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ON THE DEDEKIND SUMS AND THE QUADRATIC GAUSS SUMS[‡]

Wang Tingting[*†] and Zhang Wenpeng[*]

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Abstract

In this paper, we use the analytic method and properties of the Gauss sums to study one kind of mean value computational problem involving the Dedekind sums and the quadratic Gauss sums, and give an exact computational formula and asymptotic formula for it.

Keywords: Dedekind sums, Quadratic Gauss sums, Mean value, Computational formula, Asymptotic formula.

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1. Introduction

Let q be a natural number and h an integer prime to q. The famous Dedekind sums $S(h,q)$ are defined as

$$S(h,q) = \sum_{a=1}^{q} \left(\left(\frac{a}{q}\right)\right) \left(\left(\frac{ah}{q}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if x is not an integer;} \\ 0, & \text{if x is an integer.} \end{cases}$$

Several authors have studied various properties of $S(h,q)$, and obtained many important conclusions, see [1]–[7]. For example, L. Carlitz [2] obtained a reciprocity theorem of $S(h,k)$. That is, for any positive integers h and q with $(h,q) = 1$, we have the identity

$$S(h,q) + S(q,h) = \frac{h^2 + k^2 + 1}{12hq} - \frac{1}{4}.$$

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[*]Department of Mathematics, Northwest University, Xi'an, Shaanxi, P. R. China.
E-mail: (W. Tingting) tingtingwang126@126.com (Z. Wenpeng) wpzhang@nwu.edu.com
[†]Corresponding Author.

Tom M. Apostol [1] gave many elementary properties for $S(h,q)$; the second author [7] proved the asymptotic formula

$$\sum_{h=1}^{q}{}' |S(h,q)|^2 = \frac{5}{144} q\phi(q) \cdot \frac{\prod_{p^\alpha \| q} \left(\left(1+\frac{1}{p}\right)^2 - \frac{1}{p^{3\alpha+1}} \right)}{\prod_{p|q} \left(1+\frac{1}{p}+\frac{1}{p^2}\right)} + O\left(q \exp\left(\frac{4\ln q}{\ln\ln q} \right) \right),$$

where $p^\alpha \| q$ denotes that $p^\alpha | q$ and $p^{\alpha+1} \dagger q$.

In this paper, we consider the computational problem for the mean value

$$(1) \qquad \sum_{c=1}^{q}{}' G^2(c^2-1,q) S(c^2,q),$$

where $G(c,q)$ denotes the quadratic Gauss sums defined by $G(c,q) = \sum_{a=1}^{q}{}' e\left(\frac{ca^2}{q}\right)$, $e(y) = e^{2\pi i y}$, $\sum_{a}{}'$ denotes the summation over all a such that $(a,q)=1$.

Regarding the mean value (1), it seems that no one had studied this yet, at least we have not seen any related results. In this paper, we use the analytic method and properties of the Gauss sums to study the computational problem for (1), and give an exact computational formula and asymptotic formula for it. That is, we shall prove the following two conclusions:

1.1. Theorem. *For any prime $p \equiv 3 \bmod 4$, we have the identity*

$$\sum_{c=1}^{p-1} G^2(c^2-1,p) S(c^2,p) = \frac{(p-1)^2(p-2)}{6} - \frac{p(p-1)}{\pi^2} |L(1,\chi_2)|^2$$
$$= \frac{(p-1)^2(p-2)}{6} - (p-1) \cdot h_p^2,$$

where $\chi_2(n) = \left(\frac{n}{p}\right)$ denotes the Legendre symbol, $L(1,\chi)$ denotes the Dirichlet L-function corresponding to the character $\chi \bmod p$, and h_p denotes the class number of the quadratic field $\mathbf{Q}(\sqrt{-p})$.

1.2. Theorem. *For any prime $p \equiv 1 \bmod 4$, we have the asymptotic formula*

$$\sum_{c=1}^{p-1} G^2(c^2-1,p) S(c^2,p) = \frac{(p-1)(p-2)(p-3)}{6} + O\left(p^{\frac{3}{2}+\epsilon}\right),$$

where $\epsilon > 0$ denotes any fixed positive number.

1.3. Note. For a general integer $q \geq 3$ it is an open problem whether or not there exists an exact computational formula for $\sum_{c=1}^{q}{}' G^2(c^2-1,q) S(c^2,q)$.

