

# GENERATING MATRIX FUNCTIONS FOR CHEBYSHEV MATRIX POLYNOMIALS OF THE SECOND KIND

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## Abstract

In this paper, the generating matrix function and recurrence relations for Chebyshev matrix polynomials of the second kind are obtained. Several families of bilinear and bilateral generating matrix functions for Chebyshev matrix polynomials of the second kind are derived.

**Keywords:** Chebyshev matrix polynomials of the second kind, Generating matrix function, Recurrence relation.

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## 1. Introduction

There has been significant development in the study of orthogonal matrix polynomials. For example, some results in the theory of classical orthogonal polynomials have been extended to orthogonal matrix polynomials, see [3, 4, 7, 8, 9, 10, 11, 13, 15]. Jacobi matrix polynomials, Chebyshev matrix polynomials of the first and second kind have been introduced and studied in [2, 5, 12] for matrices in  $\mathbb{C}^{N \times N}$ . Our main aim in this paper is to prove new properties for Chebyshev matrix polynomials of the second kind. The outline of this paper is as follows. In section 2, we demonstrate some properties of the Chebyshev matrix polynomials of the second kind. We derive bilinear and bilateral generating matrix functions for Chebyshev matrix polynomials of the second kind in section 3.

Throughout this paper, for a matrix  $A$  in  $\mathbb{C}^{N \times N}$ , its spectrum  $\sigma(A)$  denotes the set of all eigenvalues of  $A$ . The two-norm of  $A$ , which will be denoted by  $\|A\|$ , is defined by

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

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where, for a vector  $y \in \mathbb{C}^N$ ,  $\|y\|_2 = (y^T y)^{1/2}$  is the Euclidean norm of  $y$ . If  $f(z)$  and  $g(z)$  are holomorphic functions of the complex variable  $z$ , which are defined in an open set  $\Omega$  of the complex plane and  $A$  is a matrix in  $\mathbb{C}^{N \times N}$  with  $\sigma(A) \subset \Omega$ , then from the properties of the matrix functional calculus in [6], it follows that:

$$f(A)g(A) = g(A)f(A).$$

Hence, if  $B \in \mathbb{C}^{N \times N}$  is a matrix for which  $\sigma(B) \subset \Omega$  and  $AB = BA$ , then

$$f(A)g(B) = g(B)f(A).$$

We say that a matrix  $A$  in  $\mathbb{C}^{N \times N}$  is a positive stable matrix if  $\operatorname{Re}(\lambda) > 0$  for all  $\lambda \in \sigma(A)$ , where  $\sigma(A)$  is the set of all eigenvalues of  $A$ . Throughout this paper, the zero matrix of  $\mathbb{C}^{N \times N}$  will be denoted by  $\mathbf{0}$ . Furthermore, the identity matrix of  $\mathbb{C}^{N \times N}$  will be denoted by  $I$ . The hypergeometric matrix function  $F(A, B; C; z)$  has been given in the form [10] as follows:

$$F(A, B; C; z) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{n!} [(C)_n]^{-1} z^n$$

for matrices  $A, B$  and  $C$  in  $\mathbb{C}^{N \times N}$  such that  $C + nI$  is invertible for all integers  $n \geq 0$  and for  $|z| < 1$ . From [10], one recalls the Pochhammer symbol in its matrix version as follows

$$(A)_n = A(A+I)(A+2I) \cdots (A+(n-1)I), \quad n \geq 1; \quad (A)_0 = I.$$

**1.1. Lemma.** [14] *If  $\|\cdot\|$  denotes any matrix norm for which  $\|I\| = 1$ , and if  $\|M\| < 1$  ( $M \in \mathbb{C}^{N \times N}$ ), then  $(I + M)^{-1}$  exists:*

$$(I + M)^{-1} = I - M + M^2 - \cdots . \quad \square$$

If  $D$  is the complex plane cut along the negative real axis and  $\log(z)$  denotes the principle logarithm of  $z$ , then  $z^{1/2}$  represents  $\exp((1/2) \log(z))$ . If  $A$  is a matrix in  $\mathbb{C}^{N \times N}$  with  $\sigma(A) \subset D$ , then  $A^{1/2} = \sqrt{A}$  denotes the image by  $z^{1/2}$  of the matrix functional calculus acting on the matrix  $A$ .

