Hacettepe Journal of Mathematics and Statistics
Volume 41 (1) (2012), 59 – 65

$[0,1]$-FUZZY β-RANK FUNCTIONS[§]

Huan Lian[*] and Xiu Xin[†‡]

Received 02 : 08 : 2010 : Accepted 09 : 08 : 2011

Abstract

In this paper, the concepts of $[0,1]$-fuzzy β-rank function and $[0,1]$-fuzzy α-rank function are presented. The set of all closed and perfect $[0,1]$-matroids (i.e. closed Goetschel-Voxman fuzzy matroids) on E and that of all $[0,1]$-fuzzy β-rank functions on E are in one-to-one correspondence. A $[0,1]$-fuzzy α-rank function on E is equivalent to a $[0,1]$-fuzzy β-rank function on E.

Keywords: $[0,1]$-matroid, Goetschel-Voxman fuzzy matroid, $[0,1]$-fuzzy β-rank function, $[0,1]$-fuzzy α-rank function.

2000 AMS Classification: 03 E 70, 05 B 35, 52 B 40.

1. Introduction

In 1988, R. Goetschel and W. Voxman introduced the concept of fuzzy matroid [1]. Subsequently many scholars researched Goetschel-Voxman fuzzy matroids [2, 4, 5].

In [8], when L is a completely distributive lattice, Shi introduced a new approach to the fuzzication of matroids, namely an L-matroid. L-matroids preserve many basic properties of matroids and can be applied to fuzzy algebras and fuzzy graphs.

In the sequel, we shall consider $L = [0,1]$. A perfect [0,1]-matroid is equivalent to a Goetschel-Voxman fuzzy matroid. In [9], we began an investigation of the $[0,1]$-fuzzy rank functions for $[0,1]$-matroids. A closed and perfect $[0,1]$-matroid can be characterized by means of its $[0,1]$-fuzzy rank function satisfying four fuzzy axioms (LR1)-(LR4).

This paper is a successor of [8] and [9]. It is shown that the $[0,1]$-fuzzy rank function for a closed and perfect $[0,1]$-matroid can be also described via four fuzzy axioms (LR1), (LR2), (LR3) and (LR4)$'$.

[§]The project is supported by the National Natural Science Foundation of China (10971242)

[*]Department of Mathematics, Beijing Institute of Technology, Beijing 100081, P.R. China. E-mail: `lianhuan369@163.com`

[†]Department of Mathematics, Tianjin University of Technology, Tianjin 300384, P.R. China. E-mail: `xinxiu518@163.com`

[‡]Corresponding Author.

2. Preliminaries

Throughout this paper, E is a nonempty finite set and we denote the power set of E by 2^E. For any $X \subseteq E$, $|X|$ denotes the cardinality of X. A fuzzy set A on E is a mapping $A : E \to [0,1]$, the set of all fuzzy sets on E is denoted by $[0,1]^E$. We often do not distinguish a crisp subset X of E from its characteristic function χ_X.

For $A \in [0,1]^E$, $a \in [0,1]$ and $\mathcal{I} \subseteq [0,1]^E$, define

$$A_{[a]} = \{e \in E : A(e) \geq a\}, \qquad A_{(a)} = \{e \in E : A(e) > a\},$$
$$\mathcal{I}[a] = \{A_{[a]} : A \in \mathcal{I}\}, \qquad \mathcal{I}(a) = \{A_{(a)} : A \in \mathcal{I}\}.$$

In [3, 6], some properties of these cut sets can be found as follows:

$$A(x) = \bigvee\{a \in (0,1] : x \in A_{[a]}\} = \bigvee\{a \in [0,1) : x \in A_{(a)}\},$$
$$A_{[a]} = \bigcap_{b<a} A_{[b]} = \bigcap_{b<a} A_{(b)},\ A_{(a)} = \bigcup_{a<b} A_{[b]} = \bigcup_{a<b} A_{(b)}.$$

For $a \in [0,1]$ and $X \subseteq E$, define two fuzzy sets $a \wedge X$ and $a \vee X$ on E as follows:

$$(a \wedge X)(e) = \begin{cases} a, & e \in X, \\ 0, & e \notin X. \end{cases} \quad (a \vee X)(e) = \begin{cases} 1, & e \in X, \\ a, & e \notin X. \end{cases}$$

For $b \in [0,1]$ and $e \in E$, the fuzzy set $b \wedge \{e\}$ is called a fuzzy point and denoted by e_b.