2. Several lemmas

In this section, we shall give two Lemmas, which are necessary in the proof of our Theorems. First we have the following:

2.1. Lemma. *Let $q > 2$ be an integer. Then for any integer a with $(a, q) = 1$, we have the identity*

$$S(a, q) = \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a) |L(1, \chi)|^2,$$

where $\sum\limits_{\substack{\chi \bmod d \\ \chi(-1)=-1}}$ *denotes summation over all odd characters* mod d *and* $L(s, \chi)$ *denotes the Dirichlet L-function corresponding to* χ mod d.

Proof. See [9, Lemma 2]. □

2.2. Lemma. *Let q be a positive integer with $q \geq 3$ and let χ be the Dirichlet character* mod q. *Then we have the estimate*

$$\sum_{a=1}^{q} \left| \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(a) \, |L(1, \chi)|^2 \right| = O\left(q^{1+\epsilon}\right),$$

where $\epsilon > 0$ denotes any fixed positive number.

Proof. See [8, Lemma 5]. □

2.3. Lemma. *Let p be an odd prime. Then we have the identity*

$$\sum_{a=1}^{p-1} \left(\frac{a^2+1}{p} \right) = -2,$$

where $\left(\frac{n}{p}\right)$ *denotes the Legendre symbol.*

Proof. Let a is an integer with $(a, p) = 1$. It is clear that $1 + \left(\frac{a}{p}\right) = 2$, if a is a quadratic residue mod p; $1 + \left(\frac{a}{p}\right) = 0$, if a is not a quadratic residue mod p. Note that $\sum\limits_{a=1}^{p} \left(\frac{a}{p}\right) = 0$, so we have

$$\begin{aligned}
\sum_{a=1}^{p-1} \left(\frac{a^2+1}{p} \right) &= \sum_{a=1}^{p-1} \left[1 + \left(\frac{a}{p} \right) \right] \left(\frac{a+1}{p} \right) \\
&= \sum_{a=1}^{p-1} \left(\frac{a+1}{p} \right) + \sum_{a=1}^{p-1} \left(\frac{a(a+1)}{p} \right) \\
&= \sum_{a=1}^{p} \left(\frac{a+1}{p} \right) + \sum_{a=1}^{p-1} \left(\frac{\overline{a}^2}{p} \right) \left(\frac{a(a+1)}{p} \right) - \left(\frac{1}{p} \right) \\
&= 0 + \sum_{a=1}^{p-1} \left(\frac{1+\overline{a}}{p} \right) - 1 \\
&= \sum_{a=1}^{p} \left(\frac{a+1}{p} \right) - 2 \\
&= -2.
\end{aligned}$$

This proves Lemma 2.3. □

2.4. Lemma. *Let p be an odd prime. Then we have the identity*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{12}\frac{(p-1)^2(p-2)}{p^2},$$

where $\sum\limits_{\substack{\chi \bmod p \\ \chi(-1)=-1}}$ *denotes summation over all* $\chi \bmod p$ *such that* $\chi(-1) = -1$.

Proof. See [9, Theorem A]. □

3. Proof of the theorems

In this section, we shall use the Lemmas from Section 2 to complete the proof of our Theorems.

Proof of Theorem 1.1. For any p with $p \equiv 3 \bmod 4$ and $\chi \bmod p$, from Lemma 2.1 and the definition and properties of the Gauss sum $\tau(\chi) = \sum\limits_{a=1}^{p-1} \chi(a)\, e\left(\frac{a}{p}\right)$ we have

$$\begin{aligned}
&\sum_{c=1}^{p-1} G^2(c^2-1,p)S(c^2,p) \\
&= \frac{1}{\pi^2}\frac{p}{p-1} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \sum_{c=1}^{p-1} G^2(c^2-1,p)\chi(c^2)\cdot |L(1,\chi)|^2 \\
&= \frac{1}{\pi^2}\frac{p}{p-1} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \sum_{c=1}^{p-1}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(c^2) e\left(\frac{(a^2+b^2)(c^2-1)}{p}\right) |L(1,\chi)|^2 \\
&= \frac{1}{\pi^2}\frac{p}{p-1} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \sum_{c=1}^{p-1}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \overline{\chi}(b^2)\chi(b^2c^2) e\left(\frac{(a^2+1)b^2(c^2-1)}{p}\right) |L(1,\chi)|^2 \\
&= \frac{1}{\pi^2}\frac{p}{p-1} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \sum_{a=1}^{p-1}\left|\sum_{c=1}^{p-1} \chi(c^2) e\left(\frac{(a^2+1)c^2}{p}\right)\right|^2 \cdot |L(1,\chi)|^2. \qquad (2)
\end{aligned}$$