Let  $A$  be a matrix in  $\mathbb{C}^{N \times N}$  where

$$A + nI \text{ is invertible for every integer } n > 0$$

and let  $\lambda$  be a complex number whose real part is positive. Then the Laguerre matrix polynomials  $L_n^{(A, \lambda)}(x)$  are defined by [8] as follows:

$$(1.1) \quad L_n^{(A, \lambda)}(x) = \sum_{k=0}^n \frac{(-1)^k}{k! (n-k)!} (A+I)_n [(A+I)_k]^{-1} (\lambda x)^k, \quad n \geq 0.$$

Let  $A$  be a matrix in  $\mathbb{C}^{N \times N}$  where

$$\operatorname{Re}(\mu) > 0 \text{ for all eigenvalue } \mu \in \sigma(A).$$

Then the Hermite matrix polynomials  $H_n(x, A)$  are defined by [7] as:

$$(1.2) \quad H_n(x, A) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{k! (n-2k)!} (x\sqrt{2A})^{n-2k}, \quad n \geq 0.$$

The Jacobi matrix polynomials  $P_n^{(A, B)}(x)$  have been given in [5] for parameter matrices  $A$  and  $B$  whose eigenvalues,  $z$ , all satisfy the condition  $\operatorname{Re}(z) > -1$ . For any natural number  $n > 0$ , the Jacobi matrix polynomials  $P_n^{(A, B)}(x)$  are defined by

$$(1.3) \quad P_n^{(A, B)}(x) = \frac{(-1)^n}{n!} F\left(A + B + nI, -nI; B + I; \frac{1+x}{2}\right) \times \Gamma^{-1}(B+I) \Gamma(B+(n+1)I).$$

In [2], the Chebyshev matrix polynomials of the second kind are defined by

$$(1.4) \quad U_n(x, A) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (n-k)! (\sqrt{2Ax})^{n-2k}}{k! (n-2k)!},$$

where  $A$  be a positive stable matrix in  $\mathbb{C}^{N \times N}$ .

## 2. Some results for Chebyshev matrix polynomials of the second kind

In this section, we derive the generating matrix function and recurrence relations for Chebyshev matrix polynomials of the second kind. We have the following main theorem.

**2.1. Theorem.** *Let  $A$  be a matrix in  $\mathbb{C}^{N \times N}$  where  $\operatorname{Re}(\lambda) > 0$  for all eigenvalue  $\lambda \in \sigma(A)$  and  $\|\sqrt{A}\| < \frac{1}{\sqrt{2}}$ . Then the generating matrix function for Chebyshev matrix polynomials of the second kind is*

$$(2.1) \quad \sum_{n=0}^{\infty} U_n(x, A) t^n = (I - \sqrt{2Ax}t + t^2 I)^{-1}, \quad |t| < 1, \quad |x| < 1.$$

*Proof.* By making use of (1.4) in the left-hand side of (2.1) and replacing  $n$  by  $n + 2k$ , we have

$$\begin{aligned} \sum_{n=0}^{\infty} U_n(x, A) t^n &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (n-k)! (\sqrt{2Ax})^{n-2k}}{k! (n-2k)!} \right) t^n \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \frac{(-1)^k (n+1)(n+2) \cdots (n+k)}{k!} t^{2k} \right) (\sqrt{2Ax})^n t^n. \end{aligned}$$

