, it can be shown that

$$\begin{aligned} G(x_{3n+2}, x_{3n+3}, x_{3n+4}) &\leq M(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\ &= G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \end{aligned}$$

and

$$\begin{aligned} G(x_{3n+3}, x_{3n+4}, x_{3n+5}) &\leq M(x_{3n+2}, x_{3n+3}, x_{3n+4}) \\ &= G(x_{3n+2}, x_{3n+3}, x_{3n+4}). \end{aligned}$$

Therefore, for all n,

$$G(x_{n+1}, x_{n+2}, x_{n+3}) \leq G(x_n, x_{n+1}, x_{n+2}),$$

and $\{G(x_{3n+1}, x_{3n+2}, x_{3n+3})\}$ is a non increasing sequence and so there exists $L \geq 0$ such that $\lim\limits_{n\to\infty} G(x_{3n+1}, x_{3n+2}, x_{3n+3}) = \lim\limits_{n\to\infty} M(x_{3n}, x_{3n+1}, x_{3n+2}) = L$. Then, by the lower semi continuity of φ,

$$\varphi(L) \leq \liminf_{n\to\infty} \varphi(M(x_{3n}, x_{3n+1}, x_{3n+2})).$$

We claim that $L = 0$. By lower semicontinuity of φ, taking the upper limits as $n \to \infty$ on both sides of

$$\psi(G(x_{3n+1}, x_{3n+2}, x_{3n+3})) \leq \psi(M(x_{3n}, x_{3n+1}, x_{3n+2})) - \varphi(M(x_{3n}, x_{3n+1}, x_{3n+2})),$$

we have

$$\begin{aligned}\psi(L) &\leq \psi\,(L) - \liminf_{n\to\infty} \varphi(M(x_{3n}, x_{3n+1}, x_{3n+2})) \\ &\leq \psi\,(L) - \varphi(L),\end{aligned}$$

i.e. $\varphi(L) \leq 0$. Thus $\varphi(L) = 0$ and we conclude that

$$\lim_{n\to\infty} G(x_{3n+1}, x_{3n+2}, x_{3n+3}) = 0. \tag{2.4}$$

Now we shall show that $\{x_n\}$ is a G- Cauchy sequence. It is sufficient to show that $\{x_{3n}\}$ is G-Cauchy in X. If it is not, there is $\varepsilon > 0$ and integers $3n_k$, $3m_k$ with $3m_k > 3n_k > k$ such that

$$G(x_{3n_k}, x_{3m_k}, x_{3m_k}) \geq \varepsilon \text{ and } G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) < \varepsilon. \tag{2.5}$$

Now (2.4) and (2.5) give

$$\begin{aligned}\varepsilon &\leq G(x_{3n_k}, x_{3m_k}, x_{3m_k}) \\ &\leq G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) + G(x_{3m_k-3}, x_{3m_k}, x_{3m_k}) \\ &\leq G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) + G(x_{3m_k-3}, x_{3m_k-1}, x_{3m_k-1}) \\ &\qquad\qquad + G(x_{3m_k-1}, x_{3m_k}, x_{3m_k}) \\ &\leq G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) + G(x_{3m_k-1}, x_{3m_k-2}, x_{3m_k-3}) \\ &\qquad\qquad + G(x_{3m_k-1}, x_{3m_k}, x_{3m_k+1}),\end{aligned}$$

which further implies that

$$\lim_{k\to\infty} G(x_{3n_k}, x_{3m_k}, x_{3m_k}) = \varepsilon. \tag{2.6}$$

Also,

$$\begin{aligned}&G(x_{3n_k}, x_{3m_k}, x_{3m_k}) \\ &\quad\leq G(x_{3n_k}, x_{3n_k+1}, x_{3n_k+1}) + G(x_{3n_k+1}, x_{3m_k}, x_{3m_k}) \\ &\quad\leq G(x_{3n_k}, x_{3n_k+1}, x_{3n_k+1}) + G(x_{3n_k+1}, x_{3m_k+2}, x_{3m_k+2}) \\ &\qquad\quad + G(x_{3m_k+2}, x_{3m_k}, x_{3m_k}) \\ &\quad\leq G(x_{3n_k}, x_{3n_k+1}, x_{3n_k+1}) + G(x_{3n_k+1}, x_{3m_k+2}, x_{3m_k+3}) \\ &\qquad\quad + G(x_{3m_k+3}, x_{3m_k+3}, x_{3m_k+2}) + G(x_{3m_k+2}, x_{3m_k}, x_{3m_k}) \\ &\quad\leq G(x_{3n_k}, x_{3n_k+1}, x_{3n_k+2}) + G(x_{3n_k+1}, x_{3m_k+2}, x_{3m_k+3}) \\ &\qquad\quad + G(x_{3m_k+1}, x_{3m_k+2}, x_{3m_k+3}) + G(x_{3m_k}, x_{3m_k+1}, x_{3m_k+2}),\end{aligned}$$

implies that $\varepsilon \leq \lim_{k\to\infty} G(x_{3n_k+1}, x_{3m_k+2}, x_{3m_k+3})$. From (2.4) and (2.6), we have