Let χ_2 denotes the quadratic character mod p, that is, χ_2 is the Legendre symbol. Then from the properties of characters we have

$$\begin{aligned}
\left|\sum_{c=1}^{p-1} \chi(c^2) e\left(\frac{(a^2+1)c^2}{p}\right)\right|^2 &= \left|\sum_{c=1}^{p-1} (1+\chi_2(c))\,\chi(c)\, e\left(\frac{c(a^2+1)}{p}\right)\right|^2 \\
&= \left|\overline{\chi}(a^2+1)\tau(\chi) + \overline{\chi}(a^2+1)\chi_2(a^2+1)\tau(\chi\chi_2)\right|^2 \\
&= |\overline{\chi}(a^2+1)|^2 \cdot \left|\tau(\chi) + \chi_2(a^2+1)\tau(\chi\chi_2)\right|^2. \qquad (3)
\end{aligned}$$

If $p \equiv 3 \bmod 4$, then $p \dagger a^2+1$ for all $1 \le a \le p-1$. So $|\overline{\chi}(a^2+1)|^2 = 1$. If $\chi(-1) = -1$ and $\chi \ne \chi_2$, then χ and $\chi\chi_2$ are two primitive characters mod p and $|\tau(\chi)| = |\tau(\chi\chi_2)| = \sqrt{p}$, so from (3) we have

$$
\begin{aligned}
&\left|\sum_{c=1}^{p-1}\chi(c^2)e\left(\frac{c^2(a^2+1)}{p}\right)\right|^2\\
&\qquad = \left|\tau\left(\chi\right)+\chi_2(a^2+1)\tau\left(\chi\chi_2\right)\right|^2\\
&\qquad = \left|\tau\left(\chi\right)\right|^2+\left|\tau\left(\chi\chi_2\right)\right|^2+\chi_2(a^2+1)\left(\overline{\tau\left(\chi\right)}\tau\left(\chi\chi_2\right)+\tau\left(\chi\right)\overline{\tau\left(\chi\chi_2\right)}\right)\\
&\qquad = 2p+\chi_2(a^2+1)\left(\overline{\tau\left(\chi\right)}\tau\left(\chi\chi_2\right)+\tau\left(\chi\right)\overline{\tau\left(\chi\chi_2\right)}\right). \qquad (4)
\end{aligned}
$$

If $\chi=\chi_2$, then $\chi\chi_2$ is the principal character mod p and $\tau(\chi\chi_2)=-1$. Note that $\tau(\chi_2)=i\sqrt{p}$, so from (3) we have

$$
\left|\sum_{c=1}^{p-1}\chi(c^2)e\left(\frac{c^2(a^2+1)}{p}\right)\right|^2=p+1. \tag{5}
$$

If $\chi(-1)=-1$ and $\chi\neq\chi_2$, then

$$
\overline{\tau\left(\overline{\chi}\right)}=\chi(-1)\tau(\chi)=-\tau(\chi)
$$

and

$$
\tau\left(\overline{\chi}\chi_2\right)=\overline{\chi(-1)\chi_2(-1)\tau(\chi\chi_2)}=\overline{\tau(\chi\chi_2)},
$$

$|L(1,\chi)|^2=|L(1,\overline{\chi})|^2$. So from (2), (3), (4), (5), Lemma 2.1, Lemma 2.3 and Lemma 2.4 we have

