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SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH THE DERIVATIVE OPERATOR

A. L. Pathak*, S. B. Joshi†‡, Preeti Dwivedi§ and R. Agarwal*

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Abstract

In the present paper, we introduce a new subclass of harmonic functions in the unit disc U by using the Derivative operator. Also, we obtain coefficient conditions, convolution conditions, convex combinations, extreme points and some other properties.

Keywords: Harmonic, Univalent functions, Derivative operator.

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1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain C if both u and v are real harmonic in C. In any simply connected domain $D \subset C$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D, see [4].

In 1984, Clunie and Sheil-Small [4] investigated the class S_H and studied some sufficient bounds. Since then, there have been several papers published related to S_H and its subclasses. In fact by introducing new subclasses Sheil-Small [13], Silverman [14], Silverman and Silvia [15], Jahangiri [6] and Ahuja [1] presented a systematic and unified study of harmonic univalent functions. Furthermore we refer to Duren [5], Ponnusamy [9] and references therein for basic results on the subject.

*Department of Mathematics, Brahmanand College, The Mall, Kanpur-208004, (U.P.), India. E-mail: (A. L. Pathak) alpathak@rediffmail.com alpathak.bnd@gmail.com (R. Agarwal) ritesh.840@ rediffmail.com

†Department of Mathematics, Walchand College of Engineering, Sangli-416415, (Maharashtra), India. E-mail: joshisb@hotmail.com

‡Corresponding Author.

§Department of Physics, V.S.S.D. College, Kanpur-208002, (U.P.), India. E-mail: priti_vdwivedi@ yahoo.com

Denote by S_H, the class of functions $f = h + \overline{g}$ that are harmonic, univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ with normalization $f(0) = h(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \; g(z) = \sum_{k=1}^{\infty} b_k z^k, \; |b_1| < 1. \tag{1.1}$$

Observe that S_H reduces to S, the class of normalized univalent functions, if the co-analytic part of f is zero. Also, denote by S_H^* the subclass of S_H consisting of functions f that map U onto a starlike domain.

For $f = h + \overline{g}$ given by (1.1), Al-Shaqsi and Darus [3] introduced the operator D_λ^n as:

$$D_\lambda^n f(z) = D_\lambda^n h(z) + (-1)^n \overline{D_\lambda^n g(z)}, \; n, \lambda \in N_0 = N \cup \{0\}, \; z \in U, \tag{1.2}$$

where $D_\lambda^n h(z) = z + \sum_{k=2}^{\infty} k^n C(\lambda, k) a_k z^k$, $D_\lambda^n g(z) = \sum_{k=1}^{\infty} k^n C(\lambda, k) b_k z^k$ and $C(\lambda, k) = \binom{k+\lambda-1}{\lambda}$.

Recently Rosy *et al.* [10] defined the subclass $G_H(\gamma) \subset S_H$ consisting of harmonic univalent functions $f(z)$ satisfying the condition

$$\operatorname{Re}\left\{(1 + e^{i\alpha}) \frac{z f'(z)}{f(z)} - e^{i\alpha}\right\} \geq \gamma, \; 0 \leq \gamma < 1, \; \alpha \in R.$$

They proved that if $f = h + \overline{g}$ is given by (1.1) and if

$$\sum_{n=1}^{\infty} \left[\frac{(2n - 1 - \gamma)}{(1 - \gamma)} |a_n| + \frac{(2n + 1 + \gamma)}{(1 - \gamma)} |b_n|\right] \leq 2, \; 0 \leq \gamma < 1, \tag{1.3}$$

then f is in $G_H(\gamma)$.

This condition is proved to be also necessary by Rosy *et al.* if h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \; g(z) = \sum_{n=1}^{\infty} |b_n| z^n. \tag{1.4}$$

Motivated by this aforementioned work, now we introduce the class $G_H(n, \lambda, \alpha, \rho)$ as the subclass of functions of the form (1.1) that satisfy the following condition

$$\operatorname{Re}\left\{(1 + \rho e^{ir}) \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} - \rho e^{ir}\right\} > \alpha, \; 0 \leq \alpha < 1, \; r \in R, \; \rho \geq 0, \tag{1.5}$$

where $D_\lambda^n f(z)$ is defined by (1.2).