By using

$$(1 + t^2)^{-n-1} = \sum_{k=0}^{\infty} \frac{(-1)^k (n+1)(n+2) \cdots (n+k)}{k!} t^{2k}; \quad |t| < 1,$$

we can write

$$\sum_{n=0}^{\infty} U_n(x, A) t^n = \sum_{n=0}^{\infty} (1 + t^2)^{-n-1} (\sqrt{2Ax})^n t^n,$$

and then by Lemma 1.1, for  $\|\sqrt{A}\| < \frac{1}{\sqrt{2}}$ , one can obtain the generating matrix function for Chebyshev matrix polynomials of the second kind.  $\square$

**2.2. Theorem.** *Let  $A$  be a matrix in  $\mathbb{C}^{N \times N}$  where  $\operatorname{Re}(\lambda) > 0$  for all eigenvalue  $\lambda \in \sigma(A)$  and  $\|\sqrt{A}\| < \frac{1}{\sqrt{2}}$ . Then recurrence relation for Chebyshev matrix polynomials of the second kind is*

$$\sqrt{2A} U_{n-1}(x, A) = U_n'(x, A) + U_{n-2}'(x, A) - \sqrt{2A} x U_{n-1}'(x, A); \quad n \geq 2, \quad |x| < 1.$$

*Proof.* By differentiating (2.1) with respect to  $x$ , making the necessary arrangements and identification of coefficients of  $t^n$ , the theorem can be proved.  $\square$

**2.3. Theorem.** *Let  $A$  be a matrix in  $\mathbb{C}^{N \times N}$  where  $\operatorname{Re}(\lambda) > 0$  for all eigenvalue  $\lambda \in \sigma(A)$  and  $\|\sqrt{A}\| < \frac{1}{\sqrt{2}}$ . Then the recurrence relation for Chebyshev matrix polynomials of the second kind is*

$$U_{n+1}(x, A) = -U_{n-1}(x, A) + \sqrt{2A} x U_n(x, A); \quad n \geq 1, \quad |x| < 1.$$

*Proof.* By differentiating (2.1) with respect to  $t$ , making the necessary arrangements and identification of coefficients of  $t^n$ , the theorem can be proved.  $\square$

**2.4. Corollary.** *Let  $A$  be a matrix in  $\mathbb{C}^{N \times N}$  where  $\operatorname{Re}(\lambda) > 0$  for all eigenvalue  $\lambda \in \sigma(A)$  and  $\|\sqrt{A}\| < \frac{1}{\sqrt{2}}$ . Then the Chebyshev matrix polynomials of the second kind satisfy*

$$U_n(-x, A) = (-1)^n U_n(x, A), \quad |x| < 1.$$

*Proof.* Taking  $(-x)$  instead of  $x$  and  $(-t)$  instead of  $t$  in (2.1), the proof is completed.  $\square$

**2.5. Corollary.** *Let  $A$  be a matrix in  $\mathbb{C}^{N \times N}$  where  $\operatorname{Re}(\lambda) > 0$  for all eigenvalue  $\lambda \in \sigma(A)$  and  $\|\sqrt{A}\| < \frac{1}{\sqrt{2}}$ . Then Chebyshev matrix polynomials of the second kind satisfy*

$$\begin{bmatrix} U_1(x, A) & -U_0(x, A) \\ U_0(x, A) & \mathbf{0} \end{bmatrix}^n = \begin{bmatrix} U_n(x, A) & -U_{n-1}(x, A) \\ U_{n-1}(x, A) & -U_{n-2}(x, A) \end{bmatrix}; \quad n \geq 1, \quad |x| < 1,$$

where  $U_{-1}(x, A) = \mathbf{0}$ .

*Proof.* By using induction and Theorem 2.3, one can obtain the desired result.  $\square$

**2.6. Theorem.** *Let  $A$  be a matrix in  $\mathbb{C}^{N \times N}$  where  $\operatorname{Re}(\lambda) > 0$  for all eigenvalue  $\lambda \in \sigma(A)$  and  $\|\sqrt{A}\| < \frac{1}{\sqrt{2}}$ . Then Chebyshev matrix polynomials of the second kind satisfy*

$$(U_n(x, A), U_{n-1}(x, A)) = (U_1(x, A), U_0(x, A))(I, \mathbf{0})^{n-1}; \quad n \geq 2,$$

where

$$(U_n(x, A), U_{n-1}(x, A))(I, \mathbf{0}) = (-U_{n-1}(x, A) + \sqrt{2Ax}U_n(x, A), U_n(x, A)); \\ n \geq 1, \quad |x| < 1.$$