2.1. Definition. [7] Let $\mathbb{N}$ denote the set of all natural numbers. A fuzzy natural number is an antitone mapping $\lambda : \mathbb{N} \to [0,1]$ satisfying

$$\lambda(0) = 1,\ \bigwedge_{n\in\mathbb{N}} \lambda(n) = 0.$$

The set of all fuzzy natural numbers is denoted by $\mathbb{N}([0,1])$.

For any $m \in \mathbb{N}$, we define $\underline{m} \in \mathbb{N}([0,1])$ as follows:

$$\underline{m}(t) = \begin{cases} 1, & \text{if } t \leq m, \\ 0, & \text{if } t \geq m+1. \end{cases}$$

If we do not distinguish m and $\underline{m}$, then $\mathbb{N}$ can be regarded as a subset of $\mathbb{N}([0,1])$.

2.2. Definition. [7] For any $\lambda, \mu \in \mathbb{N}([0,1])$, define the addition $\lambda + \mu$ of λ and μ as follows:

$$\forall n \in \mathbb{N},\ (\lambda + \mu)(n) = \bigvee_{k+l=n} (\lambda(k) \wedge \mu(l)).$$

2.3. Remark. For crisp sets we may consider the correspondence $n \mapsto \{0,1,2,\ldots,n\}$ which associates with each natural number n the subset $\{0,1,2,\ldots,n\}$ of the natural numbers. This correspondence is clearly a bijection between the set of natural numbers and the set of sets of the given form under which n is one less than the cardinality of the set to which it corresponds. In the sequel, we shall not distinguish $\{0,1,\ldots,n\}$ from n.

2.4. Lemma. [7] *Let $\lambda, \mu \in \mathbb{N}([0,1])$. It holds that $(\lambda+\mu)_{(a)} = \lambda_{(a)} + \mu_{(a)}$ for any $a \in [0,1)$.* □

2.5. Definition. [8] Let $A \in [0,1]^E$. Then $|A| \in \mathbb{N}([0,1])$ defined by

$$\forall n \in \mathbb{N},\ |A|(n) = \bigvee\{a \in [0,1] : |A_{[a]}| \geq n\}$$

is called the *fuzzy cardinality* of A.

2.6. Lemma. [8] *Let $A \in [0,1]^E$. It holds that $|A|_{(a)} = |A_{(a)}|$ for any $a \in [0,1)$.* □

2.7. Definition. [8] If $\mathfrak{I} \subseteq [0,1]^E$ satisfies the following conditions:

(LI1) $\mathfrak{I}$ is nonempty;
(LI2) If $A, B \in [0,1]^E$, $A \leq B$ and $B \in \mathfrak{I}$, then $A \in \mathfrak{I}$;
(LI3) If $A, B \in \mathfrak{I}$ and $b = |B|(n) > |A|(n)$ for some $n \in \mathbb{N}$, then there exists $e \in F(A,B)$ such that $(b \wedge A_{[b]}) \vee e_b \in \mathfrak{I}$, where $F(A,B) = \{e \in E : A(e) < b \leq B(e)\}$,

then the pair $(E, \mathfrak{I})$ is called a $[0,1]$-*matroid*.

