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SEPARATE CONTRACTION AND EXISTENCE OF PERIODIC SOLUTIONS IN TOTALLY NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract

In this study, we employ the fixed point theorem of Krasnoselskii and the concepts of separate and large contractions to show the existence of a periodic solution of a highly nonlinear delay differential equation. Also, we give a classification theorem providing sufficient conditions for an operator to be a large contraction, and hence, a separate contraction. Finally, under slightly different conditions, we obtain the existence of a positive periodic solution.

Keywords: Krasnoselskii's fixed point theorem, Large contraction, Periodic solution, Positive periodic solution, Separate contraction, Totally nonlinear delay differential equations.

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1. Introduction

In this paper, we use Krasnoselskii's fixed point theorem and show the highly nonlinear delay differential equation

$$x'(t) = -a(t)h(x(t)) + G(t, x(t - r(t))) \tag{1.1}$$

has a non-zero periodic solution and a positive periodic solution.

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To show existence of the non-zero periodic solution for Eq. (1.1), throughout the paper we assume that

$$G(t,0) \neq a(t)h(0) \text{ for some } t \in \mathbb{R}.$$

On the other hand, since we are dealing with periodic solutions of Eq. (1.1), it is natural to ask that all functions be continuous on their respective domains and that for a least positive real number T, we have

$$a(t+T) = a(t),\ r(t+T) = r(t),\ G(t,\cdot) = G(t+T,\cdot). \tag{1.2}$$

Since the equation (1.1) is totally nonlinear, to convert it into an integral equation problem, we will have to add and subtract a linear term. This process destroys the traditional contraction property for one of the mappings that is necessary for the use of Krasnoselskii's theorem. But the process replaces it with what is called a "large contraction". For more on the existence of periodic solutions we refer the reader to [1, 2, 3, 6], and the references therein. Next we state Krasnoselskii's fixed point theorem. For its proof we refer the reader to [8].

1.1. Theorem. [Krasnoselskiĭ] *Let $\mathcal{M}$ be a bounded convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that A and B map $\mathcal{M}$ into $\mathbb{B}$ such that*

i. *$x, y \in \mathcal{M}$, implies $Ax + By \in \mathcal{M}$,*
ii. *A is compact and continuous,*
iii. *B is a contraction mapping.*

Then there exists $z \in \mathcal{M}$ with $z = Az + Bz$. □

Concerning the terminology of compact mappings used in this theorem, we mean the following. Let A be a mapping from a set $\mathcal{M}$ into a topological space X. If $A(\mathcal{M})$ is contained in a compact subset of X, we say that A is compact.

In [4], Liu and Li reformulated the theorem of Krasnoselskii by replacing the contraction with what they called a separate contraction, which we define below.

1.2. Definition. [4, Definition 1.2] Let $(\mathcal{M}, d)$ be a metric space and $B : \mathcal{M} \to \mathcal{M}$. B is said to be a *separate contraction* if there exists two functions $\varphi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

i. $\phi(0) = 0$, ϕ is strictly increasing,
ii. $d(B(x), B(y)) \leq \varphi(d(x,y))$,
iii. $\phi(r) \leq r - \varphi(r)$ for $r > 0$.

One may easily verify that every contraction mapping is a separate contraction mapping. Next, we define another kind of contraction mapping called a large contraction.

1.3. Definition. Let $(\mathcal{M}, d)$ be a metric space and $B : \mathcal{M} \to \mathcal{M}$. B is said to be a *large contraction* if for $\phi, \varphi \in \mathcal{M}$ with $\phi \neq \varphi$ we have $d(B\phi, B\varphi) \leq d(\phi, \varphi)$ and if for all $\varepsilon > 0$, there exists a $\delta < 1$ such that

$$[\phi, \varphi \in \mathcal{M},\ d(\phi,\varphi) \geq \varepsilon] \Rightarrow d(B\phi, B\varphi) \leq \delta d(\phi, \varphi).$$

The following lemma provides a relationship between the classes of separate and large contraction mappings.

1.4. Lemma. [4, Lemma 1.1] *If B is a large contraction mapping then B is a separate contraction mapping.* □

It should be noted that the converse of the statement in Lemma 1.4 is not true in general (see [5, Example 4.3]).

The next theorem requires a separate contraction instead of a contraction.

1.5. Theorem. [4, Theorem 2.3] *Let $\mathcal{M}$ be a bounded convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that A and B map $\mathcal{M}$ into $\mathbb{B}$ such that*

i. *$x, y \in \mathcal{M}$, implies $Ax + By \in \mathcal{M}$,*
ii. *A is continuous and compact,*
iii. *B is a separate contraction mapping.*

Then there exists $z \in \mathcal{M}$ with $z = Az + Bz$. □

Hence, the next result can be derived as an application of Lemma 1.4 and Theorem 1.5.