Let $\overline{G}_H(n, \lambda, \alpha, \rho)$ denote that the subclasses of $G_H(n, \lambda, \alpha, \rho)$ which consists of harmonic functions $f_n = h + \overline{g}_n$ such that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \; g_n(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k. \tag{1.6}$$

It is clear that the class $G_H(n, \lambda, \alpha, \rho)$ includes a variety of well-known subclasses of S_H, such as,

(i) $G_H(0, 0, \alpha, 0) \equiv S_H^*(\alpha)$, Jahangiri [6],

(ii) $G_H(0, 1, \alpha, 0) \equiv HK(\alpha)$, Jahangiri [6],

(iii) $\overline{G}_H(n, 0, \alpha, 0) \equiv M\overline{H}(n, 0, \alpha)$, Jahangiri *et al.* [7],

(iv) $\overline{G}_H(0,\lambda,\alpha,0) \equiv M_{\overline{H}}(0,\lambda,\alpha)$, Murugusundaramoorthy and Vijya [8],

(v) $\overline{G}_H(n,\lambda,\alpha,0) \equiv M_{\overline{H}}(n,\lambda,\alpha)$, Al-Shaqsi and Darus [2],

(vi) $\overline{G}_H(0,1,\gamma,1) \equiv \overline{G}_H(\gamma)$,Rosy *et al.* [10].

In this paper, we will give sufficient condition for functions $f = h + \overline{g}$, where h and g are given by (1.1), to be in the class $G_H(n,\lambda,\alpha,\rho)$ and it is shown that this coefficient condition is also necessary for functions in the class $\overline{G}_H(n,\lambda,\alpha,\rho)$. Also, we obtain distortion theorems and characterize the extreme points and convolution conditions for functions in $\overline{G}_H(n,\lambda,\alpha,\rho)$.

Closure theorems and an application of neighborhoods are also obtained.

2. Coefficient bound

We begin with a sufficient coefficient condition for functions in $G_H(n,\lambda,\alpha,\rho)$.

2.1. Theorem. *Let $f = h + \overline{g}$ be given by (2.1). If*

$$\sum_{k=1}^{\infty} \left[\{k(1+\rho) - (\alpha+\rho)\} |a_k| + \{k(1+\rho) + (\alpha+\rho)\} |b_k| \right] \times k^n C(\lambda,k) \leq 2(1-\alpha), \tag{2.1}$$

where $a_1 = 1$, $n,\lambda \in N_0, C(\lambda,k) = \binom{k+\lambda-1}{\lambda}$, $\rho \geq 0$ and $0 \leq \alpha < 1$, then f is sense-preserving, harmonic univalent in U and $f \in G_H(n,\lambda,\alpha,\rho)$.

Proof. If $z_1 \neq z_2$, then

$$\begin{aligned}
\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| \\
&= 1 - \left| \frac{\sum_{k=1}^{\infty} b_k (z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)} \right| \\
&> 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=2}^{\infty} k|a_k|} \\
&\geq 1 - \frac{\sum_{k=1}^{\infty} \frac{[k(1+\rho) + (\alpha+\rho)] k^n C(\lambda,k) |b_k|}{1-\alpha}}{1 - \sum_{k=2}^{\infty} \frac{[k(1+\rho) - (\alpha+\rho)] k^n C(\lambda,k) |a_k|}{1-\alpha}} \\
&\geq 0,
\end{aligned} \tag{2.2}$$

which proves univalence. Note that f is sense-preserving in U. This is because

$$
\begin{aligned}
|h'(z)| &\geq 1-\sum_{k=2}^{\infty}k|a_k||z|^{k-1}\\
&> 1-\sum_{k=2}^{\infty}\frac{\{k(1+\rho)-(\alpha+\rho)\}k^nC(\lambda,k)|a_k|}{1-\alpha}\\
&\geq \sum_{k=1}^{\infty}\frac{\{k(1+\rho)+(\alpha+\rho)\}k^nC(\lambda,k)|b_k|}{1-\alpha} \qquad (2.3)\\
&> \sum_{k=1}^{\infty}\frac{\{k(1+\rho)+(\alpha+\rho)\}k^nC(\lambda,k)|b_k||z|^{k-1}}{1-\alpha}\\
&\geq \sum_{k=1}^{\infty}k|b_k||z|^{k-1}\geq |g'(z)|.
\end{aligned}
$$

Using the fact that $\mathrm{Re}w > \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$ it suffices to show that

$$
\begin{aligned}
&|(1-\alpha)+(1+\rho e^{ir})\frac{D_\lambda^{n+1}f(z)}{D_\lambda^n f(z)}-\rho e^{ir}|\\
&-|(1+\alpha)-(1+\rho e^{ir})\frac{D_\lambda^{n+1}f(z)}{D_\lambda^n f(z)}+\rho e^{ir}|\geq 0.
\end{aligned} \tag{2.4}
$$