2.8. Theorem. [8] *If $(E, \mathfrak{I})$ is a $[0,1]$-matroid, then*

(1) *$(E, \mathfrak{I}[a])$ is a matroid for each $a \in (0,1]$.*
(2) *$(E, \mathfrak{I}(a))$ is a matroid for each $a \in [0,1)$.* □

2.9. Theorem. [9] *Let $(E, \mathfrak{I})$ be a $[0,1]$-matroid. Then there is a finite sequence $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ such that*

(1) *If $a_i < a, b < a_{i+1}$, then $\mathfrak{I}[a] = \mathfrak{I}[b]$, $0 \leq i \leq n-1$;*
(2) *If $a_i < a < a_{i+1} < b < a_{i+2}$, then $\mathfrak{I}[a] \supset \mathfrak{I}[b]$, $0 \leq i \leq n-2$.*

The sequence $a_0, a_1, \ldots, a_n$ is called the fundamental sequence *for $(E, \mathfrak{I})$.* □

2.10. Definition. [9]

(1) A $[0,1]$-matroid $(E, \mathfrak{I})$ with the fundamental sequence $a_0, a_1, \ldots, a_n$ is called a *closed* $[0,1]$-*matroid* if whenever $a_{i-1} < a \leq a_i$ $(1 \leq i \leq n)$, then $\mathfrak{I}[a] = \mathfrak{I}[a_i]$.
(2) A $[0,1]$-matroid $(E, \mathfrak{I})$ is called a *perfect* $[0,1]$-*matroid*, if it satisfies the following condition:
(LI4) $\forall A \in [0,1]^E$, if $a \wedge A_{[a]} \in \mathfrak{I}$ for all $a \in (0,1]$, then $A \in \mathfrak{I}$.

2.11. Theorem. [9] *Let $\mathfrak{I} \subseteq [0,1]^E$ satisfy* (LI2) *and* (LI4). *Then the following conditions are equivalent:*

(1) *$(E, \mathfrak{I})$ is a $[0,1]$-matroid;*
(2) *$(E, \mathfrak{I}[a])$ is a matroid for all $a \in (0,1]$.* □

3. $[0,1]$-fuzzy β-rank functions

In this section, the concept of $[0,1]$-fuzzy β-rank functions on E is presented. There is a one-to-one correspondence between the set of all closed and perfect $[0,1]$-matroids (i.e. closed Goetschel-Voxman fuzzy matroids) on E and that of all $[0,1]$-fuzzy β-rank functions on E.

3.1. Definition. [8] Let $(E, \mathfrak{I})$ be a $[0,1]$-matroid. The mapping $R_{\mathfrak{I}} : [0,1]^E \to \mathbb{N}([0,1])$ defined by

$$R_{\mathfrak{I}}(A) = \bigvee \{|B| : B \leq A, B \in \mathfrak{I}\}$$

is called the $[0,1]$-*fuzzy rank function* for $(E, \mathfrak{I})$.

3.2. Remark. Definition 3.1 gives a new definition of fuzzy rank function for Goetschel-Voxman fuzzy matroids (i.e. perfect $[0,1]$-matroids). Based on this, the theory of Goetschel-Voxman fuzzy matroids achieves further development.

3.3. Theorem. [7] *Let $(E, \mathfrak{I})$ be a $[0,1]$-matroid and $R_{\mathfrak{I}}$ the $[0,1]$-fuzzy rank function for $(E, \mathfrak{I})$. $\forall a \in [0,1)$, let $R_{\mathfrak{I}(a)}$ denote the rank function for $(E, \mathfrak{I}(a))$. Then $R_{\mathfrak{I}}(A)_{(a)} = R_{\mathfrak{I}(a)}(A_{(a)})$ for each $A \in [0,1]^E$.* □

3.4. Theorem. *Let $(E, \mathfrak{I})$ be a $[0,1]$-matroid and $R_{\mathfrak{I}}$ the $[0,1]$-fuzzy rank function for $(E, \mathfrak{I})$. Then $R_{\mathfrak{I}}$ satisfies the following conditions: $\forall A, B \in [0,1]^E$, $a \in [0,1)$,*

(LR1) *$\underline{0} \leq R_{\mathfrak{I}}(A) \leq |A|$;*
(LR2) *$A \leq B \Rightarrow R_{\mathfrak{I}}(A) \leq R_{\mathfrak{I}}(B)$;*

(LR3) $R_{\mathcal{I}}(A) + R_{\mathcal{I}}(B) \geq R_{\mathcal{I}}(A \vee B) + R_{\mathcal{I}}(A \wedge B)$;
(LR4)$'$ $R_{\mathcal{I}}(A_{(a)})_{(a)} = R_{\mathcal{I}}(A)_{(a)}$.