1.6. Theorem. *Let $\mathcal{M}$ be a bounded convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that A and B map $\mathcal{M}$ into $\mathbb{B}$ such that*

i. *$x, y \in \mathcal{M}$ implies $Ax + By \in \mathcal{M}$,*
ii. *A is continuous and compact,*
iii. *B is a large contraction mapping.*

Then there exists $z \in \mathcal{M}$ with $z = Az + Bz$. □

In the present paper we employ Theorem 1.6 which involves a large contraction to study the existence of periodic and positive periodic solutions of (1.1).

2. Existence of periodic solutions

Define $P_T = \{\varphi \in C(\mathbb{R}, \mathbb{R}) : \varphi(t + T) = \varphi(t)\}$, where $C(\mathbb{R}, \mathbb{R})$ is the space of all real valued continuous functions on $\mathbb{R}$. Then P_T is a Banach space when endowed with the supremum norm

$$\|x\| = \sup_{t\in[0,T]} |x(t)| = \sup_{t\in\mathbb{R}} |x(t)|.$$

The next Lemma is essential for the construction of our mapping that is required for the application of Theorem 1.6. To have a well behaved mapping we must assume that

$$\int_0^T a(s)ds \neq 0. \tag{2.1}$$

Let the mapping H be defined by

$$H(x(u)) = x(u) - h(x(u)), \tag{2.2}$$

where the function h is as in Eq. (1.1).

2.1. Lemma. *If $x \in P_T$, then $x(t)$ is a solution of equation* (1.1) *if and only if*

$$x(t) = \int_t^{t+T} \frac{e^{-\int_u^{t+T} a(s)\,ds}}{1 - e^{-\int_0^T a(s)\,ds}} \left(a(u)H(x(u)) + G(u, x(u - r(u)))\right) du. \tag{2.3}$$

Proof. Let $x \in P_T$ be a solution of (1.1). We rewrite (1.1) in the form

$$x'(t) + a(t)x(t) = a(t)H(x(u)) + G(t, x(t - r(t))).$$

Next we multiply both sides of the resulting equation with $e^{\int_0^t a(s)\,ds}$, and then integrate from t to $t + T$ to obtain

$$x(t + T)e^{\int_0^{t+T} a(s)ds} - x(t)e^{\int_0^t a(s)\,ds} = \int_t^{t+T} \left(a(u)H(x(u)) + G(u, x(u - r(u)))\right) e^{\int_0^u a(s)\,ds}\, du.$$

Using the fact that $x(t + T) = x(t)$ and $e^{\int_t^{t+T} a(s)\,ds} = e^{\int_0^T a(s)\,ds}$, the above expression can be put in the form of Eq. (2.3). One may verify that $x(t)$ given by (2.3) satisfies Eq. (1.1). This completes the proof. □

First we note that for $t \in [0, T]$ and $u \in [t, t+T]$ we have

$$(2.4)\qquad \frac{e^{-\int_u^{t+T} a(s)ds}}{1 - e^{-\int_0^T a(s)ds}} \le \frac{e^{\int_0^{2T} |a(s)|ds}}{|1 - e^{-\int_0^T a(s)ds}|} := M.$$

Let J be a positive constant. Define the set

$$(2.5)\qquad \mathbb{M}_J = \{\varphi \in P_T : ||\varphi|| \le J\}.$$

Obviously, the set $\mathbb{M}_J$ is a bounded and convex subset of the Banach space P_T. Let the mapping $A : \mathbb{M}_J \to P_T$ be defined by

$$(2.6)\qquad (A\varphi)(t) = \int_t^{t+T} \frac{e^{-\int_u^{t+T} a(s)\,ds}}{1 - e^{-\int_0^T a(s)ds}} G(u, \varphi(u - r(u)))\,du,\ t \in \mathbb{R}.$$

Similarly we define the mapping $B : \mathbb{M}_J \to P_T$ by

$$(2.7)\qquad (B\psi)(t) = \int_t^{t+T} \frac{e^{-\int_u^{t+T} a(s)\,ds}}{1 - e^{-\int_0^T a(s)ds}} a(u)H(\psi(u))\,du,\ t \in \mathbb{R}.$$

It is clear from (2.6) and (2.7) that $A\varphi$ and $B\psi$ are T-periodic in t.

2.2. Lemma. *Suppose there exists a function $\xi \in P_T$ such that*

$$(2.8)\qquad |G(t, \varphi(t - r(t)))| \le \xi(t) \text{ for all } t \in \mathbb{R} \text{ and } \varphi \in \mathbb{M}_J.$$

Then the mapping A, defined by (2.6), is continuous in $\varphi \in \mathbb{M}_J$.