Substituting the value of $D_\lambda^n f(z)$ in (2.4) yields, by (2.1),

$$
\begin{aligned}
&|(1-\alpha-\rho e^{ir})D_\lambda^n f(z)+(1+\rho e^{ir})D_\lambda^{n+1}f(z)|\\
&\qquad -|-(1+\alpha+\rho e^{ir})D_\lambda^n f(z)+(1+\rho e^{ir})D_\lambda^{n+1}f(z)|\\
&=|(2-\alpha)z+\sum_{k=2}^{\infty}\{k(1+\rho e^{ir})+(1-\alpha-\rho e^{ir})\}k^nC(\lambda,k)\\
&\qquad\times a_kz^k-(-1)^n\sum_{k=1}^{\infty}\{k(1+\rho e^{ir})-(1-\alpha-\rho e^{ir})\}k^nC(\lambda,k)b_kz^k|\\
&\qquad -|-\alpha z+\sum_{k=2}^{\infty}\{k(1+\rho e^{ir})-(1+\alpha+\rho e^{ir})\}k^nC(\lambda,k)a_kz^k\\
&\qquad -(-1)^n\sum_{k=1}^{\infty}\{k(1+\rho e^{ir})+(1+\alpha+\rho e^{ir})\}k^nC(\lambda,k)\overline{b_kz^k}\\
&\geq 2(1-\alpha)|z|\Bigg[1-\sum_{k=2}^{\infty}\frac{\{k(1+\rho)-(\alpha+\rho)\}k^nC(\lambda,k)|a_k||z|^k}{1-\alpha}\\
&\qquad -\sum_{k=1}^{\infty}\frac{\{k(1+\rho)+(\alpha+\rho)\}k^nC(\lambda,k)|b_k||z|^k}{1-\alpha}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\geq 2(1-\alpha)\Bigg[1-\sum_{k=2}^{\infty}\frac{\{k(1+\rho)-(\alpha+\rho)\}k^nC(\lambda,k)|a_k|}{1-\alpha}\\
&\qquad -\sum_{k=1}^{\infty}\frac{\{k(1+\rho)+(\alpha+\rho)\}k^nC(\lambda,k)|b_k|}{1-\alpha}\Bigg].
\end{aligned} \tag{2.5}
$$

This last expressions is non-negative by (2.1), and so the proof is complete. □

The harmonic function

$$f(z) = z + \sum_{k=2}^{\infty} \frac{(1-\alpha)}{\{k(1+\rho)-(\alpha+\rho)\}k^n C(\lambda,k)} x_k z^k + \sum_{k=1}^{\infty} \frac{(1-\alpha)}{\{k(1+\rho)+(\alpha+\rho)\}k^n C(\lambda,k)} \overline{y_k z^k} \tag{2.6}$$

where $n, \lambda \in N_0$, $o \leq \rho \leq 1$ and $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$ shows that the coefficient bound given by (2.1) is sharp. The functions of the form (2.6) are in $G_H(n,\lambda,\alpha,\rho)$ because

$$\sum_{k=1}^{\infty} \left[\frac{k(1+\rho)-(\alpha+\rho)}{1-\alpha}|a_k| + \frac{k(1+\rho)+(\alpha+\rho)}{1-\alpha}|b_k| \right] k^n C(\lambda,k) = 1 + \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 2. \tag{2.7}$$

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f_n = h + \overline{g}_n$, where h and, g_n are of the form (1.6).

2.2. Theorem. *Let $f_n = h + \overline{g}_n$ be given by* (1.6). *Then $f_n \in \overline{G}_H(n,\lambda,\alpha,\rho)$ if and only if*

$$\sum_{k=1}^{\infty} [\{k(1+\rho)-(\alpha+\rho)\}|a_k| + \{k(1+\rho)+(\alpha+\rho)\}|b_k|] k^n C(\lambda,k) \leq 2(1-\alpha) \tag{2.8}$$

where $a_1 = 1,\ n, \lambda \in N_0,\ C(\lambda,k) = \binom{k+\lambda-1}{\lambda}, \rho \geq 0, 0 \leq \alpha < 1.$