Proof. By [8, Theorem 3.14], $R_{\mathcal{I}}$ satisfies (LR1)–(LR3). We only need to check that $R_{\mathcal{I}}$ satisfies (LR4)$'$. For any $A \in [0,1]^E$ and $a \in [0,1)$, by Theorem 3.3,

$$R_{\mathcal{I}}(A)_{(a)} = R_{\mathcal{I}(a)}(A_{(a)}) = R_{\mathcal{I}(a)}((A_{(a)})_{(a)}) = R_{\mathcal{I}}(A_{(a)})_{(a)}. \qquad \square$$

3.5. Definition. A mapping $R : [0,1]^E \to \mathbb{N}([0,1])$ satisfying conditions (LR1)–(LR3) and (LR4)$'$ is called a $[0,1]$-*fuzzy* β-*rank function* on E.

In the following, we will discuss the relation between the set of all $[0,1]$-matroids on E and that of all $[0,1]$-fuzzy β-rank functions on E.

3.6. Lemma. *Let R^β be a $[0,1]$-fuzzy β-rank function on E. Define $R_a^\beta : 2^E \to \mathbb{N}$ as follows:*

$$\forall a \in [0,1),\ R_a^\beta(A) = R^\beta(A)_{(a)}.$$

Then R_a^β satisfies the following conditions (R1)–(R3)*: $\forall\, A, B \in 2^E$,*

(R1) $0 \leq R_a^\beta(A) \leq |A|$;
(R2) $A \subseteq B \Rightarrow R_a^\beta(A) \leq R_a^\beta(B)$;
(R3) $R_a^\beta(A) + R_a^\beta(B) \geq R_a^\beta(A \cup B) + R_a^\beta(A \cap B)$.

Hence there exists a matroid $(E, \mathcal{I}_{R_a^\beta})$ such that R_a^β is the rank function for $(E, \mathcal{I}_{R_a^\beta})$, where $\mathcal{I}_{R_a^\beta} = \{A \in 2^E : R_a^\beta(A) = |A|\}$.

Proof. (R1). By (LR1), $\underline{0} \leq R^\beta(A) \leq |\chi_A|$, then $\{0\} \subseteq R^\beta(A)_{(a)}$, thus $0 \leq R_a^\beta(A)$. Moreover, by Lemma 2.6, we know that

$$n \leq R_a^\beta(A) \Leftrightarrow n \in R^\beta(A)_{(a)} \Rightarrow n \in |\chi_A|_{(a)} \Leftrightarrow n \leq |\chi_{A_{(a)}}| = |A|.$$

Hence $0 \leq R_a^\beta(A) \leq |A|$.

(R2). Since $A \subseteq B$, hence $R^\beta(A) \leq R^\beta(B)$ by (LR2). Then $\forall n \in \mathbb{N}$,

$$n \leq R_a^\beta(A) \Leftrightarrow n \in R^\beta(A)_{(a)} \Rightarrow n \in R^\beta(B)_{(a)} \Leftrightarrow n \leq R_a^\beta(B).$$

This shows $R_a^\beta(A) \leq R_a^\beta(B)$.