Proof. To see that A is a continuous mapping, let $\{\varphi_i\}_{i\in\mathbb{N}}$ be a sequence of functions in $\mathbb{M}_J$ such that $\varphi_i \to \varphi$ as $i \to \infty$. By (2.4), (2.8), and the the continuity of G, the Dominated Convergence theorem yields

$$\begin{aligned}
\lim_{i\to\infty} \|A\varphi_i - A\varphi\| &\le M \lim_{i\to\infty} \int_0^T \big|G(u, \varphi_i(u - r(u))) - G(u, \varphi(u - r(u)))\big|\,du \\
&\le M \int_0^T \lim_{i\to\infty} \big|G(u, \varphi_j(u - r(u))) - G(u, \varphi(u - r(u)))\big|\,du \\
&= 0.
\end{aligned}$$

This shows the continuity of the mapping A. □

In the next example, we display such a function satisfying (2.8).

2.3. Example. If we assume that $G(t, x)$ satisfies the Lipschitz condition in x, i.e., there is a positive constant k such that

$$(2.9)\qquad |G(t, z) - G(t, w)| \le k\|z - w\|, \text{ for } z, w \in P_T,$$

then for $\varphi \in \mathbb{M}_J$ we obtain the following

$$\begin{aligned}
|G(t, \varphi(t - r(t)))| &= |G(t, \varphi(t - r(t))) - G(t, 0) + G(t, 0)| \\
&\le |G(t, \varphi(t - r(t))) - G(t, 0)| + |G(t, 0)| \\
&\le kJ + |G(t, 0)|.
\end{aligned}$$

In this case we may choose ξ as

$$(2.10)\qquad \xi(t) = kJ + |G(t, 0)|\,.$$

Another possible ξ satisfying (2.8) is the following

$$(2.11)\qquad \xi(t) = |g(t)| + |p(t)|\,|y(t)|^n\,,$$

where $n = 1, 2, \ldots$, g and p are continuous functions on $\mathbb{R}$, and $y \in \mathbb{M}_J$.

2.4. Remark. Condition (2.9) is strong since it requires the function G to be globally Lipschitz. A lesser condition is (2.8) in which ξ can be directly chosen as in (2.10) or (2.11).

In the next two results we assume that for all $t \in \mathbb{R}$ and $\psi \in \mathbb{M}_J$,

$$\eta \int_t^{t+T} [|a(u)||H(\psi(u))| + \xi(t)]\, e^{-\int_u^{t+T} a(s)\, ds}\, du \le J, \tag{2.12}$$

where J and ξ are defined by (2.5) and (2.8), respectively, and

$$\eta := \left|(1 - e^{-\int_0^T a(s) ds})^{-1}\right|. \tag{2.13}$$

2.5. Lemma. *Suppose* (2.8) *and* (2.12). *Then A is continuous in $\varphi \in \mathbb{M}_J$ and maps $\mathbb{M}_J$ into a compact subset of $\mathbb{M}_J$.*

Proof. Let $\varphi \in \mathbb{M}_J$. Continuity of A for $\varphi \in \mathbb{M}_J$ follows from Lemma 2.2. Now, by (2.8) and (2.12) we have $|(A\varphi)(t)| < J$. Thus, $A\varphi \in \mathbb{M}_J$. Let $\varphi_i \in \mathbb{M}_J,\ i = 1, 2, \ldots$. Then from the above discussion we conclude that

$$||A\varphi_i|| \le J,\ i = 1, 2, \ldots .$$

This shows $A(\mathbb{M}_J)$ is uniformly bounded. It remins to show that $A(\mathbb{M}_J)$ is equicontinuous. Since ξ is continuous and T-periodic, (2.8) and the differentiation of (2.6) with respect to t yield

$$\begin{aligned}\left|(A\varphi_j)'(t)\right| &= |G(t, \varphi_j(t - r(t))) - a(t)(A\varphi_i)(t)| \\ &\le \xi(t) + |a(t)||(A\varphi_i)(t)| \\ &\le \xi(t) + ||a||\, ||A\varphi_i|| \le L,\end{aligned}$$

for some positive constant L. Thus the estimation on $|(A\varphi_i)'(t)|$ implies that $A(\mathbb{M}_J)$ is equicontinuous. Then the Arzela–Ascoli theorem yields the compactness of the mapping A. The proof is complete. □

2.6. Theorem. *Suppose* (2.8) *and* (2.12). *If B is a large contraction on $\mathbb{M}_J$, then (1.1) has a periodic solution in $\mathbb{M}_J$.*

Proof. Let A and B be defined by (2.6) and (2.7), respectively. By Lemma 2.5, the mapping A is compact and continuous. Then using (2.12) and the periodicity of A and B, we have

$$A\varphi + B\psi : \mathbb{M}_J \to \mathbb{M}_J$$

for $\varphi, \psi \in \mathbb{M}_J$. Hence an application of Theorem 1.6 implies the existence of a periodic solution in $\mathbb{M}_J$. This completes the proof. □

The next result shows the relationship between the mappings H and B in the sense of large contractions.