$$\begin{aligned}
&G(x_{3n_k+1}, x_{3m_k+2}, x_{3m_k+3}) \\
&\qquad \leq G(x_{3n_k+1}, x_{3n_k}, x_{3n_k}) + G(x_{3n_k}, x_{3m_k+2}, x_{3m_k+3}) \\
&\qquad \leq G(x_{3n_k+1}, x_{3n_k}, x_{3n_k}) + G(x_{3n_k}, x_{3m_k}, x_{3m_k}) \\
&\qquad\qquad + G(x_{3m_k}, x_{3m_k+2}, x_{3m_k+3}) \\
&\qquad \leq G(x_{3n_k}, x_{3n_k+1}, x_{3n_k+2}) + G(x_{3n_k}, x_{3m_k}, x_{3m_k}) \\
&\qquad\qquad + G(x_{3m_k}, x_{3m_k+2}, x_{3m_k+2}) + G(x_{3m_k+2}, x_{3m_k+2}, x_{3m_k+3}) \\
&\qquad \leq G(x_{3n_k}, x_{3n_k+1}, x_{3n_k+2}) + G(x_{3n_k}, x_{3m_k}, x_{3m_k}) \\
&\qquad\qquad + G(x_{3m_k}, x_{3m_k+1}, x_{3m_k+2}) + G(x_{3m_k+1}, x_{3m_k+2}, x_{3m_k+3}),
\end{aligned}$$

which gives that $\lim_{k\to\infty} G(x_{3n_k+1}, x_{3m_k+2}, x_{3m_k+3}) \leq \varepsilon$, and hence

$$\lim_{k\to\infty} G(x_{3n_k+1}, x_{3m_k+2}, x_{3m_k+3}) = \varepsilon. \tag{2.7}$$

Now,

$$\begin{aligned}
G(x_{3n_k}, x_{3m_k}, x_{3m_k}) &\leq G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+1}) + G(x_{3m_k+1}, x_{3m_k}, x_{3m_k}) \\
&\leq G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}) + G(x_{3m_k}, x_{3m_k+1}, x_{3m_k+2}),
\end{aligned}$$

gives that $\varepsilon \leq \lim_{k\to\infty} G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2})$, and

$$\begin{aligned}
G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}) &\leq G(x_{3n_k}, x_{3m_k+1}, x_{3n_k}) + (x_{3n_k}, x_{3n_k}, x_{3m_k+2}) \\
&\leq G(x_{3n_k}, x_{3n_k+1}, x_{3n_k+2}) + G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}).
\end{aligned}$$

By using (2.4) and (2.7), we get

$$\lim_{k\to\infty} G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}) \leq \varepsilon,$$

and hence

$$\lim_{k\to\infty} G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}) = \varepsilon. \tag{2.8}$$

Also,

$$\begin{aligned}
G(x_{3n_k}, x_{3m_k}, x_{3m_k}) &\leq G(x_{3n_k}, x_{3m_k+2}, x_{3m_k+2}) + G(x_{3m_k+2}, x_{3m_k}, x_{3m_k}) \\
&\leq G(x_{3n_k}, x_{3n_k+1}, x_{3m_k+2}) + G(x_{3m_k}, x_{3m_k+1}, x_{3m_k+2})
\end{aligned}$$

yields $\varepsilon \leq \lim_{k\to\infty} G(x_{3n_k}, x_{3n_k+1}, x_{3m_k+2})$ and

$$\begin{aligned}
G(x_{3n_k}, x_{3n_k+1}, x_{3m_k+2}) &\leq G(x_{3n_k}, x_{3n_k+1}, x_{3n_k}) + G(x_{3n_k}, x_{3n_k}, x_{3m_k+2}) \\
&\leq G(x_{3n_k}, x_{3n_k+1}, x_{3n_k+2}) + G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}).
\end{aligned}$$

By using (2.4) and (2.8), we get

$$\lim_{k\to\infty} G(x_{3n_k}, x_{3n_k+1}, x_{3m_k+2}) \leq \varepsilon$$

and hence

$$\lim_{k\to\infty} G(x_{3n_k}, x_{3n_k+1}, x_{3m_k+2}) = \varepsilon. \tag{2.9}$$

Now from the definition of M and from (2.4), (2.7), (2.8), (2.9) we have

$$\begin{aligned}
&M(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2})\\
&\quad = \max\big\{G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}), G(x_{3n_k}, x_{3n_k}, fx_{3n_k}),\\
&\qquad G(x_{3m_k+1}, x_{3m_k+1}, gx_{3m_k+1}), G(x_{3m_k+2}, x_{3m_k+2}, hx_{3m_k+2}),\\
&\qquad G(x_{3n_k}, fx_{3n_k}, gx_{3m_k+1}), G(x_{3m_k+1}, gx_{3m_k+1}, hx_{3m_k+2}),\\
&\qquad G(x_{3m_k+2}, hx_{3m_k+2}, fx_{3n_k})\big\}\\
&\quad = \max\big\{G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}), G(x_{3n_k}, x_{3n_k}, x_{3n_k+1}),\\
&\qquad G(x_{3m_k+1}, x_{3m_k+1}, x_{3m_k+2}), G(x_{3m_k+2}, x_{3m_k+2}, x_{3m_k+3}),\\
&\qquad G(x_{3n_k}, x_{3n_k+1}, x_{3m_k+2}), G(x_{3m_k+1}, x_{3m_k+2}, x_{3m_k+3}),\\
&\qquad G(x_{3m_k+2}, x_{3m_k+3}, x_{3n_k+1})\big\}.
\end{aligned}$$

Thus

$$\lim_{k\to\infty} M(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}) = \max\{\varepsilon, 0, 0, 0, 0, \varepsilon, \varepsilon\} = \varepsilon.$$

From (2.1), we obtain

$$\begin{aligned}
\psi(G(x_{3n_k+1}, x_{3m_k+2}, x_{3m_k+3})) &= \psi(G(fx_{3n_k}, gx_{3m_k+1}, hx_{3m_k+2}))\\
&\leq \psi(M(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}))\\
&\qquad - \varphi(M(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2})),
\end{aligned}$$

which on taking the limit as $k \to \infty$ implies

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon),$$

a contradiction as $\varepsilon > 0$.