(R3). By Lemma 2.4 and (LR3), we have the following implications:

$$\begin{aligned} n \leq R_a^\beta(A \cup B) + R_a^\beta(A \cap B) &\iff n \leq R^\beta(A \cup B)_{(a)} + R^\beta(A \cap B)_{(a)} \\ &\iff n \in (R^\beta(A \cup B) + R^\beta(A \cap B))_{(a)} \\ &\implies n \in (R^\beta(A) + R^\beta(B))_{(a)} \\ &\iff n \leq R^\beta(A)_{(a)} + R^\beta(B)_{(a)} \\ &\iff n \leq R_a^\beta(A) + R_a^\beta(B). \end{aligned}$$

Therefore, $R_a^\beta(A) + R_a^\beta(B) \geq R_a^\beta(A \cup B) + R_a^\beta(A \cap B)$. $\square$

3.7. Lemma. *If $a \in [0,1)$, then $\mathcal{I}_{R_a^\beta} = \bigcup \{\mathcal{I}_{R_b^\beta} : a < b\}$.*

Proof. $\forall a \in [0,1)$, we have

$$\begin{aligned} A \in \mathcal{I}_{R^\beta_a} &\iff R^\beta(A)_{(a)} = R^\beta_a(A) = |A| \\ &\iff \bigcup_{a<b} R^\beta_b(A) = \bigcup_{a<b} R^\beta(A)_{(b)} = |A| \\ &\iff \text{ there exists } b > a \text{ such that } R^\beta_b(A) = |A| \\ &\iff \text{ there exists } b > a \text{ such that } A \in \mathcal{I}_{R^\beta_b}. \end{aligned}$$

This implies that $\mathcal{I}_{R^\beta_a} = \bigcup \{\mathcal{I}_{R^\beta_b} : a < b\}$. □

3.8. Theorem. [7] *Let $\{(E, \mathcal{I}_a) : a \in [0,1)\}$ be a family of matroids. If $\mathcal{I}_a = \bigcup\{\mathcal{I}_b : a < b\}$ for all $a \in [0,1)$, then there exists a fuzzifying matroid $(E, \mathcal{I})$ such that $\mathcal{I}_{(a)} = \mathcal{I}_a$.* □

By the definitions of R^β_a and $\mathcal{I}_{R^\beta_a}$, Lemma 3.7 and Theorem 3.8, we have the following result.

3.9. Lemma. *Let R^β be a $[0,1]$-fuzzy β-rank function on E. Define $\mathcal{I}_{R^\beta} : 2^E \to [0,1]$ as follows:*

$$\forall\, A \in 2^E,\ \mathcal{I}_{R^\beta}(A) = \bigvee\{a \in [0,1) : A \in \mathcal{I}_{R^\beta_a}\}.$$

Then

(1) *$(E, \mathcal{I}_{R^\beta})$ is a fuzzifying matroid.*
(2) *$\forall a \in [0,1)$, $(\mathcal{I}_{R^\beta})_{(a)} = \mathcal{I}_{R^\beta_a}$.* □

3.10. Theorem. *Let R^β be a $[0,1]$-fuzzy β-rank function on E. Define*

$$\mathcal{I}^\beta{}_{R^\beta} = \{A \in [0,1]^E : \forall a \in [0,1), A_{(a)} \in \mathcal{I}_{R^\beta_a}\}.$$

Then $(E, \mathcal{I}^\beta{}_{R^\beta})$ is a closed and perfect $[0,1]$-matroid.

Proof. Obviously, $\mathcal{I}^\beta{}_{R^\beta}$ satisfies (LI1) and (LI2). Let $A \in [0,1]^E$. If $a \wedge A_{[a]} \in \mathcal{I}^\beta{}_{R^\beta}$ for all $0 < a \leq 1$, then for any $b \in [0,1)$, there exists $a > b$ such that $A_{(b)} = A_{[a]}$, hence $A_{(b)} = A_{[a]} = (a \wedge A_{[a]})_{(b)} \in \mathcal{I}_{R^\beta_b}$, thus $A \in \mathcal{I}^\beta{}_{R^\beta}$. This implies that $\mathcal{I}^\beta{}_{R^\beta}$ satisfies (LI4).

$\forall a \in (0,1]$, let $A \in \mathcal{I}^\beta{}_{R^\beta}[a]$. Then $a \wedge A \in \mathcal{I}^\beta{}_{R^\beta}$, thus $\forall b < a$, $A = (a \wedge A)_{(b)} \in \mathcal{I}_{R^\beta_b}$, hence $A \in \bigcap_{b<a} \mathcal{I}_{R^\beta_b}$. This means that $\mathcal{I}^\beta{}_{R^\beta}[a] \subseteq \bigcap_{b<a} \mathcal{I}_{R^\beta_b}$.