2.7. Lemma. *Let a be a positive valued function. If H is a large contraction on $\mathbb{M}_J$, then so is the mapping B.*

Proof. If H is large contraction on $\mathbb{M}_J$, then for $x, y \in \mathbb{M}_J$, with $x \neq y$, we have $||Hx - Hy|| \leq ||x-y||$. Hence,

$$\begin{aligned}|Bx(t) - By(t)| &\leq \int_t^{t+T} \frac{e^{-\int_u^{t+T} a(s)\,ds}}{1 - e^{-\int_0^T a(s)\,ds}} a(u) |H(x)(u) - H(y)(u)|\,du \\ &\leq \frac{||x-y||}{1 - e^{-\int_0^T a(s)ds}} \int_t^{t+T} e^{-\int_u^{t+T} a(s)\,ds} a(u)\,du \\ &= ||x-y||.\end{aligned}$$

Taking the supremum over the set $[0,T]$, we get that $\|Bx - By\| \leq ||x-y||$. One may also show in a similar way that

$$\|Bx - By\| \leq \delta ||x-y||$$

holds if we know the existence of a $0 < \delta < 1$ such that for all $\varepsilon > 0$,

$$[x, y \in \mathbb{M}_J, ||x-y|| \geq \varepsilon] \implies ||Hx - Hy|| \leq \delta ||x-y||.$$

The proof is complete. □

From Theorem 2.6 and Lemma 2.7, we deduce the following result.

2.8. Corollary. *In addition to the assumptions of Theorem 2.6, suppose also that a is a positive valued function. If H is a large contraction on $\mathbb{M}_J$, then* (1.1) *has a periodic solution in $\mathbb{M}_J$.* □

3. Examples and a classification

It was shown in [1, Example 1.2.7] that if $h(u) = u^3$, then the mapping H in (2.2) defines a large contraction on the set

$$\mathbb{M}_{\sqrt{3}/3} = \{\phi : [0,\infty) \to \mathbb{R} \text{ continuous and } \|\phi\| \leq \sqrt{3}/3\}.$$

In this section, we give two additional examples to show that different types of functions can be chosen in order for the mapping H to be a large contraction, and hence, a separate contraction (see Lemma 1.4). Moreover, we state a theorem classifying the functions h so that the mapping H given by (2.2) is a large contraction on a given set.

First, we begin with the case $h(u) = u^5$.

3.1. Example. Let $\|\cdot\|$ denote the supremum norm. If

$$\mathbb{M}_{5^{-1/4}} = \{\phi : \phi \in C(\mathbb{R}, \mathbb{R}) \text{ and } \|\phi\| \leq 5^{-1/4}\},$$

and $h(u) = u^5$, then the mapping H defined by (2.2) is a large contraction on the set $\mathbb{M}_{5^{-1/4}}$.

Proof. For any reals a and b we have

$$0 \leq (a+b)^4 = a^4 + b^4 + ab(4a^2 + 6ab + 4b^2),$$

and therefore,

$$-ab(a^2 + ab + b^2) \leq \frac{a^4 + b^4}{4} + \frac{a^2 b^2}{2} \leq \frac{a^4 + b^4}{2}.$$

If $x, y \in \mathbb{M}_{5^{-1/4}}$ with $x \neq y$ then $[x(t)]^4 + [y(t)]^4 < 1$. Hence, we arrive at

$$\begin{aligned} |H(u) - H(v)| &\leq |u - v| \left| 1 - \left(\frac{u^5 - v^5}{u - v} \right) \right| \\ &= |u - v| \left[1 - u^4 - v^4 - uv(u^2 + uv + v^2) \right] \\ &\leq |u - v| \left[1 - \frac{(u^4 + v^4)}{2} \right] \leq |u - v|, \end{aligned} \tag{3.1}$$

where we used the notations $u = x(t)$ and $v = y(t)$. Now, we are ready to show that H is a large contraction on $\mathbb{M}_{5^{-1/4}}$. For a given $\varepsilon \in (0, 1)$, suppose $x, y \in \mathbb{M}_{5^{-1/4}}$ with $\|x - y\| \geq \varepsilon$. There are two cases:

(a) $\frac{\varepsilon}{2} \leq |x(t) - y(t)|$ for some $t \in \mathbb{R}$, and **(b)** $|x(t) - y(t)| \leq \frac{\varepsilon}{2}$ for some $t \in \mathbb{R}$.