Proof. Since $\overline{G}_H(n,\lambda,\alpha,\rho) \subset G_H(n,\lambda,\alpha,\rho)$ we only need to prove the "only if" part of Theorem 2.2. To this end, for functions f_n of the form (1.6), we notice that the condition (1.5) is equivalent to

$$\text{Re}\left\{ (1+\rho e^{ir}) \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} - (\rho e^{ir} + \alpha) \right\} \geq 0$$

$\Longrightarrow$

$$\text{Re} \frac{\{(1+\rho e^{ir}) D_\lambda^{n+1} f(z) - (\rho e^{ir} + \alpha) D_\lambda^n f(z)\}}{D_\lambda^n f(z)} \geq 0$$

$\Longrightarrow$

$$\text{Re} \left\{ \frac{(1+\rho e^{ir})\Big(z - \sum_{k=2}^{\infty} k^{n+1} C(\lambda,k)|a_k| z^k + (-1)^{2n+1} \sum_{k=1}^{\infty} k^{n+1} |b_k| C(\lambda,k) \overline{z}^k\Big)}{z - \sum_{k=2}^{\infty} k^n C(\lambda,k)|a_k| z^k + (-1)^{2n} \sum_{k=1}^{\infty} k^n C(\lambda,k)|b_k| \overline{z}^k} - \frac{(\rho e^{ir} + \alpha)(z - \sum_{k=2}^{\infty} k^n C(\lambda,k)|a_k| z^k + (-1)^{2n} \sum_{k=1}^{\infty} k^n |b_k| C(\lambda,k) \overline{z}^k)}{z - \sum_{k=2}^{\infty} k^n C(\lambda,k)|a_k| z^k + (-1)^{2n} \sum_{k=1}^{\infty} k^n C(\lambda,k)|b_k| \overline{z}^k} \right\} \geq 0$$

$\Longrightarrow$

$$\operatorname{Re}\left\{\frac{(1-\alpha)z-\sum_{k=2}^{\infty}k^n[k(1+\rho e^{ir})-(\rho e^{ir}+\alpha)]C(\lambda,k)|a_k|z^k}{z-\sum_{k=2}^{\infty}k^nC(\lambda,k)|a_k|z^k+(-1)^{2n}\sum_{k=1}^{\infty}k^nC(\lambda,k)|b_k|\overline{z}^k}\right.$$
$$\left.+\frac{(-1)^{2n+1}\sum_{k=1}^{\infty}k^n[k(1+\rho e^{ir})+(\rho e^{ir}+\alpha)]C(\lambda,k)|b_k|\overline{z}^k}{z-\sum_{k=2}^{\infty}k^nC(\lambda,k)|a_k|z^k+(-1)^{2n}\sum_{k=1}^{\infty}k^nC(\lambda,k)|b_k|\overline{z}^k}\right\}\geq 0$$

$\Longrightarrow$

$$\operatorname{Re}\left\{\frac{(1-\alpha)-\sum_{k=2}^{\infty}k^n[k(1+\rho e^{ir})-(\rho e^{ir}+\alpha)]C(\lambda,k)|a_k|z^{k-1}}{1-\sum_{k=2}^{\infty}k^nC(\lambda,k)|a_k|z^{k-1}+\frac{\overline{z}}{z}(-1)^{2n}\sum_{k=1}^{\infty}k^nC(\lambda,k)|b_k|\overline{z}^{k-1}}\right.$$
$$\left.-\frac{\frac{\overline{z}}{z}(-1)^{2n}\sum_{k=1}^{\infty}k^n[k(1+\rho e^{ir})+(\rho e^{ir}+\alpha)]C(\lambda,k)|b_k|\overline{z}^{k-1}}{1-\sum_{k=2}^{\infty}k^nC(\lambda,k)|a_k|z^{k-1}+\frac{\overline{z}}{z}(-1)^{2n}\sum_{k=1}^{\infty}k^nC(\lambda,k)|b_k|\overline{z}^{k-1}}\right\}\geq 0 \tag{2.9}$$