Conversely, let $A \in \bigcap_{b<a} \mathcal{I}_{R^\beta_b}$. It is obvious that $(a \wedge A)_{(c)} = A \in \mathcal{I}_{R^\beta_c}$ for each $c < a$ and $(a \wedge A)_{(c)} = \emptyset \in \mathcal{I}_{R^\beta_c}$ for each $c \geq a$, thus $a \wedge A \in \mathcal{I}^\beta{}_{R^\beta}$, hence $A = (a \wedge A)_{[a]} \in \mathcal{I}^\beta{}_{R^\beta}[a]$. This means that $\bigcap_{b<a} \mathcal{I}_{R^\beta_b} \subseteq \mathcal{I}^\beta{}_{R^\beta}[a]$. Therefore, $\mathcal{I}^\beta{}_{R^\beta}[a] = \bigcap_{b<a} \mathcal{I}_{R^\beta_b}$.

By Lemma 3.9, $\mathcal{I}^\beta{}_{R^\beta}[a] = \bigcap_{b<a} \mathcal{I}_{R^\beta_b} = \bigcap_{b<a} (\mathcal{I}_{R^\beta})_{(b)} = (\mathcal{I}_{R^\beta})_{[a]}$. Therefore, $(E, \mathcal{I}^\beta{}_{R^\beta})$ is a closed and perfect $[0,1]$-matroid by Definition 2.10 and Theorem 2.11. □

3.11. Lemma. *Let R^β be a $[0,1]$-fuzzy β-rank function on E. Then*

$$\forall\, a \in [0,1),\ \mathcal{I}^\beta{}_{R^\beta}(a) = \mathcal{I}_{R^\beta_a}.$$

Proof. For each $a \in [0,1)$, let $A_{(a)} \in \mathcal{I}^\beta{}_{R^\beta}(a)$, where $A \in \mathcal{I}^\beta{}_{R^\beta}$. By the definition of $\mathcal{I}^\beta{}_{R^\beta}$, $A_{(a)} \in \mathcal{I}_{R^\beta_a}$. This implies that $\mathcal{I}^\beta{}_{R^\beta}(a) \subseteq \mathcal{I}_{R^\beta_a}$.

Conversely, let $A \in \mathcal{I}_{R_a^\beta}$, then $\bigvee_{a<b} R_b^\beta(A) = \bigcup_{a<b} R^\beta(A)_{(b)} = R^\beta(A)_{(a)} = R_a^\beta(A) = |A|$, thus there exists $b > a$ such that $R_b^\beta(A) = |A|$, i.e. $A \in \mathcal{I}_{R_b^\beta}$. $\forall\, c \in [0,1)$, $(b \wedge A)_{(c)} = A \in \mathcal{I}_{R_b^\beta} \subseteq \mathcal{I}_{R_c^\beta}$ for each $c < b$, $(b \wedge A)_{(c)} = \emptyset \in \mathcal{I}_{R_c^\beta}$ for each $c \geq b$. Thus $b \wedge A \in {\mathcal{I}^\beta}_{R^\beta}$, hence $A = (b \wedge A)_{(a)} \in {\mathcal{I}^\beta}_{R^\beta}(a)$. This implies that $\mathcal{I}_{R_a^\beta} \subseteq {\mathcal{I}^\beta}_{R^\beta}(a)$. Therefore, ${\mathcal{I}^\beta}_{R^\beta}(a) = \mathcal{I}_{R_a^\beta}$ for each $a \in [0,1)$. □

3.12. Theorem. *Let R^β be a $[0,1]$-fuzzy β-rank function on E. Then R^β is the $[0,1]$-fuzzy rank function for $(E, \mathcal{I}^\beta)$, i.e. $R_{{\mathcal{I}^\beta}_{R^\beta}} = R^\beta$.*