In case **(a)** we have

$$(\varepsilon/2)^4 \leq |x(t) - y(t)|^4 \leq 8(x(t)^4 + y(t)^4),$$

so

$$x(t)^4 + y(t)^4 \geq \frac{\varepsilon^4}{2^7}.$$

For all such t, we get by (3.1) that

$$|H(x(t)) - H(y(t))| \leq |x(t) - y(t)| \left(1 - \frac{\varepsilon^4}{2^8} \right).$$

On the other hand in case **(b)** we find by using (3.1) that

$$|H(x(t)) - H(y(t))| \leq |x(t) - y(t)| \leq \frac{1}{2} \|x - y\|.$$

Hence, in both cases we have

$$|H(x(t)) - H(y(t))| \leq \max \left\{ 1 - \frac{\varepsilon^4}{2^8}, \frac{1}{2} \right\} \|x - y\|.$$

Thus, H is a large contraction on the set $\mathbb{M}_{5^{-1/4}}$ with $\delta = \max \left\{ 1 - \frac{\varepsilon^4}{2^7}, \frac{1}{2} \right\}$. The proof is complete. □

Next, we make use of Example 3.1 and Corollary 2.8 to show that

$$4(5^{-5/4}) + \eta \int_t^{t+T} \left(5^{-5/4} |b(u)| + |c(u)| \right) e^{-\int_u^{t+T} a(s)\, ds}\, du \leq 5^{-1/4} \tag{3.2}$$

is the condition guaranteeing that the totally nonlinear delay differential equation

$$x'(t) = -a(t)x(t)^5 + b(t)x(t - r(t))^5 + c(t) \tag{3.3}$$

has a T-periodic solution, where η is as in (2.13) and $c\,(\neq 0) \in P_T$.

3.2. Example. Suppose that a, b, and $c\,(\neq 0)$ are in P_T and that $a(t) > 0$ for all t. If (3.2) holds for all $t \in \mathbb{R}$, then Eq. (3.3) has a T-periodic solution in $\mathbb{M}_{5^{-1/4}}$.

Proof. Let

$$G(u, \psi(u - r(u))) = b(u)\psi(u - r(u))^5 + c(u)$$

and

$$h(\psi(u)) = \psi(u)^5.$$

Define the mapping B as in (2.7). For $x \in \mathbb{M}_{5^{-1/4}}$, we have

$$|x(t)|^5 \leq 5^{-5/4},$$

and therefore,

$$\begin{aligned} G(u, x(u - r(u))) &= b(u)x(u - r(u))^5 + c(u) \\ &\leq 5^{-5/4}|b(u)| + |c(u)|. \end{aligned} \tag{3.4}$$

This is (2.8) with $\xi(u) = 5^{-5/4}|b(u)| + |c(u)|$. Since $a(t) > 0$ and

$$|H(x(t))| = |x(t) - x(t)^5| \leq 4(5^{-5/4}) \text{ for all } x \in \mathbb{M},$$

(3.2) implies

$$\begin{aligned} &\eta \int_t^{t+T} [|a(u)|\,|H(x(u))| + \xi(t)]\, e^{-\int_u^{t+T} a(s)\,ds}\, du \\ &\leq \frac{4(5^{-5/4})}{1 - e^{-\int_0^T a(s)ds}} \int_t^{t+T} a(u) e^{-\int_u^{t+T} a(s)\,ds}\, du + \eta \int_t^{t+T} \xi(t) e^{-\int_u^{t+T} a(s)\,ds}\, du \\ &= 4(5^{-5/4}) + \eta \int_t^{t+T} \xi(u) e^{-\int_u^{t+T} a(s)\,ds}\, du \\ &\leq 5^{-1/4}. \end{aligned}$$

Thus, (2.12) is satisfied. The proof is completed by making use of Corollary 2.8. □

3.3. Example. In addition to the assumptions of Example 3.2, suppose also that there is a positive constant L such that

$$L\left\{5^{-5/4}|b(u)| + |c(u)|\right\} \leq a(u), \text{ for all } u \in \mathbb{R} \tag{3.5}$$

and

$$4(5^{-5/4}) + \frac{1}{L} \leq 5^{-1/4}. \tag{3.6}$$

Then (3.3) has a periodic solution in $\mathbb{M}_{5^{-1/4}}$.

Proof. Let ξ be defined by $\xi(u) = 5^{-5/4}|b(u)| + |c(u)|$. Combining (3.4)-(3.6), we find

$$\begin{aligned} 4(5^{-5/4}) + \eta \int_t^{t+T} \xi(u) e^{-\int_u^{t+T} a(s)\,ds}\, du & \\ &\leq 4(5^{-5/4}) + \frac{\eta}{L} \int_t^{t+T} a(u) e^{-\int_u^{t+T} a(s)\,ds}\, du \\ &\leq 4(5^{-5/4}) + \frac{1}{L} \leq 5^{-1/4}. \end{aligned}$$

That is, (3.2) holds. The proof follows from the result of Example 3.2. □

By Example 3.1, we observe that the properties of the function h in (2.2) play a substantial role in obtaining a large contraction H on a convenient set.