The above condition (2.9) must hold for all values of z on the positive real axes, where, $0\leq|z|=\gamma<1$, we must have

$$\operatorname{Re}\left\{\frac{(1-\alpha)-\sum_{k=2}^{\infty}k^n(k-\alpha)C(\lambda,k)|a_k|\gamma^{k-1}}{1-\sum_{k=2}^{\infty}k^nC(\lambda,k)|a_k|\gamma^{k-1}+(-1)^{2n}\sum_{k=1}^{\infty}k^nC(\lambda,k)|b_k|\gamma^{k-1}}\right.$$
$$-\frac{(-1)^{2n}\sum_{k=1}^{\infty}k^n(k+\alpha)C(\lambda,k)|b_k|\gamma^{k-1}-\rho e^{ir}\sum_{k=2}^{\infty}k^n(k-1)C(\lambda,k)|a_k|\gamma^{k-1}}{1-\sum_{k=2}^{\infty}k^nC(\lambda,k)|a_k|\gamma^{k-1}+(-1)^{2n}\sum_{k=1}^{\infty}k^nC(\lambda,k)|b_k|\gamma^{k-1}}$$
$$\left.-\frac{(-1)^{2n}\rho e^{ir}\sum_{k=1}^{\infty}k^n(k+1)C(\lambda,k)|b_k|\gamma^{k-1}}{1-\sum_{k=2}^{\infty}k^nC(\lambda,k)|a_k|\gamma^{k-1}+(-1)^{2n}\sum_{k=1}^{\infty}k^nC(\lambda,k)|b_k|\gamma^{k-1}}\right\}\geq 0.$$

Since $\mathrm{Re}(-e^{ir}) \geq -|e^{ir}| = -1$, the above inequality reduce to

$$\frac{(1-\alpha)-\sum_{k=2}^{\infty} k^n\{(k(1+\rho)-(\rho+\alpha))\}C(\lambda,k)|a_k|\gamma^{k-1}}{1-\sum_{k=2}^{\infty} k^n C(\lambda,k)|a_k|\gamma^{k-1}+\sum_{k=1}^{\infty} k^n C(\lambda,k)|b_k|\gamma^{k-1}} - \frac{\sum_{k=1}^{\infty} k^n\{(k(1+\rho)+(\rho+\alpha))\}C(\lambda,k)|b_k|\gamma^{k-1}}{1-\sum_{k=2}^{\infty} k^n C(\lambda,k)|a_k|\gamma^{k-1}+\sum_{k=1}^{\infty} k^n C(\lambda,k)|b_k|\gamma^{k-1}} \geq 0. \tag{2.10}$$

If the condition (2.8) does not hold, then the numerator in (2.10) is negative for γ sufficiently close to 1. Hence there exists a $z_0 = \gamma_0$ in $(0,1)$ for which the quotient in (2.10) is negative. This contradicts the condition for $f_n \in \overline{G}_H(n,\lambda,\alpha,\rho)$ and so the proof is complete. □

3. Distortion bounds

In this section, we will obtain distortion bounds for functions in $\overline{G}_H(n,\lambda,\alpha,\rho)$.

3.1. Theorem. *Let $f_n \in \overline{G}_H(n,\lambda,\alpha,\rho)$. Then for $|z| = \gamma < 1$, we have*

$$|f_n(z)| \leq (1+|b_1|)\gamma + \frac{(1-\alpha)}{2^n[2(1+\rho)-(\rho+\alpha)](\lambda+1)}\left[1-\frac{1+2\rho+\alpha}{1-\alpha}|b_1|\right]\gamma^2.$$

$$|f_n(z)| \geq (1-|b_1|)\gamma - \frac{(1-\alpha)}{2^n[2(1+\rho)-(\rho+\alpha)](\lambda+1)}\left[1-\frac{1+2\rho+\alpha}{1-\alpha}|b_1|\right]\gamma^2.$$

Proof. We only prove the left-hand inequality. The proof for the right-hand inequality is similar and is thus omitted. Let $f_n \in \overline{G}_H(n,\lambda,\alpha,\rho)$. Taking the absolute value of f_n, we obtain

$$\begin{aligned}
&|f_n(z)| \\
&\quad = \left|z-\sum_{k=2}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k \overline{z}^k\right| \\
&\quad \leq (1+|b_1|)\gamma + \sum_{k=2}^{\infty}(|a_k|+|b_k|)\gamma^k \\
&\quad \leq (1+|b_1|)\gamma + \sum_{k=2}^{\infty}(|a_k|+|b_k|)\gamma^2 \\
&\quad \leq (1+|b_1|)\gamma + \frac{1-\alpha}{(2(1+\rho)-(\rho+\alpha))2^n(\lambda+1)} \\
&\qquad\qquad \times \sum_{k=2}^{\infty}\left(\frac{(2(1+\rho)-(\rho+\alpha))2^n(\lambda+1)}{1-\alpha}|a_k|\right. \\
&\qquad\qquad\qquad \left. + \frac{(2(1+\rho)-(\rho+\alpha))2^n(\lambda+1)}{1-\alpha}|b_k|\right)\gamma^2
\end{aligned}$$