It follows that $\{x_{3n}\}$ is a G-Cauchy sequence and by the G-completeness of X, there exists $u \in X$ such that $\{x_n\}$ converges to u as $n \to \infty$. We claim that $fu = u$. For this, consider

$$\begin{aligned}
\psi(G(fu, x_{3n+2}, x_{3n+3})) &= \psi(G(fu, gx_{3n+1}, hx_{3n+2}))\\
&\leq \psi(M(u, x_{3n+1}, x_{3n+2})) - \varphi(M(u, x_{3n+1}, x_{3n+2})),
\end{aligned}$$

where

$$\begin{aligned}
&M(u, x_{3n+1}, x_{3n+2})\\
&\quad = \max\{G(u, x_{3n+1}, x_{3n+2}), G(u, u, fu), G(x_{3n+1}, x_{3n+1}, gx_{3n+1}),\\
&\qquad G(x_{3n+2}, x_{3n+2}, hx_{3n+2}), G(u, fu, gx_{3n+1}), G(x_{3n+1}, gx_{3n+1}, hx_{3n+2}),\\
&\qquad G(x_{3n+2}, hx_{3n+2}, fu)\}\\
&\quad = \max\{G(u, x_{3n+1}, x_{3n+2}), G(u, u, fu), G(x_{3n+1}, x_{3n+1}, x_{3n+2}),\\
&\qquad G(x_{3n+2}, x_{3n+2}, x_{3n+3}), G(u, fu, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3}),\\
&\qquad G(x_{3n+2}, x_{2n+3}, fu)\}.
\end{aligned}$$

On taking the limit as $n \to \infty$, we obtain that

$$\psi(G(fu, u, u)) \leq \psi(G(fu, u, u)) - \varphi(G(fu, u, u)),$$

a contradiction. Hence $fu = u$. Similarly it can be shown that $gu = u$ and $hu = u$.

Now we prove the uniqueness of the common fixed point. Suppose that v is another common fixed point of f, g and h. Then

$$\psi(G(u, v, v)) = \psi(G(fu, gv, hv)) \leq \psi(M(u, v, v)) - \varphi(M(u, v, v)),$$

where

$$\begin{aligned} M(u,v,v) &= \max\{G(u,v,v), G(u,u,fu), G(v,v,gv), G(v,v,hv), \\ &\qquad G(u,fu,v), G(v,gv,hv), G(v,hv,fu)\} \\ &= \max\{G(u,v,v), G(u,u,u), G(v,v,v), G(v,v,v) \\ &\qquad G(u,u,v), G(v,v,v), G(v,v,u)\} \\ &= \max\{G(u,v,v), G(u,u,v)\}. \end{aligned}$$

If $M(u,v,v) = G(u,v,v)$. then

$$\psi(G(u,v,v)) \leq \psi(G(u,v,v)) - \varphi(G(u,v,v)),$$

a contradiction.

If $M(u,v,v) = G(u,u,v)$, then

$$\begin{aligned} \psi\left(G(u,v,v)\right) &\leq \psi(G(u,u,v)) - \varphi(G(u,u,v)) \\ &\leq \psi(G(u,u,v)). \end{aligned}$$

Again applying (2.1), we have

$$\psi(G(u,u,v)) = \psi(G(fu,gu,hv)) \leq \psi(M(u,u,v)) - \varphi(M(u,u,v)),$$

where

$$M(u,u,v) = \max\{G(u,u,v), G(u,v,v)\}. \tag{2.10}$$

If $M(u,u,v) = G(u,u,v)$, then we obtain $u = v$. Otherwise, we have

$$\psi(G(u,u,v)) \leq \psi(G(u,v,v)) - \varphi(G(u,v,v)),$$

a contradiction. Hence u is a unique common fixed point of f, g and h.

Now suppose that for some p in X, we have $f(p) = p$. We claim that $p = g(p) = h(p)$. If not then in the case when $p \neq g(p)$ and $p \neq h(p)$ we obtain

$$\psi(G(p,gp,hp)) = \psi(G(fp,gp,hp)) \leq \psi(M(p,p,p)) - \varphi(M(p,p,p)),$$

where

$$\begin{aligned} M(p,p,p) &= \max\big\{G(p,p,p), G(p,p,fp), G(p,p,gp), G(p,p,hp), \\ &\qquad G(p,fp,gp), G(p,gp,hp), G(p,hp,fp)\big\} \\ &= \max\big\{0, G(p,p,gp), G(p,p,hp), G(p,gp,hp)\big\} \\ &= G(p,gp,hp). \end{aligned}$$