Let $\alpha \in (0, 1]$ be a fixed real number. Define the set $\mathbb{M}_\alpha$ by

$$\mathbb{M}_\alpha = \{\phi : \phi \in C(\mathbb{R}, \mathbb{R}) \text{ and } \|\phi\| \leq \alpha\}. \tag{3.7}$$

We deduce by the next theorem that the following are sufficient conditions implying that the mapping H given by (2.2) is a large contraction on the set $\mathbb{M}_\alpha$.

H.1. $h : \mathbb{R} \to \mathbb{R}$ is continuous on $[-\alpha, \alpha]$ and differentiable on $(-\alpha, \alpha)$,

H.2. The function h is strictly increasing on $[-\alpha, \alpha]$,

H.3. $\sup_{t \in (-\alpha, \alpha)} h'(t) \leq 1$.

3.4. Theorem. *Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying (H.1–H.3). Then the mapping H in* (2.2) *is a large contraction on the set $\mathbb{M}_\alpha$.*

Proof. Let $\phi, \varphi \in \mathbb{M}_\alpha$ with $\phi \neq \varphi$. Then $\phi(t) \neq \varphi(t)$ for some $t \in \mathbb{R}$. Let us denote the set of all such t by $D(\phi, \varphi)$, i.e.,

$$D(\phi, \varphi) = \{t \in \mathbb{R} : \phi(t) \neq \varphi(t)\}.$$

For all $t \in D(\phi, \varphi)$, we have

$$\begin{aligned} |H\phi(t) - H\varphi(t)| &= |\phi(t) - h(\phi(t)) - \varphi(t) + h(\varphi(t))| \\ &= |\phi(t) - \varphi(t)| \left| 1 - \left(\frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} \right) \right|. \end{aligned} \tag{3.8}$$

Since h is a strictly increasing function we have

$$\frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} > 0 \text{ for all } t \in D(\phi, \varphi). \tag{3.9}$$

For each fixed $t \in D(\phi, \varphi)$ define the interval $U_t \subset [-\alpha, \alpha]$ by

$$U_t = \begin{cases} (\varphi(t), \phi(t)) & \text{if } \phi(t) > \varphi(t), \\ (\phi(t), \varphi(t)) & \text{if } \phi(t) < \varphi(t). \end{cases}$$

The Mean Value Theorem implies that for each fixed $t \in D(\phi, \varphi)$ there exists a real number $c_t \in U_t$ such that

$$\frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} = h'(c_t).$$

By (H.2–H.3) we have

$$0 \leq \inf_{u \in (-\alpha, \alpha)} h'(u) \leq \inf_{u \in U_t} h'(u) \leq h'(c_t) \leq \sup_{u \in U_t} h'(u) \leq \sup_{u \in (-\alpha, \alpha)} h'(u) \leq 1. \tag{3.10}$$

Hence, by (3.8–3.10) we obtain

$$|H\phi(t) - H\varphi(t)| \leq \left| 1 - \inf_{u \in (-\alpha, \alpha)} h'(u) \right| |\phi(t) - \varphi(t)|. \tag{3.11}$$

for all $t \in D(\phi, \varphi)$. This implies a large contraction in the supremum norm. To see this, choose a fixed $\varepsilon \in (0, 1)$ and assume that ϕ and φ are two functions in $\mathbb{M}_\alpha$ satisfying

$$\varepsilon \leq \sup_{t \in D(\phi, \varphi)} |\phi(t) - \varphi(t)| = \|\phi - \varphi\|.$$

If $|\phi(t) - \varphi(t)| \leq \frac{\varepsilon}{2}$ for some $t \in D(\phi, \varphi)$, then we get by (3.10) and (3.11) that

$$|H(\phi(t)) - H(\varphi(t))| \leq |\phi(t) - \varphi(t)| \leq \frac{1}{2} \|\phi - \varphi\|. \tag{3.12}$$

Since h is continuous and strictly increasing, the function $h\left(u + \frac{\varepsilon}{2}\right) - h(u)$ attains its minimum on the closed and bounded interval $[-\alpha, \alpha]$. Thus, if $\frac{\varepsilon}{2} < |\phi(t) - \varphi(t)|$ for some $t \in D(\phi, \varphi)$, then by (H.2) and (H.3) we conclude that

$$1 \geq \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} > \lambda,$$

where

$$\lambda := \frac{1}{2\alpha} \min\{h(u + \varepsilon/2) - h(u) : u \in [-\alpha, \alpha]\} > 0.$$