$$\leq (1+|b_1|)\gamma + \frac{1-\alpha}{(2(1+\rho)-(\rho+\alpha))2^n(\lambda+1)} \times \sum_{k=2}^{\infty}\left[\frac{(k(1+\rho)-(\rho+\alpha))k^nC(\lambda,k)}{1-\alpha}|a_k| + \frac{(k(1+\rho)+(\rho+\alpha))k^nC(\lambda,k)}{1-\alpha}|b_k|\right]\gamma^2$$

$$\leq (1+|b_1|)\gamma + \frac{1-\alpha}{(2(1+\rho)-(\rho+\alpha))2^n(\lambda+1)}\left[1-\frac{((1+\rho)+(\rho+\alpha))}{1-\alpha}|b_1|\right]\gamma^2$$

$$\leq (1+|b_1|)\gamma + \frac{1-\alpha}{(2(1+\rho)-(\rho+\alpha))2^n(\lambda+1)}\left[1-\frac{1+2\rho+\alpha}{1-\alpha}|b_1|\right]\gamma^2.$$

The functions

$$f(z) = z + |b_1|\overline{z} + \frac{1}{2^n(\lambda+1)}\left[\frac{1-\alpha}{2(1+\rho)-(\rho+\alpha)} - \frac{1+2\rho+\alpha}{2(1+\rho)-(\rho+\alpha)}|b_1|\right]\overline{z}^2,$$

$$f(z) = (1-|b_1|)z - \frac{1}{2^n(\lambda+1)}\left[\frac{1-\alpha}{2(1+\rho)-(\rho+\alpha)} - \frac{1+2\rho+\alpha}{2(1+\rho)-(\rho+\alpha)}|b_1|\right]z^2$$

for $|b_1| \leq \frac{1-\alpha}{1+2\rho+\alpha}$ show that the bounds given in Theorem 3.1 are sharp. □

The following covering result follows from the left-hand inequality in Theorem 3.1.

3.2. Corollary. *If the function* $f_n = h + \overline{g}_n$*, where* h *and* g *given by* (1.4) *are in* $\overline{G}_H(n,\lambda,\alpha,\rho)$*, then*

$$\left\{w : |w| < \frac{(2^n(\lambda+1)(\rho+2) - 1 - (2^n(\lambda+1)-1)\alpha)}{2^n(\lambda+1)(2(1+\rho)-(\rho+\alpha))} - \frac{2^n(\lambda+1)(\rho+2) - (2\rho+1) - (2^n(\lambda+1)+1)\alpha|b_1|}{2^n(\lambda+1)(2(1+\rho)-(\rho+\alpha))}\right\} \subset f_n(U) \tag{3.1}$$ □

4. Convolution, convex combinations and extreme points

In this section, we show the class $\overline{G}_H(n,\lambda,\alpha,\rho)$ is invariant under convolution and convex combination.

For harmonic functions

$$f_n(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k \overline{z}^k$$

and

$$F_n(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^n \sum_{k=1}^{\infty} B_k \overline{z}^k,$$

the convolution of f_n and F_n is given by

$$(f_n * F_n)(z) = f_n(z) * F_n(z) = z - \sum_{k=2}^{\infty} a_k A_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k B_k \overline{z}^k. \tag{4.1}$$

4.1. Theorem. *For $0 \leq \beta \leq \alpha < 1$, let $f_n \in \overline{G}_H(n,\lambda,\alpha,\rho)$ and $F_n \in \overline{G}_H(n,\lambda,\beta,\rho)$. Then $f_n * F_n \in \overline{G}_H(n,\lambda,\alpha,\rho) \subset \overline{G}_H(n,\lambda,\beta,\rho)$.*