Thus

$$\psi(G(p,gp,hp)) \leq \psi(G(p,gp,hp)) - \varphi(G(p,gp,hp)),$$

a contradiction. Similarly, when $p \neq g(p)$ and $p = h(p)$, or $p \neq h(p)$ and $p = g(p)$, we arrive at a contradiction following a similar argument to that given above. Therefore in all cases, we conclude that, $f(p) = g(p) = h(p) = p$. Hence, every fixed point of f is a fixed point of g and h, and conversely. □

2.2. Corollary. *Let f, g and h be self maps on a complete G-metric space X satisfying*

$$\psi(G(f^m x, g^m y, h^m z)) \leq \psi(M(x,y,z)) - \varphi(M(x,y,z)), \tag{2.11}$$

where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$\begin{aligned} M(x,y,z) = \max\big\{&G(x,y,z), G(f^m x,x,x), G(y,g^m y,y), G(z,z,h^m z), \\ &G(x,f^m x,g^m y), G(y,g^m y,h^m z), G(z,h^m z,f^m x)\big\} \end{aligned}$$

for all $x, y, z \in X$ and $m \in N$. Then f. g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h, and conversely.

Proof. It follows from Theorem 2.1 that f^m, g^m and h^m have a unique common fixed point p. Now $f(p) = f(f^m(p)) = f^{m+1}(p) = f^m(f(p))$, $g(p) = g(g^m(p)) = g^{m+1}(p) = g^m(g(p))$ and $h(p) = h(h^m(p)) = h^{m+1}(p) = h^m(h(p))$ imply that $f(p)$, $g(p)$ and $h(p)$ are also fixed points for f^m, g^m and h^m. Hence f, g and h have a unique common fixed point.

Now suppose that for some p in X, we have $f(p) = p$. We claim that $p = g(p) = h(p)$. If not then for the case when $p \neq g(p)$ and $p \neq h(p)$ we obtain

$$\begin{aligned}\psi(G(p, gp, hp)) &= \psi(G(f^m p, g^m(gp), h^m(hp))) \\ &\leq \psi(M(p, gp, hp)) - \varphi(M(p, gp, hp)),\end{aligned}$$

where

$$\begin{aligned}M(p, gp, hp) = \max\big\{&G(p, gp, hp), G(f^m p, p, p), G(gp, g^m(gp), gp), \\ &G(hp, hp, h^m(hp)), G(p, f^m p, g^m(gp)), \\ &G(gp, g^m(gp), h^m(hp)), G(hp, h^m(hp), f^m p)\big\} \\ = \max\big\{&G(p, gp, hp), G(p, p, p), G(gp, gp, gp), G(hp, hp, hp), \\ &G(p, p, gp), G(gp, gp, hp), G(hp, hp, p)\big\} \\ = G(p&, gp, hp).\end{aligned}$$

Thus

$$\psi(G(p, gp, hp)) \leq \psi(G(p, gp, hp)) - \varphi(G(p, gp, hp)),$$

which is a contradiction as $\varphi(G(p, gp, hp)) > 0$. Similarly, when $p \neq g(p)$ and $p = h(p)$, or $p \neq h(p)$ and $p = g(p)$, we arrive at a contradiction by following a similar argument. Therefore in all cases, we conclude that $f(p) = g(p) = h(p) = p$. Hence, every fixed point of f is a fixed point of g and h, and conversely. □

2.3. Corollary. *Let f, g and h be self maps on a complete G-metric space X satisfying*

$$\begin{aligned}G(fx, gy, hz) \leq k \max\big\{&G(x, y, z), G(x, x, fx), G(y, y, gy), G(z, z, hz), \\ &G(x, fx, gy), G(y, gy, hz), G(z, hz, fx)\big\}\end{aligned} \tag{2.12}$$

for all $x, y, z \in X$, where $k \in [0, 1)$. Then f, g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h, and conversely.

Proof. Define $\varphi, \psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t$ and $\varphi(t) = (1 - k)t$ for all $t \in [0, \infty)$, where $k \in [0, 1)$. Then it is clear that $\psi \in \Psi$ and $\varphi \in \Phi$. The result now follows from Theorem 2.1. □

2.4. Corollary. *Let f, g and h be self maps on a complete G-metric space X satisfying*

$$\psi(G(fx, gy, hz)) \leq \psi(G(x, y, z)) - \varphi(G(x, y, z)) \tag{2.13}$$

for all $x, y, z \in X$, where $\psi \in \Psi$, $\varphi \in \Phi$. Then f, g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h, and conversely. □

2.5. Corollary. *Let f. g and h be self maps on a complete G-metric space X satisfying*

$$G(fx, gy, hz) \leq \frac{G(x, y, z)}{1 + G(x, y, z)} \tag{2.14}$$

for all $x, y, z \in X$. Then f, g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h, and conversely.

Proof. Define $\varphi, \psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t$ and $\varphi(t) = \dfrac{1}{1+t}$ for all $t \in [0,\infty)$. Then it is clear that $\psi \in \Psi$ and $\varphi \in \Phi$. The result now follows from Corollary 2.4. $\square$

2.6. Theorem. *Let f, g and h be self maps on a complete G-metric space X satisfying*

$$\psi(G(fx, gy, hz)) \le \psi(M(x, y, z)) - \varphi(M(x, y, z)), \tag{2.15}$$

where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$M(x, y, z) = \max\{G(x, y, z), G(x, fx, fx), G(y, gy, gy), G(z, hz, hz)\}$$

for all $x, y, z \in X$. Then f, g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h, and conversely.