*Proof.* By using induction and Theorem 2.3, the theorem can be proved.  $\square$

### 3. Multilinear and multilateral generating matrix functions for Chebyshev matrix polynomials of the second kind

In this section, we derive several families of bilinear and bilateral generating matrix functions for Chebyshev matrix polynomials of the second kind generated by (2.1).

We first state our result as follows.

**3.1. Theorem.** *Corresponding to a non-vanishing function  $\Omega_\mu(y_1, \dots, y_s)$  consisting of  $s$  complex variables  $y_1, \dots, y_s$  ( $s \in \mathbb{N}$ ) and of complex order  $\mu$ , let*

$$(3.1) \quad \Lambda_{\mu, \nu}(y_1, \dots, y_s; z) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1, \dots, y_s) z^k$$

and for  $(a_k \neq 0, \mu, \nu \in \mathbb{C})$ ,

$$(3.2) \quad \Theta_{n, p, \mu, \nu}(x; y_1, \dots, y_s; \zeta) := \sum_{k=0}^{[n/p]} a_k U_{n-pk}(x, A) \Omega_{\mu+\nu k}(y_1, \dots, y_s) \zeta^k,$$

where  $A$  is a positive stable matrix in  $\mathbb{C}^{N \times N}$  with  $\|\sqrt{A}\| < \frac{1}{\sqrt{2}}$ ,  $n, p \in \mathbb{N}$  and (as usual)  $[\lambda]$  represents the greatest integer in  $\lambda \in \mathbb{R}$ . Then we have

$$(3.3) \quad \sum_{n=0}^{\infty} \Theta_{n, p, \mu, \nu} \left( x; y_1, \dots, y_s; \frac{\eta}{t^p} \right) t^n = (I - \sqrt{2Ax}t + t^2I)^{-1} \Lambda_{\mu, \nu}(y_1, \dots, y_s; \eta),$$

provided that each member of (3.3) exists for  $|t| < 1$ ,  $|x| < 1$ .

*Proof.* For convenience, let  $S$  denote the first member of the assertion (3.3) of Theorem 3.1. Then, plugging the polynomials

$$\Theta_{n,p,\mu,\nu} \left( x; y_1, \dots, y_s; \frac{\eta}{t^p} \right)$$

which come from (3.2) into the left-hand side of (3.3), we obtain

$$(3.4) \quad S = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} a_k U_{n-pk}(x, A) \Omega_{\mu+\nu k}(y_1, \dots, y_s) \eta^k t^{n-pk}.$$

Upon changing the order of summation in (3.4), if we replace  $n$  by  $n + pk$ , we can write

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k U_n(x, A) \Omega_{\mu+\nu k}(y_1, \dots, y_s) \eta^k t^n \\ &= \left( \sum_{n=0}^{\infty} U_n(x, A) t^n \right) \left( \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1, \dots, y_s) \eta^k \right) \\ &= (I - \sqrt{2Ax}t + t^2I)^{-1} \Lambda_{\mu,\nu}(y_1, \dots, y_s; \eta), \end{aligned}$$

which completes the proof of Theorem 3.1.  $\square$

By expressing the multivariable function  $\Omega_{\mu+\nu k}(y_1, \dots, y_s)$ , ( $k \in \mathbb{N}_0$ ,  $s \in \mathbb{N}$ ) in terms of a simpler function of one or more variables, we can give further applications of Theorem 3.1. For example, consider the case of

$$s = 1 \text{ and } \Omega_{\mu+\nu k}(y) = L_{\mu+\nu k}^{(B,\lambda)}(y)$$

in Theorem 3.1. Here the Laguerre matrix polynomials  $L_n^{(B,\lambda)}(y)$  are defined by (1.1) as:

$$L_n^{(B,\lambda)}(y) = \sum_{k=0}^n \frac{(-1)^k \lambda^k}{k! (n-k)!} (B+I)_n [(B+I)_k]^{-1} y^k$$

in which  $B$  is a matrix in  $\mathbb{C}^{N \times N}$ ,  $B + nI$  is invertible for every integer  $n > 0$  and  $\lambda$  is a complex number with  $\text{Re}(\lambda) > 0$ . Notice that the Laguerre matrix polynomials are generated as follows:

$$(3.5) \quad \sum_{n=0}^{\infty} L_n^{(B,\lambda)}(y) t^n = (1-t)^{-(B+I)} \exp\left(\frac{-\lambda y t}{1-t}\right),$$

$|t| < 1$ ,  $-\infty < y < \infty$ . Then we obtain the following result which provides a class of bilateral generating matrix functions for the Chebyshev matrix polynomials of the second kind and the Laguerre matrix polynomials.

**3.2. Corollary.** Let  $\Lambda_{\mu,\nu}(y; z) := \sum_{k=0}^{\infty} a_k L_{\mu+\nu k}^{(B,\lambda)}(y) z^k$ , where ( $a_k \neq 0$ ,  $\mu, \nu \in \mathbb{N}_0$ ) and

$$\Theta_{n,p,\mu,\nu}(x; y; \zeta) := \sum_{k=0}^{[n/p]} a_k U_{n-pk}(x, A) L_{\mu+\nu k}^{(B,\lambda)}(y) \zeta^k,$$

where  $n, p \in \mathbb{N}$  and  $A$  and  $B$  are matrices in  $\mathbb{C}^{N \times N}$  satisfying the condition  $\text{Re}(\lambda) > 0$  for all eigenvalues  $\lambda \in \sigma(A)$ , and  $B + nI$  is invertible for every integer  $n > 0$ . Then we have

$$(3.6) \quad \sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu} \left( x; y; \frac{\eta}{t^p} \right) t^n = (I - \sqrt{2Ax}t + t^2I)^{-1} \Lambda_{\mu,\nu}(y; \eta)$$

provided that each member of (3.6) exists for  $\|\sqrt{A}\| < \frac{1}{\sqrt{2}}$  and  $|t| < 1$ ,  $|x| < 1$ .  $\square$

**3.3. Remark.** Using the generating matrix function (3.5) for the Laguerre matrix polynomials and taking  $a_k = 1$ ,  $\mu = 0$ ,  $\nu = 1$ , we have

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} U_{n-pk}(x, A) L_k^{(B, \lambda)}(y) \eta^k t^{n-pk} \\ = (I - \sqrt{2Ax}t + t^2I)^{-1} (1 - \eta)^{-(B+I)} \exp\left(\frac{-\lambda y \eta}{1 - \eta}\right), \end{aligned}$$

where  $|\eta| < 1$ ,  $-\infty < y < \infty$ .

Set  $s = 1$  and  $\Omega_{\mu+\nu k}(y) = P_{\mu+\nu k}^{(B, C)}(y)$  in Theorem 3.1, where the Jacobi matrix polynomials  $P_n^{(B, C)}(y)$  are defined by (1.3) as:

$$\begin{aligned} P_n^{(B, C)}(y) = \frac{(-1)^n}{n!} F\left(B + C + (n+1)I, -nI; C + I; \frac{1+y}{2}\right) \\ \times \Gamma^{-1}(C + I) \Gamma(C + (n+1)I), \end{aligned}$$

where  $B$  and  $C$  are matrices in  $\mathbb{C}^{N \times N}$  whose eigenvalues,  $z$ , all satisfy the condition  $\operatorname{Re}(z) > -1$ . Here the Jacobi matrix polynomials are generated by

$$\begin{aligned} \sum_{n=0}^{\infty} (B + C + I)_n P_n^{(B, C)}(y) [(C + I)_n]^{-1} r^n \\ (3.7) \quad = (1+r)^{-(B+C+I)} F\left(\frac{B+C+I}{2}, \frac{B+C+2I}{2}; C + I; \frac{2r(y+1)}{(1+r)^2}\right) \\ |r| < 1, |y| < 1, \end{aligned}$$

which was given in [1]. Then we obtain the following result which provides a class of bilateral generating matrix functions for Chebyshev matrix polynomials of the second kind and the Jacobi matrix polynomials.