Proof. We wish to show that the coefficient of $f_n * F_n$ satisfies the required condition given in Theorem 2.2. For $F_n \in \overline{G}_H(n,\lambda,\beta,\rho)$, we note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Now, for the convolution function $f_n * F_n$, we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{\{k(1+\rho)-(\beta+\rho)\}k^n C(\lambda,k)}{1-\beta}|a_k||A_k| \\
&\qquad\qquad + \sum_{k=1}^{\infty} \frac{\{k(1+\rho)+(\beta+\rho)\}k^n C(\lambda,k)}{1-\beta}|b_k||B_k| \\
&\quad \leq \sum_{k=2}^{\infty} \frac{\{k(1+\rho)-(\beta+\rho)\}k^n C(\lambda,k)}{1-\beta}|a_k| \\
&\qquad\qquad + \sum_{k=1}^{\infty} \frac{\{k(1+\rho)+(\beta+\rho)\}k^n C(\lambda,k)}{1-\beta}|b_k| \\
&\quad \leq \sum_{k=2}^{\infty} \frac{\{k(1+\rho)-(\alpha+\rho)\}k^n C(\lambda,k)}{1-\alpha}|a_k| \\
&\qquad\qquad + \sum_{k=1}^{\infty} \frac{\{k(1+\rho)+(\alpha+\rho)\}k^n C(\lambda,k)}{1-\alpha}|b_k| \\
&\quad \leq 1.
\end{aligned}$$

Since $0 \leq \beta \leq \alpha < 1$ and $f_n \in \overline{G}_H(n,\lambda,\alpha,\rho)$, then $f_n * F_n \in \overline{G}_H(n,\lambda,\alpha,\rho) \subset \overline{G}_H(n,\lambda,\beta,\rho)$. □

We now examine convex combinations of $\overline{G}_H(n,\lambda,\alpha,\rho)$.

Let the functions $f_{n_j}(z)$ be defined, for $j = 1,2,\ldots,m$, by

$$f_{n_j}(z) = z - \sum_{k=2}^{\infty} |a_{k,j}|z^k + (-1)^n \sum_{k=1}^{\infty} |b_{k,j}|\overline{z}^k. \tag{4.2}$$

4.2. Theorem. *Let the functions $f_{n_j}(z)$ defined by (4.2) be in the class $\overline{G}_H(n,\lambda,\alpha,\rho)$ for every $j = 1,2,\ldots,m$. Then the functions $t_j(z)$ defined by*

$$t_j(z) = \sum_{j=1}^{m} c_j f_{n_j}(z),\ 0 \leq c_j \leq 1, \tag{4.3}$$

are also in the class $\overline{G}_H(n,\lambda,\alpha,\rho)$, where $\sum_{j=1}^{m} c_j = 1$.

Proof. According to the definition of t_j, we can write

$$t_j(z) = z - \sum_{k=2}^{\infty}\left(\sum_{j=1}^{m} c_j|a_{k,j}|\right)z^k + (-1)^n \sum_{k=1}^{\infty}\left(\sum_{j=1}^{m} c_j|b_{n,j}|\right)\overline{z}^k.$$

Further, since $f_{n_j}(z)$ are in $\overline{G}_H(n,\lambda,\alpha,\rho)$ for every $j=1,2,\dots,m$, then

$$\begin{aligned}
&\sum_{k=1}^{\infty}\Bigg\{\bigg[(k(1+\rho)-(\alpha+\rho))\bigg(\sum_{j=1}^{m}c_j|a_{k,j}|\bigg)\\
&\qquad\qquad + (k(1+\rho)+(\alpha+\rho))\bigg(\sum_{j=1}^{m}c_j|b_{k,j}|\bigg)\bigg]k^nC(\lambda,k)\Bigg\}\\
&=\sum_{j=1}^{m}c_j\bigg(\sum_{k=1}^{\infty}[(k(1+\rho)-(\alpha+\rho))|a_{n,j}|\\
&\qquad\qquad + (k(1+\rho)+(\alpha+\rho))|b_{n,j}|]k^nC(\lambda,k)\bigg)\\
&\le\sum_{j=1}^{m}c_j2(1-\alpha)\le 2(1-\alpha).
\end{aligned}$$

Hence Theorem 4.2 follows. □

4.3. Corollary. *The class $\overline{G}_H(n,\lambda,\alpha,\rho)$ is closed under convex linear combinations.*

Proof. Let the functions $f_{n_j}(z)(j=1,2\dots,m)$ defined by (4.2) be in the class $\overline{G}_H(n,\lambda,\alpha,\rho)$. Then the function $\Psi(z)$ defined by

(4.4) $$\Psi(z)=\mu f_{n_j}(z)+(1-\mu)f_{n_j}(z),\ 0\le\mu\le 1$$

is in the class $\overline{G}_H(n,\lambda,\alpha,\rho)$. Also, by taking $m=2$, $t_1=\mu$ and $t_2=1-\mu$ in Theorem 4.1. □

Next we determine the extreme points of closed convex hulls of $\overline{G}_H(n,\lambda,\alpha,\rho)$, denoted by clco $\overline{G}_H(n,\lambda,\alpha,\rho)$.