Proof. Suppose that x_0 is an arbitrary point in X. Define $\{x_n\}$ by $x_{3n+1} = fx_{3n}$, $x_{3n+2} = gx_{3n+1}$, $x_{3n+3} = hx_{3n+2}$. We suppose that $G(x_{3n}, x_{3n+1}, x_{3n+2}) > 0$ for every n. If not, then by similar arguments to those given in Theorem 2.1, we obtain that x_{3n} is the common fixed point of f, g and h.

Now, by taking $G(x_{3n}, x_{3n+1}, x_{3n+2}) > 0$ for each n, and from (2.15), we have

$$\begin{aligned}\psi(G(x_{3n+1}, x_{3n+2}, x_{3n+3})) &= \psi(G(fx_{3n}, gx_{3n+1}, hx_{3n+2}))\\ &\le \psi(M(x_{3n}, x_{3n+1}, x_{3n+2})) - \varphi(M(x_{3n}, x_{3n+1}, x_{3n+2}))\end{aligned}$$

for $n = 0, 1, 2, \ldots$, where

$$\begin{aligned}&M(x_{3n}, x_{3n+1}, x_{3n+2})\\ &\qquad = \max\big\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, fx_{3n}, fx_{3n}),\\ &\qquad\qquad G(x_{3n+1}, gx_{3n+1}, gx_{3n+1}), G(x_{3n+2}, hx_{3n+2}, hx_{3n+2})\big\}\\ &\qquad = \max\big\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n+1}, x_{3n+1}),\\ &\qquad\qquad G(x_{3n+1}, x_{3n+2}, x_{3n+2}), G(x_{3n+2}, x_{3n+3}, x_{3n+3})\big\}\\ &\qquad \le G(x_{3n}, x_{3n+1}, x_{3n+2}) \le M(x_{3n}, x_{3n+1}, x_{3n+2}).\end{aligned}$$

Therefore, we must have

$$\begin{aligned}\psi(G(x_{3n+1}, x_{3n+2}, x_{3n+3})) &\le \psi(M(x_{3n}, x_{3n+1}, x_{3n+2})) - \varphi(M(x_{3n}, x_{3n+1}, x_{3n+2}))\\ &< \psi(M(x_{3n}, x_{3n+1}, x_{3n+2}))\\ &= \psi(G(x_{3n}, x_{3n+1}, x_{3n+2})),\end{aligned}$$

and since the control function ψ is nondecreasing, it follows that

$$\begin{aligned}G(x_{3n+1}, x_{3n+2}, x_{3n+3}) &\le M(x_{3n}, x_{3n+1}, x_{3n+2})\\ &= G(x_{3n}, x_{3n+1}, x_{3n+2}).\end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned}G(x_{3n+2}, x_{3n+3}, x_{3n+4}) &\le M(x_{3n+1}, x_{3n+2}, x_{3n+3})\\ &= G(x_{3n+1}, x_{3n+2}, x_{3n+3})\end{aligned}$$

and

$$\begin{aligned}G(x_{3n+3}, x_{3n+4}, x_{3n+5}) &\le M(x_{3n+2}, x_{3n+3}, x_{3n+4})\\ &= G(x_{3n+2}, x_{3n+3}, x_{3n+4}).\end{aligned}$$

Therefore, for all n, $G(x_{n+1}, x_{n+2}, x_{n+3}) \le G(x_n, x_{n+1}, x_{n+2})$ and $\{G(x_{3n+1}, x_{3n+2}, x_{3n+3})\}$ is a non increasing sequence and so there exists $L \ge 0$ such that

$$\lim_{n\to\infty} G(x_{3n+1}, x_{3n+2}, x_{3n+3}) = \lim_{n\to\infty} M(x_{3n}, x_{3n+1}, x_{3n+2}) = L.$$

Following similar arguments to those given in Theorem 2.1, we conclude that

(2.16) $\lim_{n\to\infty} G(x_{3n+1}, x_{3n+2}, x_{3n+3}) = 0.$

Now, we shall show that $\{x_n\}$ is a G-Cauchy sequence. It is sufficient to show that $\{x_{3n}\}$ is G-Cauchy in X. If not, there is $\varepsilon > 0$ and there exist integers $3n_k$ and $3m_k$ with $3m_k > 3n_k > k$ such that

(2.17) $G(x_{3n_k}, x_{3m_k}, x_{3m_k}) \geq \varepsilon$ and $G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) < \varepsilon.$

Now (2.16) and (2.17) imply that

$$\begin{aligned}\varepsilon &\leq G(x_{3n_k}, x_{3m_k}, x_{3m_k})\\ &\leq G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) + G(x_{3m_k-3}, x_{3m_k}, x_{3m_k})\\ &\leq G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) + G(x_{3m_k-3}, x_{3m_k-1}, x_{3m_k-1})\\ &\qquad + G(x_{3m_k-1}, x_{3m_k}, x_{3m_k})\\ &\leq G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) + G(x_{3m_k-1}, x_{3m_k-2}, x_{3m_k-3})\\ &\qquad + G(x_{3m_k-1}, x_{3m_k}, x_{3m_k+1}),\end{aligned}$$

and hence

(2.18) $\lim_{k\to\infty} G(x_{3n_k}, x_{3m_k}, x_{3m_k}) = \varepsilon.$

The inequalities

$$\begin{aligned}\varepsilon &\leq G(x_{3n_k}, x_{3m_k}, x_{3m_k})\\ &\leq G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+1}) + G(x_{3m_k+1}, x_{3m_k}, x_{3m_k})\\ &\leq G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}) + G(x_{3m_k}, x_{3m_k+1}, x_{3m_k+2}),\end{aligned}$$

give that $\varepsilon \leq \lim_{k\to\infty} G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2})$, while (2.16), (2.18) and the inequality

$$G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}) \leq G(x_{3n_k}, x_{3m_k}, x_{3m_k}) + G(x_{3m_k}, x_{3m_k+1}, x_{3m_k+2})$$

yields $\lim_{k\to\infty} G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}) \leq \varepsilon$ and hence