**3.4. Corollary.** Let  $\Lambda_{\mu, \nu}(y; z) := \sum_{k=0}^{\infty} a_k (B + C + I)_k P_{\mu+\nu k}^{(B, C)}(y) [(C + I)_k]^{-1} z^k$  where ( $a_k \neq 0$ ,  $\mu, \nu \in \mathbb{N}_0$ ) and

$$\Theta_{n, p, \mu, \nu}(x; y; \zeta) := \sum_{k=0}^{[n/p]} a_k U_{n-pk}(x, A) (B + C + I)_k P_{\mu+\nu k}^{(B, C)}(y) [(C + I)_k]^{-1} \zeta^k,$$

where  $n, p \in \mathbb{N}$  and  $A, B$  and  $C$  are matrices in  $\mathbb{C}^{N \times N}$  satisfying the condition  $\operatorname{Re}(\lambda) > 0$  for all eigenvalues  $\lambda \in \sigma(A)$ ,  $\operatorname{Re}(\gamma) > -1$  for all eigenvalues  $\gamma \in \sigma(B)$  and  $\operatorname{Re}(\xi) > -1$  for all eigenvalues  $\xi \in \sigma(C)$ . Then we have

$$(3.8) \quad \sum_{n=0}^{\infty} \Theta_{n, p, \mu, \nu}\left(x; y; \frac{\eta}{t^p}\right) t^n = (I - \sqrt{2Ax}t + t^2I)^{-1} \Lambda_{\mu, \nu}(y; \eta)$$

provided that each member of (3.8) exists for  $\|\sqrt{A}\| < \frac{1}{\sqrt{2}}$  and  $|t| < 1$ ,  $|x| < 1$ .  $\square$

**3.5. Remark.** Using the generating matrix function (3.7) for the Jacobi matrix polynomials and taking  $a_k = 1$ ,  $\mu = 0$ ,  $\nu = 1$ , we have

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} U_{n-pk}(x, A) (B + C + I)_k P_k^{(B, C)}(y) [(C + I)_k]^{-1} \eta^k t^{n-pk} \\ = (I - \sqrt{2Ax}t + t^2I)^{-1} \\ \times (1 + \eta)^{-(B+C+I)} F\left(\frac{B+C+I}{2}, \frac{B+C+2I}{2}; C + I; \frac{2\eta(y+1)}{(1+\eta)^2}\right), \end{aligned}$$

where  $|\eta| < 1$ ,  $|y| < 1$ .

In Theorem 3.1 choose

$$s = 1 \text{ and } \Omega_{\mu+\nu k}(y) = U_{\mu+\nu k}(y, B),$$

where  $B$  is a positive stable matrix in  $\mathbb{C}^{N \times N}$  and  $\mu, \nu \in \mathbb{N}_0$ . Then we obtain the following class of bilinear generating matrix function for the Chebyshev matrix polynomials of the second kind.

**3.6. Corollary.** Let  $\Lambda_{\mu,\nu}(y; z) := \sum_{k=0}^{\infty} a_k U_{\mu+\nu k}(y, B) z^k$ , where  $(a_k \neq 0, \mu, \nu \in \mathbb{N}_0)$  and

$$\Theta_{n,p,\mu,\nu}(x; y; \zeta) := \sum_{k=0}^{[n/p]} a_k U_{n-pk}(x, A) U_{\mu+\nu k}(y, B) \zeta^k,$$

where  $n, p \in \mathbb{N}$  and  $B$  is a positive stable matrix in  $\mathbb{C}^{N \times N}$ . Then we have

$$(3.9) \quad \sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu} \left( x; y; \frac{\eta}{t^p} \right) t^n = (I - \sqrt{2A}xt + t^2 I)^{-1} \Lambda_{\mu,\nu}(y; \eta)$$

provided that each member of (3.9) exists for  $\|\sqrt{A}\| < \frac{1}{\sqrt{2}}$  and  $|t| < 1$ ,  $|x| < 1$ .  $\square$