Hence, (3.8) implies

$$|H\phi(t) - H\varphi(t)| \leq (1 - \lambda) \|\phi(t) - \varphi(t)\|. \tag{3.13}$$

Consequently, combining (3.12) and (3.13) we obtain

$$|H\phi(t) - H\varphi(t)| \leq \delta \|\phi - \varphi\|,$$

where

$$\delta = \max\left\{\frac{1}{2}, 1-\lambda\right\} < 1.$$

The proof is complete. □

3.5. Example. Let $\alpha \in (0,1)$ and $k \in \{1,2,\ldots\}$ be two constants.

i. The function $h_1(u) = \frac{1}{2k}u^{2k}$, $u \in (-1,1)$, does not satisfy the condition (H.2).
ii. The function $h_2(u) = \frac{1}{2k+1}u^{2k+1}$, $u \in (-1,1)$, satisfies (H.1–H.3).

Proof. Since $h_1'(u) = u^{2k-1} < 0$ for $-1 < u < 0$, the condition (H.2) is not satisfied for h_1. Evidently, (H.1–H.2) hold for h_2. Finally, (H.3) follows from the fact that $h_2'(u) \leq \alpha^{2k}$ and $\alpha \in (0,1)$. □

3.6. Example. Let the function h be defined by $h(u) = 2^{u-1}$, $u \in [-1,1]$. One may easily verify the conditions (H.1–H.3). Define the set

$$\mathbb{M}_1 = \{\phi : \phi \in C(\mathbb{R},\mathbb{R}) \text{ and } \|\phi\| \leq 1\}.$$

By Theorem 3.4, the mapping $Hx = x - h \circ x$ defines a large contraction on the set $\mathbb{M}_1$. As in Example 3.2, one may show that

$$\eta \int_t^{t+T} \{|b(u)| + |c(u)|\} \exp\left(-\int_u^{t+T} a(s)\,ds\right) du \leq 1$$

is the condition guaranteeing the existence of a non-zero periodic solution of the nonlinear delay differential equation

$$x'(t) = -a(t)2^{x(t)-1} + b(t)2^{x(t-r(t))-1} + c(t), \ t \in \mathbb{R},$$

where a, b, and c are T-periodic continuous functions with $a \neq b$, $a(t) > 0$, and $c \neq 0$.

4. Positive periodic solutions

This section is concerned with the existence of a positive periodic solution of (1.1). Again, we arrive at our result by using Theorem 1.6. Since we are looking for the existence of positive periodic solutions, some of the conditions in previous sections will have to be modified accordingly. Theorem 1.6 was first used in [7] by Raffoul to show the existence of positive periodic solutions of neutral differential equations with functional delay, where the notion of contraction instead of large contraction is used. However, our problem which is given by (1.1), is totally nonlinear and the mapping B is a large contraction. We begin with the modification of (2.1) and ask that,

$$\int_0^T a(s)\,ds > 0. \tag{4.1}$$

For the sake of simplicity, we let

$$M = \frac{e^{\int_0^{2T} |a(s)| ds}}{1 - e^{-\int_0^T a(s) ds}}, \tag{4.2}$$

and

$$m = \frac{e^{-\int_0^{2T} |a(s)| ds}}{1 - e^{-\int_0^T a(s) ds}}. \tag{4.3}$$

It is easy to see that for all $(t,s) \in [0,2T] \times [0,2T]$,

$$m \leq \frac{e^{-\int_u^{t+T} a(s)ds}}{1 - e^{-\int_0^T a(s)ds}} \leq M.$$

For some non-negative constant L and a positive constant K we define the set

$$\mathbb{Q} = \{\phi \in P_T : L \leq \phi(t) \leq K \text{ for all } t \in [0,T]\},$$

which is a closed, convex, and bounded subset of the Banach space P_T. In addition we assume that for all $u \in \mathbb{R}$, $x \in \mathbb{Q}$,

$$\frac{L}{mT} \leq G(u,(u)) + a(u)(x(u) - h(x(u))) \leq \frac{K}{MT}, \tag{4.4}$$

where M and m are defined by (4.2) and (4.3), respectively. If we assume $G(t,x)$ satisfies (2.8), then the continuity and compactness of A can be easily shown, as before.