(2.19) $\lim_{k\to\infty} G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}) = \varepsilon.$

Now from the definition of M and from (2.16) and (2.18), we have

$$\begin{aligned}&M(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2})\\ &\qquad = \max\big\{G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}), G(x_{3n_k}, fx_{3n_k}, fx_{3n_k}),\\ &\qquad\qquad G(x_{3m_k+1}, gx_{3m_k+1}, gx_{3m_k+1}), G(x_{3m_k+2}, hx_{3m_k+2}, hx_{3m_k+2})\big\}\\ &\qquad = \max\big\{G(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}), G(x_{3n_k}, x_{3n_k+1}, x_{3n_k+1}),\\ &\qquad\qquad G(x_{3m_k+1}, x_{3m_k+2}, x_{3m_k+2}), G(x_{3m_k+2}, x_{3m_k+3}, x_{3m_k+3})\big\}.\end{aligned}$$

Thus

$$\lim_{k\to\infty} M(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2}) = \max\{\varepsilon, 0, 0, 0\} = \varepsilon.$$

From (2.15), we obtain

$$\begin{aligned}&\psi(G(x_{3n_k+1}, x_{3m_k+2}, x_{3m_k+3}))\\ &\qquad = \psi(G(fx_{3n_k}, gx_{3m_k+1}, hx_{3m_k+2}))\\ &\qquad \leq \psi(M(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2})) - \varphi(M(x_{3n_k}, x_{3m_k+1}, x_{3m_k+2})).\end{aligned}$$

Taking the limit as $k \to \infty$ implies that

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon),$$

which is a contradiction as $\varepsilon > 0$. Hence $\{x_{3n}\}$ is a G-Cauchy sequence. By the G-completeness of X, there exists some $u \in X$ such that $\{x_n\}$ converges to u as $n \to \infty$. We claim that $fu = u$. If not, then consider

$$\begin{aligned}\psi(G(fu, fu, x_{3n+3})) &\leq \psi(G(fu, x_{3n+2}, x_{3n+3})) \\ &= \psi(G(fu, gx_{3n+1}, hx_{3n+2})) \\ &\leq M(u, x_{3n+1}, x_{3n+2}) - \varphi(M(u, x_{3n+1}, x_{3n+2})),\end{aligned}$$

where

$$\begin{aligned}M(u, x_{3n+1}, x_{3n+2}) = \max\{&G(u, x_{3n+1}, x_{3n+2}), G(u, fu, fu), \\ &G(x_{3n+1}, gx_{3n+1}, gx_{3n+1}), G(x_{3n+2}, hx_{3n+2}, hx_{3n+2})\} \\ = \max\{&G(u, x_{3n+1}, x_{3n+2}), G(u, fu, fu), \\ &G(x_{3n+1}, x_{3n+2}, x_{3n+2}), G(x_{3n+2}, x_{3n+3}, x_{3n+3})\}.\end{aligned}$$

On taking the limit as $n \to \infty$ we obtain that

$$\begin{aligned}\psi(G(fu, fu, u)) &\leq \psi(G(u, fu, fu)) - \varphi(G(u, fu, fu)) \\ &< \psi(G(fu, fu, u)),\end{aligned}$$

a contradiction. Hence $fu = u$. Similarly it can be shown that $gu = u$ and $hu = u$.

Now we prove the uniqueness of the common fixed point. If not, suppose that if v is another common fixed point of f, g and h. Then

$$\psi(G(u, v, v)) = \psi(G(fu, gv, hv)) \leq \psi(M(u, v, v)) - \varphi(M(u, v, v)),$$

where

$$\begin{aligned}M(u, v, v) &= \max\{G(u, v, v), G(u, fu, fu), G(v, gv, gv), G(v, hv, hv)\} \\ &= G(u, v, v).\end{aligned}$$

Hence

$$\psi(G(u, v, v)) \leq \psi(G(u, v, v)) - \varphi(G(u, v, v)),$$

a contradiction. Hence u is a unique common fixed point of f, g and h.