**3.7. Remark.** Using Corollary 3.4 and taking  $a_k = 1$ ,  $\mu = 0$ ,  $\nu = 1$ , we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} U_{n-pk}(x, A) U_k(y, B) \eta^k t^{n-pk} \\ &= (I - \sqrt{2A}xt + t^2 I)^{-1} (I - \sqrt{2B}y\eta + \eta^2 I)^{-1}, \end{aligned}$$

where  $|\eta| < 1$ ,  $|y| < 1$  and  $\|\sqrt{B}\| < \frac{1}{\sqrt{2}}$ .

Furthermore, for every suitable choice of the coefficients  $a_k$  ( $k \in \mathbb{N}_0$ ), if the multi-variable function  $\Omega_{\mu+\psi k}(y_1, \dots, y_s)$ , ( $s \in \mathbb{N}$ ), is expressed as an appropriate product of several simpler families, the assertions of Theorem 3.1 can be applied in order to derive various families of multilinear and multilateral generating matrix functions for the Chebyshev matrix polynomials of the second kind.

We set

$$s = 2 \text{ and } \Omega_{\mu+\nu k}(y, z) = H_{\mu+\nu k}(y, z, B)$$

in Theorem 3.1. Here the two-variable Hermite matrix polynomials  $H_n(y, z, B)$  are defined by means of the generating matrix function in [2] as follows:

$$(3.10) \quad \exp \left( yt\sqrt{2B} - zt^2 I \right) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(y, z, B) t^n; \quad |t| < \infty,$$

where  $B$  is a positive stable matrix in  $\mathbb{C}^{N \times N}$ . Then we obtain the following result which provides a class of multilateral generating matrix functions for the two-variable Hermite matrix polynomials and the Chebyshev matrix polynomials of the second kind defined by (1.4).

**3.8. Corollary.** Let  $\Lambda_{\mu,\psi}(y, z; r) := \sum_{k=0}^{\infty} a_k H_{\mu+\nu k}(y, z, B) r^k$ , where  $(a_k \neq 0, \mu, \nu \in \mathbb{N}_0)$

and

$$\Theta_{n,p,\mu,\psi}(x; y, z; \zeta) := \sum_{k=0}^{[n/p]} a_k U_{n-pk}(x, A) H_{\mu+\nu k}(y, z, B) \zeta^k,$$

where  $n, p \in \mathbb{N}$  and  $A$  and  $B$  are positive stable matrices in  $\mathbb{C}^{N \times N}$ . Then we have

$$(3.11) \quad \sum_{n=0}^{\infty} \Theta_{n,p,\mu,\psi} \left( x; y, z; \frac{\eta}{t^p} \right) t^n = (I - \sqrt{2Ax}t + t^2I)^{-1} \Lambda_{\mu,\psi}(y, z; \eta)$$

provided that each member of (3.11) exists for  $\|\sqrt{A}\| < \frac{1}{\sqrt{2}}$  and  $|t| < 1$ ,  $|x| < 1$ .  $\square$

**3.9. Remark.** Using the generating matrix function (3.10) for the two-variable Hermite matrix polynomials and taking  $a_k = \frac{1}{k!}$ ,  $\mu = 0$ ,  $\nu = 1$ , we have

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} U_{n-pk}(x, A) \frac{H_k(y, z, B)}{k!} \eta^k t^{n-pk} \\ = (I - \sqrt{2Ax}t + t^2I)^{-1} \exp \left( y\eta\sqrt{2B} - z\eta^2I \right), \end{aligned}$$

where  $|\eta| < \infty$ .

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