4.1. Theorem. *Suppose* (2.8), (2.12), (4.1), *and* (4.4) *hold. If B is a large contraction on $\mathbb{Q}$, then* (1.1) *has a positive periodic solution in the set $\mathbb{Q}$.*

Proof. We proceed by Theorem 1.6. Let A and B be defined by (2.6) and (2.7), respectively. The conditions (2.8) and (2.12) guarantee the continuity and compactness of A. Let the constants M and m be defined by (4.2) and (4.3), respectively. We get by (4.1) and (4.4) that

$$\begin{aligned}(B\psi)(t) + (A\varphi)(t) &= \int_t^{t+T} \gamma(t,u)\big[a(u)H(\psi(u)) + G(u,\varphi(u - r(u)))\big]\,du \\ &\leq TM\frac{K}{TM} = K,\end{aligned}$$

and

$$\begin{aligned}(B\psi)(t) + (A\varphi)(t) &= \int_t^{t+T} \gamma(t,u)\big[a(u)H(\psi(u)) + G(u,\varphi(u - r(u)))\big]\,du \\ &\geq Tm\frac{L}{Tm} = L,\end{aligned}$$

for $\psi, \varphi \in \mathbb{Q}$, where

$$\gamma(t,u) = e^{-\int_u^{t+T} a(s)\,ds}\left(1 - e^{-\int_0^T a(s)\,ds}\right)^{-1}.$$

This shows that $B\psi + A\varphi \in \mathbb{Q}$. All the hypothesis of Theorem 1.6 are satisfied and therefore equation (1.1) has a positive periodic solution, say z residing in $\mathbb{Q}$. The proof is complete. □

4.2. Example. In the following five steps we will construct an example so that the assumptions of Theorem 4.1 hold.

Step 1. Let the function a be defined by

$$a(t) = 11 - 10\left\{\sin\left(\pi\frac{12}{\ln(3/2)}t\right)\right\}^2. \tag{4.5}$$

It is obvious that the function a is T-periodic with

$$T = \frac{1}{6}\ln(3/2) \tag{4.6}$$

and satisfies the following inequality

$$1 \leq a(t) \leq 11. \tag{4.7}$$

Furthermore, we have

$$\int_0^T a(t)\,dt = \ln(3/2) > 0.$$

This shows that condition (4.1) holds.

Step 2. Let

$$C(u,x(u)) := G(u,x(u-r(u))) + a(u)H(x(u)),$$

where

$$G(u,x(u)) = -a(u)x(u) + \frac{3}{2}a(u)h(x(u)) + x(u-r(u)) + 3. \tag{4.8}$$

For all $x \in \mathbb{Q}$, we have

$$G(u+T,\cdot) = G(u,\cdot),$$

i.e., (1.2) holds.

Step 3. Let $K = 11$ and $L = 1$ to define the set

$$\mathbb{Q} := \{\phi \in P_T : 1 \le \phi(u) \le 11 \text{ for all } t \in [0,T]\},$$

where T is as in (4.6). Evidently, $a \in \mathbb{Q}$. To see that the condition (4.4) holds we need to calculate

$$M = \frac{\exp\left(\int_0^{2T} a(t)\,dt\right)}{1 - \exp\left(-\int_0^T a(t)\,dt\right)}$$

and

$$m = \frac{\exp\left(-\int_0^{2T} a(t)dt\right)}{1 - \exp\left(-\int_0^T a(t)\,dt\right)}.$$

Step 4. Using Maple 6.0 we approximate the numbers m, M, L/mT and K/MT as follows:

$$M = 6.75000, \quad m = 1.33333, \quad \frac{L}{mT} = 3.70674, \quad \frac{K}{MT} = 22.81481.$$

Setting

$$h(t) = \frac{1}{10}\frac{1}{12^4}t^5, \tag{4.9}$$

we conclude that

$$\frac{1}{103680} \le h(t) \le \frac{161051}{103680} \tag{4.10}$$

for all $t \in [1,11]$. Since $1 \le x(u) \le 11$ for all $x \in \mathbb{Q}$ we have

$$\frac{L}{mT} = 3.706674 \le C(u,x) = \frac{a(u)h(x(u))}{2} + x(u-r(u)) + 3 \le \frac{K}{MT} = 22.81481.$$

This means the condition (4.4) holds.

Step 5. As in the proof of Example 3.5, one may show that h satisfies (H.1–H.4). Theorem 3.4 implies that the operator $Hx = x - h \circ x$ is a large contraction on the set $\mathbb{Q}$. Since the function a is positive valued, from Lemma 2.7 the operator B given by (2.7) defines a large contraction on the set $\mathbb{Q}$. By (4.5)–(4.10), one may also verify that (2.8) and (2.12) hold. Therefore, Theorem 4.1 implies that the equation

$$x'(t) = -a(t)h(x(t)) + G(t,x(t-r(t))) = \frac{1}{2}a(t)h(x(t)) - a(t)x(t) + x(t-r(t)) + 3$$

has a periodic solution in the set $\mathbb{Q}$, where a and h are defined by (4.5) and (4.9), respectively.

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