Now suppose that for some p in X we have $f(p) = p$. We claim that $p = g(p) = h(p)$. If not, then in case when $p \neq g(p)$ and $p \neq h(p)$ we obtain

$$\psi(G(p, gp, hp)) = \psi(G(fp, gp, hp)) \leq \psi(M(p, p, p)) - \varphi(M(p, p, p)),$$

where

$$\begin{aligned}M(p, p, p) &= \max\{G(p, p, p), G(p, fp, fp), G(p, gp, gp), G(p, hp, hp)\} \\ &= \max\{0, G(p, gp, gp), G(p, hp, hp)\} \\ &\leq G(p, gp, hp).\end{aligned}$$

Thus

$$\psi(G(p, gp, hp)) \leq \psi(G(p, gp, hp)) - \varphi(G(p, gp, hp)),$$

a contradiction. Similarly, when $p \neq g(p)$ and $p = h(p)$, or $p \neq h(p)$ and $p = g(p)$, we arrive at a contradiction by using a similar argument. Therefore in all cases, we conclude that, $f(p) = g(p) = h(p) = p$. Hence, every fixed point of f is a fixed point of g and h, and conversely. $\square$

2.7. Corollary. *Let f. g and h be self maps on a complete G-metric space X satisfying*

$$\psi(G(f^m x, g^m y, h^m z)) \leq \psi(M(x,y,z)) - \varphi(M(x,y,z)), \tag{2.20}$$

where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$M(x,y,z) = \max\left\{G(x,y,z), G(x, f^m x, f^m x), G(y, g^m y, g^m y), G(z, h^m z, h^m z)\right\}$$

for all $x, y, z \in X$. Then f, g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h, and conversely.

Proof. It follows from Theorem 2.6 that f^m, g^m and h^m have a unique common fixed point p. Now $f(p) = f(f^m(p)) = f^{m+1}(p) = f^m(f(p))$, $g(p) = g(g^m(p)) = g^{m+1}(p) = g^m(g(p))$ and $h(p) = h(h^m(p)) = h^{m+1}(p) = h^m(h(p))$ imply that $f(p)$, $g(p)$ and $h(p)$ are also fixed points for f^m, g^m and h^m. Hence f, g and h have a unique common fixed point.

Now suppose that for some p in X, we have $f(p) = p$. We claim that $p = g(p) = h(p)$. If not, then in the case where $p \neq g(p)$ and $p \neq h(p)$ we obtain

$$\begin{aligned}\psi(G(p, gp, hp)) &= \psi(G(f^m p, g^m\,(gp)\,, h^m\,(hp)))\\ &\leq \psi(M(p, gp, hp)) - \varphi(M(p, gp, hp)),\end{aligned}$$

where

$$\begin{aligned}M(p, gp, hp) &= \max\left\{G(p, gp, hp), G(p, f^m p, f^m p), G(gp, g^m (gp), g^m\,(gp)),\right.\\ &\qquad\qquad\qquad\qquad\qquad\qquad \left. G(hp, h^m\,(hp)\,, h^m (hp))\right\}\\ &= \max\{G(p, gp, hp), G(p, p, p), G(gp, gp, gp), G(hp, hp, hp)\}\\ &= G(p, gp, hp),\end{aligned}$$

that is

$$\psi(G(p, gp, hp)) \leq \psi(G(p, gp, hp)) - \varphi(G(p, gp, hp)),$$

which is a contradiction. Similarly, when $p \neq g(p)$ and $p = h(p)$, or $p \neq h(p)$ and $p = g(p)$, we arrive at a contradiction following a similar argument to the above. Therefore, in all cases we conclude that $f(p) = g(p) = h(p) = p$. Hence, every fixed point of f is a fixed point of g and h, and conversely. □

2.8. Corollary. *Let f, g and h be self maps on a complete G-metric space X satisfying*

$$G(fx, gy, hz) \leq \lambda(\max\{G(x,y,z), G(x, fx, fx), G(y, gy, gy), G(z, hz, hz)), \tag{2.21}$$

for all $x, y, z \in X$, where $\lambda \in [0,1)$. Then f, g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h, and conversely.

Proof. Define $\varphi, \psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t$ and $\varphi(t) = (1-\lambda)t$ for all $t \in [0,\infty)$, where $\lambda \in [0,1)$. Then it is clear that $\psi \in \Psi$ and $\varphi \in \Phi$. The result now follows from Theorem 2.6. □

2.9. Example. Let $X = [0,1]$ and $G(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\}$ be a G-metric on X. Define $f, g, h : X \to X$ by

$$f(x) = \begin{cases} \dfrac{x}{12} & \text{for } x \in [0, \frac{1}{2}),\\ \dfrac{x}{9} & \text{for } x \in [\frac{1}{2}, 1],\end{cases}$$

$$g(x) = \begin{cases} \dfrac{x}{8} & \text{for } x \in [0, \frac{1}{2}),\\ \dfrac{x}{6} & \text{for } x \in [\frac{1}{2}, 1],\end{cases}$$

and

$$h(x) = \begin{cases} \dfrac{x}{6} & \text{for } x \in [0, \frac{1}{2}), \\ \dfrac{x}{3} & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

Note that f, g and h are discontinuous maps. And also, $gh\Big(\dfrac{1}{2}\Big) = \dfrac{1}{48} \neq \dfrac{1}{72} = hg\Big(\dfrac{1}{2}\Big)$ and $fh(\dfrac{1}{2}) = \dfrac{1}{72} \neq \dfrac{1}{108} = hf(\dfrac{1}{2})$, which shows that f, g and h do not commute.