Proof. $\forall\, A \in [0,1]^E$, $a \in (0,1]$. By Theorem 3.3, Theorem 3.10, Lemma 3.11 and (LR4)$'$, $R_{{\mathcal{I}^\beta}_{R^\beta}}(A)_{(a)} = R_{{\mathcal{I}^\beta}_{R^\beta}(a)}(A_{(a)}) = R_{\mathcal{I}_{R_a^\beta}}(A_{(a)}) = R_a^\beta(A_{(a)}) = R^\beta(A_{(a)})_{(a)} = R^\beta(A)_{(a)}$. Hence $R_{{\mathcal{I}^\beta}_{R^\beta}}(A) = R^\beta(A)$ for each $A \in [0,1]^E$, thus $R_{{\mathcal{I}^\beta}_{R^\beta}} = R^\beta$. □

3.13. Lemma. *Let R^β be a $[0,1]$-fuzzy β-rank function on E. Then*

$${\mathcal{I}^\beta}_{R^\beta} = \{A \in [0,1]^E : R^\beta(A) = |A|\}.$$

Proof. Let $A \in {\mathcal{I}^\beta}_{R^\beta}$. Then for any $a \in [0,1)$, $A_{(a)} \in \mathcal{I}_{R_a^\beta}$, i.e. $R_a^\beta(A_{(a)}) = |A_{(a)}| = |A|_{(a)}$. By (LR4)$'$, we have $R^\beta(A)_{(a)} = R^\beta(A_{(a)})_{(a)} = R_a^\beta(A_{(a)}) = |A|_{(a)}$. This implies that $R^\beta(A) = |A|$.

Conversely, let $A \in [0,1]^E$ with $R^\beta(A) = |A|$. Then for any $a \in [0,1)$, $R_a^\beta(A_{(a)}) = R^\beta(A_{(a)})_{(a)} = R^\beta(A)_{(a)} = |A|_{(a)} = |A_{(a)}|$, thus $A_{(a)} \in \mathcal{I}_{R_a^\beta}$. This implies that $A \in {\mathcal{I}^\beta}_{R^\beta}$. Therefore, ${\mathcal{I}^\beta}_{R^\beta} = \{A \in [0,1]^E : R^\beta(A) = |A|\}$. □

3.14. Lemma. [9] *Let $(E, \mathcal{I})$ be a closed and perfect $[0,1]$-matroid and $R_\mathcal{I}$ the $[0,1]$-fuzzy rank function for $(E, \mathcal{I})$. Then $A \in \mathcal{I} \Leftrightarrow R_\mathcal{I}(A) = |A|$.* □

3.15. Theorem. *Let $(E, \mathcal{I})$ be a closed and perfect $[0,1]$-matroid, then ${\mathcal{I}^\beta}_{R_\mathcal{I}} = \mathcal{I}$.*

Proof. By Lemma 3.13 and Lemma 3.14, we have

$$A \in {\mathcal{I}^\beta}_{R_\mathcal{I}} \iff R_\mathcal{I}(A) = |A| \iff A \in \mathcal{I}.$$

Therefore, ${\mathcal{I}^\beta}_{R_\mathcal{I}} = \mathcal{I}$. □

By Theorem 3.4, Theorem 3.10, Theorem 3.12 and Theorem 3.15, we have the following result.

3.16. Theorem. *The set of all closed and perfect $[0,1]$-matroids on E and that of all $[0,1]$-fuzzy β-rank functions on E are in one-to-one correspondence.* □

4. $[0,1]$-fuzzy α-rank functions

4.1. Theorem. [9] *Let $(E, \mathcal{I})$ be a $[0,1]$-matroid and $R_\mathcal{I}$ the $[0,1]$-fuzzy rank function for $(E, \mathcal{I})$. Then $R_\mathcal{I}$ satisfies* (LR1)–(LR3) *and the following condition:*

(LR4) $\forall\, A \in [0,1]^E$ *and* $a \in (0,1]$, $R_\mathcal{I}(a \wedge A_{[a]})_{[a]} = R_\mathcal{I}(A)_{[a]}$. □

4.2. Remark. (LR4) $\iff \forall\, A \in [0,1]^E$ and $a \in (0,1]$, $R_\mathcal{I}(A_{[a]})_{[a]} = R_\mathcal{I}(A)_{[a]}$.