4.4. Theorem. *Let f_n be given by (1.6). Then $f_n\in\overline{G}_H(n,\lambda,\alpha,\rho)$ if and only if*

$$f_n(z)=\sum_{k=1}^{\infty}(X_kh_k(z)+Y_kg_{n_k}(z)),$$

where

$$h_1(z)=z,\ h_k(z)=z-\left(\frac{1-\alpha}{(k(1+\rho)-(\alpha+\rho))k^nC(\lambda,k)}\right)z^k,\ k=2,3\dots,$$

$$g_{n_k}(z)=z+(-1)^n\left(\frac{1-\alpha}{(k(1+\rho)+(\alpha+\rho))k^nC(\lambda,k)}\right)\overline{z}^k,\ k=1,2,3\dots$$

and $\sum_{k=1}^{\infty}(X_k+Y_k)=1$, $X_k\ge 0$, $Y_k\ge 0$. *In particular, the extreme points of $\overline{G}_H(n,\lambda,\alpha,\rho)$ are $\{h_k\}$ and $\{g_{n_k}\}$.*

Proof. For the function f_n of the form (4.7), we have

$$\begin{aligned}
f_n(z)&=\sum_{k=1}^{\infty}(X_kh_k(z)+Y_kg_{n_k}(z))\\
&=\sum_{k=1}^{\infty}(X_k+Y_k)z-\sum_{k=2}^{\infty}\frac{1-\alpha}{(k(1+\rho)-(\alpha+\rho))k^nC(\lambda,k)}X_kz^k\\
&\qquad + (-1)^n\sum_{k=1}^{\infty}\frac{1-\alpha}{(k(1+\rho)+(\alpha+\rho))k^nC(\lambda,k)}Y_k\overline{z}^k.
\end{aligned}$$

Then

$$\sum_{k=2}^{\infty}\frac{(k(1+\rho)-(\alpha+\rho))k^nC(\lambda,k)}{1-\alpha}|a_k| \\ +\sum_{k=1}^{\infty}\frac{(k(1+\rho)+(\alpha+\rho))k^nC(\lambda,k)}{1-\alpha}|b_k| \\ =\sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k=1-X_1\le 1, \tag{4.5}$$

and so $f_n \in \operatorname{clco}\overline{G}_H(n,\lambda,\alpha,\rho)$.

Conversely, suppose that $f_n \in \operatorname{clco}\overline{G}_H(n,\lambda,\alpha,\rho)$. Setting

$$X_k=\frac{(k(1+\rho)-(\alpha+\rho))k^nC(\lambda,k)}{1-\alpha}|a_k|,\ 0\le X_k\le 1\ k=2,3,\dots, \\ Y_k=\frac{(k(1+\rho)+(\alpha+\rho))k^nC(\lambda,k)}{1-\alpha}|b_k|,\ 0\le Y_k\le 1\ k=1,2,3,\dots, \tag{4.6}$$

and $X_1=1-\sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k$ then f_n can be written as

$$f_n(z)=z-\sum_{k=2}^{\infty}|a_k|z^k+(-1)^n\sum_{k=1}^{\infty}|b_k|\overline{z}^k \\ =z-\sum_{k=2}^{\infty}\frac{(1-\alpha)X_k}{(k(1+\rho)-(\alpha+\rho))k^nC(\lambda,k)}z^k \\ +(-1)^n\sum_{k=1}^{\infty}\frac{(1-\alpha)Y_k}{(k(1+\rho)+(\alpha+\rho))k^nC(\lambda,k)}\overline{z}^k \\ =z+\sum_{k=2}^{\infty}(h_k(z)-z)X_k+\sum_{k=1}^{\infty}(g_{n_k}(z)-z)Y_k \\ =\sum_{k=2}^{\infty}h_k(z)X_k+\sum_{k=1}^{\infty}g_{n_k}(z)Y_k+z\left(1-\sum_{k=2}^{\infty}X_k-\sum_{k=1}^{\infty}Y_k\right) \\ =\sum_{k=1}^{\infty}(h_k(z)X_k+g_{n_k}(z)Y_k),\ \text{as required.} \tag{4.7}$$

Using Corollary 4.3 we have $\operatorname{clco}\overline{G}_H(n,\lambda,\alpha,\rho)=\overline{G}_H(n,\lambda,\alpha,\rho)$. Then the statement of Theorem 4.4 is true for $f\in\overline{G}_H(n,\lambda,\alpha,\rho)$. □

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