We take $\psi(t) = t$ and $\varphi(t) = \frac{1}{10}t$ for $t \in [0, \infty)$, so that

$$\psi(M(x, y, z)) - \varphi(M(x, y, z)) = \tfrac{9}{10}M(x, y, z).$$

For $x, y, z \in [0, \frac{1}{2})$,

$$G(x, fx, fx) = \tfrac{11x}{12},\ G(y, gy, gy) = \tfrac{7y}{8} \text{ and } G(z, hz, hz) = \tfrac{5z}{6}.$$

Since,

$$\begin{aligned} M(x, y, z) &= \max\{G(x, y, z), G(x, fx, fx), G(y, gy, gy), G(z, hz, hz)\} \\ &= \max\big\{ \max\{|x-y|, |y-z|, |z-x|\}, \tfrac{11x}{12}, \tfrac{7y}{8}, \tfrac{5z}{6}\big\}, \end{aligned}$$

so that

$$\begin{aligned} G(fx, gy, hz) &= \max\big\{ \big|\tfrac{x}{12} - \tfrac{y}{8}\big|, \big|\tfrac{y}{8} - \tfrac{z}{6}\big|, \big|\tfrac{z}{6} - \tfrac{x}{12}\big| \big\} \\ &= \tfrac{1}{6}\max\big\{ \big|\tfrac{x}{2} - \tfrac{3y}{4}\big|, \big|\tfrac{3y}{4} - z\big|, \big|z - \tfrac{x}{2}\big| \big\} \\ &\leq \tfrac{9}{10}\max\big\{ \max\{|x-y|, |y-z|, |z-x|\}, \tfrac{11x}{12}, \tfrac{7y}{8}, \tfrac{5z}{6}\big\} \\ &= \tfrac{9}{10}M(x, y, z) \\ &= \psi\left(M(x, y, z)\right) - \varphi(M(x, y, z)). \end{aligned}$$

For $x, y, z \in [\frac{1}{2}, 1]$,

$$G(x, fx, fx) = \tfrac{8x}{9},\ G(y, gy, gy) = \tfrac{5y}{6} \text{ and } G(z, hz, hz) = \tfrac{2z}{3},$$

so that

$$\begin{aligned} M(x, y, z) &= \max\{G(x, y, z), G(x, fx, fx), G(y, gy, gy), G(z, hz, hz)\} \\ &= \max\big\{ \max\{|x-y|, |y-z|, |z-x|\}, \tfrac{8x}{9}, \tfrac{5y}{6}, \tfrac{2z}{3}\big\}. \end{aligned}$$

Now

$$\begin{aligned} G(fx, gy, hz) &= \max\{\big|\tfrac{x}{9} - \tfrac{y}{6}\big|, \big|\tfrac{y}{6} - \tfrac{z}{3}\big|, \big|\tfrac{z}{3} - \tfrac{x}{9}\big|\} \\ &= \tfrac{1}{3}\max\big\{ \big|\tfrac{x}{3} - \tfrac{y}{2}\big|, \big|\tfrac{y}{3} - z\big|, \big|z - \tfrac{x}{3}\big| \big\} \\ &\leq \tfrac{9}{10}\max\big\{ \max\{|x-y|, |y-z|, |z-x|\}, \tfrac{8x}{9}, \tfrac{5y}{6}, \tfrac{2z}{3}\big\} \\ &= \tfrac{9}{10}M(x, y, z) \\ &= \psi\left(M(x, y, z)\right) - \varphi(M(x, y, z)). \end{aligned}$$

Now for $x \in [0, \frac{1}{2})$. $y, z \in [\frac{1}{2}, 1]$.

$$G(x, fx, fx) = \tfrac{11x}{12},\ G(y, gy, gy) = \tfrac{5y}{6} \text{ and } G(z, hz, hz) = \tfrac{2z}{3},$$

so that

$$\begin{aligned} M(x, y, z) &= \max\{G(x, y, z), G(x, fx, fx), G(y, gy, gy), G(z, hz, hz)\} \\ &= \max\big\{ \max\{|x-y|, |y-z|, |z-x|\}, \tfrac{11x}{12}, \tfrac{5y}{6}, \tfrac{2z}{3}\big\}. \end{aligned}$$

Now

$$
\begin{aligned}
G(fx, gy, hz) &= \max\left\{ \left|\tfrac{x}{12} - \tfrac{y}{6}\right|, \left|\tfrac{y}{6} - \tfrac{z}{3}\right|, \left|\tfrac{z}{3} - \tfrac{x}{12}\right| \right\} \\
&= \tfrac{1}{3} \max\left\{ \left|\tfrac{x}{4} - \tfrac{y}{2}\right|, \left|\tfrac{y}{2} - z\right|, \left|z - \tfrac{x}{4}\right| \right\} \\
&\le \tfrac{9}{10} \max\left\{ \max\{|x-y|, |y-z|, |z-x|\}, \tfrac{11x}{12}, \tfrac{5y}{6}, \tfrac{2z}{3} \right\} \\
&= \tfrac{9}{10} M(x,y,z) \\
&= \psi\left(M(x,y,z)\right) - \varphi(M(x,y,z)).
\end{aligned}
$$

The remaining cases are follow similarly as above. So the axioms of Theorem 2.6 are satisfied, and 0 is the unique common fixed point of f, g and h. Moreover, each fixed point of f is a fixed point of g and h, and conversely.

2.10. Remark. The following results can be viewed as special cases of our results.

(a) Theorem 2.1 generalizes Theorem 2.1 and Theorem 2.3 of [9]; and Theorem 3.1, Corollary 3.3, Corollary 3.4 and Corollary 3.5 of [15] into three maps.

(b) Corollary 2.5 generalizes Corollary 3.5 of [15] into three maps.

(c) Theorem 2.6 generalizes Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.5, Corollary 2.6, Corollary 2.7 and Corollary2.8 of [11].

(d) Corollary 2.8 generalizes Corollary 3.4 of [15] into three maps.

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