4.3. Definition. A mapping $R : [0,1]^E \to \mathbb{N}([0,1])$ satisfying (LR1)–(LR4) is called a *$[0,1]$-fuzzy α-rank function* on E.

4.4. Theorem. *A $[0,1]$-fuzzy α-rank function on E is equivalent to a $[0,1]$-fuzzy β-rank function on E. That is, the four fuzzy axioms* (LR1)–(LR4) *are equivalent to* (LR1)–(LR3) *and* (LR4)$'$.

Proof. Let R^α be a $[0,1]$-fuzzy α-rank function on E. Then $(E, \mathcal{I}^\alpha{}_{R^\alpha})$ is a closed and perfect $[0,1]$-matroid and $R^\alpha = R_{\mathcal{I}^\alpha{}_{R^\alpha}}$ by [9, Theorem 4.12], thus $R^\alpha = R_{\mathcal{I}^\alpha{}_{R^\alpha}}$ satisfies (LR4)$'$ by Theorem 3.4. This implies that R^α is a $[0,1]$-fuzzy β-rank function on E.

Conversely, let R^β be a $[0,1]$-fuzzy β-rank function on E. Then $(E, \mathcal{I}^\beta{}_{R^\beta})$ is a closed and perfect $[0,1]$-matroid by Theorem 3.10, thus $R^\beta = R_{\mathcal{I}^\beta{}_{R^\beta}}$ satisfies (LR4) by Theorem 3.12 and [9, Theorem 4.7]. This implies that R^β is a $[0,1]$-fuzzy α-rank function on E.

Therefore, a $[0,1]$-fuzzy α-rank function on E is equivalent to a $[0,1]$-fuzzy β-rank function on E. □

4.5. Remark. By [9, Theorem 4.15], Theorem 3.16 and Theorem 4.4, the $[0,1]$-fuzzy rank function for a closed and perfect $[0,1]$-matroid is determined by the conditions (LR1)–(LR3) and (LR4)$'$, or (LR1)–(LR4) completely. Therefore, (LR1)–(LR3) and (LR4)$'$ or (LR1)–(LR4) are called the $[0,1]$-fuzzy rank function axioms for a closed and perfect $[0,1]$-matroid (i.e. closed Goetschel-Voxman fuzzy matroid).

References

[1] Goetschel, R. and Voxman, W. *Fuzzy matroids*, Fuzzy Sets and Systems **27**, 291–302, 1988.

[2] Hsueh, I.-C. *On fuzzication of matroids*, Fuzzy Sets and Systems **53**, 319–327, 1993.

[3] Huang, H. L. and Shi, F.-G. *L-fuzzy numbers and their properties*, Information Sciences **178**, 1141–1151, 2008.

[4] Novak, L. A. *On fuzzy independence set systems*, Fuzzy Sets and Systems **91**, 365–374, 1997.

[5] Novak, L. A. *On Goetschel and Voxman fuzzy matroids*, Fuzzy Sets and Systems **117**, 407–412, 2001.

[6] Shi, F.-G. *L-fuzzy relation and L-fuzzy subgroup*, Journal of Fuzzy Mathematics **8**, 491–499, 2000.

[7] Shi, F.-G. *A new approach to the fuzzification of matroids*, Fuzzy Sets and Systems **160**, 696–705, 2009.

[8] Shi, F.-G. (L, M)-*fuzzy matroids*, Fuzzy Sets and Systems **160**, 2387–2400, 2009.

[9] Xin, X. and Shi, F.-G. *Rank functions for closed and perfect* $[0,1]$*-matroids*, Hacet. J. Math. Stat. **39** (1), 31–39, 2010.