$$
\begin{aligned}
&\sum_{c=1}^{p-1} G^2(c^2-1,p)S(c^2,p)\\
&\quad=\frac{1}{\pi^2}\frac{p}{p-1}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\sum_{a=1}^{p-1}\left|\sum_{c=1}^{p-1}\chi(c^2)e\left(\frac{(a^2+1)c^2}{p}\right)\right|^2\cdot|L(1,\chi)|^2\\
&\quad=\frac{1}{\pi^2}\frac{p(p+1)}{p-1}(p-1)\cdot|L(1,\chi_2)|^2+\frac{1}{\pi^2}\frac{2p^2(p-1)}{p-1}\sum_{\substack{\chi\neq\chi_2\\ \chi(-1)=-1}}|L(1,\chi)|^2\\
&\qquad+\frac{p}{\pi^2(p-1)}\sum_{\substack{\chi\neq\chi_2\\ \chi(-1)=-1}}\sum_{a=1}^{p-1}\chi_2(a^2+1)\left(\overline{\tau\left(\chi\right)}\tau\left(\chi\chi_2\right)+\tau\left(\chi\right)\overline{\tau\left(\chi\chi_2\right)}\right)\\
&\qquad\qquad\cdot|L(1,\chi)|^2\\
&\quad=\frac{p(1-p)}{\pi^2}|L(1,\chi_2)|^2+\frac{2p^2}{\pi^2}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2\\
&\qquad-\frac{2p}{\pi^2(p-1)}\sum_{\substack{\chi\neq\chi_2\\ \chi(-1)=-1}}\left(\overline{\tau\left(\overline{\chi}\right)}\tau\left(\overline{\chi}\chi_2\right)+\tau\left(\chi\right)\overline{\tau\left(\chi\chi_2\right)}\right)\cdot|L(1,\chi)|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{p(1-p)}{\pi^2}|L(1,\chi_2)|^2 + \frac{2p^2}{\pi^2} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 \\
&\quad - \frac{2p}{\pi^2(p-1)} \sum_{\substack{\chi\neq\chi_2 \\ \chi(-1)=-1}} \left(-\tau(\chi)\overline{\tau(\chi\chi_2)} + \tau(\chi)\overline{\tau(\chi\chi_2)}\right)\cdot|L(1,\chi)|^2 \\
&= \frac{p(1-p)}{\pi^2}|L(1,\chi_2)|^2 + \frac{2p^2}{\pi^2} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 \\
&= \frac{(p-1)^2(p-2)}{6} - \frac{p(p-1)}{\pi^2}|L(1,\chi_2)|^2 \\
&= \frac{(p-1)^2(p-2)}{6} - (p-1)\cdot h_p^2.
\end{aligned}
$$

This proves Theorem 1.1. □

Proof of Theorem 1.2. For any prime $p \equiv 1 \bmod 4$, note that the Legendre symbol χ_2 satisfies $\chi_2(-1) = 1$, $\tau(\chi_2) = \sqrt{p}$, so

$$
\begin{aligned}
\tau(\chi)\overline{\tau(\chi\chi_2)} &= \sum_{b=1}^{p-1}\sum_{c=1}^{p-1} \chi(b)\overline{\chi}(c)\chi_2(c) e\left(\frac{b+c}{p}\right) \\
&= \sum_{b=1}^{p-1}\sum_{c=1}^{p-1} \chi(b)\chi_2(c) e\left(\frac{c(b+1)}{p}\right) \\
&= \tau(\chi_2)\sum_{b=1}^{p-1} \chi(b)\chi_2(b+1) \\
&= \sqrt{p}\sum_{b=1}^{p-1} \chi(b)\chi_2(b+1),
\end{aligned}
$$

and

$$\chi(a^2+1) = 0 \text{ if } p \mid a^2+1.$$

So from (3), Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4 we have

$$
\begin{aligned}
&\sum_{c=1}^{p-1} G^2(c^2-1,p)S(c^2,p) \\
&\quad = \frac{1}{\pi^2}\frac{p}{p-1} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \sum_{a=1}^{p-1} \left|\sum_{c=1}^{p-1}\chi(c^2) e\left(\frac{(a^2+1)c^2}{p}\right)\right|^2 \cdot |L(1,\chi)|^2 \\
&\quad = \frac{2p^2}{\pi^2(p-1)}(p-3) \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 + \frac{p}{\pi^2(p-1)} \\
&\qquad \cdot \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \sum_{a=1}^{p-1} \chi_2(a^2+1)\left(\overline{\tau(\chi)}\tau(\chi\chi_2) + \tau(\chi)\overline{\tau(\chi\chi_2)}\right)\cdot|L(1,\chi)|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2p^2}{\pi^2}\frac{\pi^2}{12}\frac{(p-1)(p-2)(p-3)}{p^2} \\
&\quad - \frac{1}{\pi^2}\frac{4p^{\frac{3}{2}}}{p-1}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\sum_{b=1}^{p-1}\chi_2(b+1)\chi(b)\left|L(1,\chi)\right|^2 \\
&= \frac{(p-1)(p-2)(p-3)}{6} - \frac{1}{\pi^2}\frac{4p^{\frac{3}{2}}}{p-1}\sum_{b=1}^{p-1}\chi_2(b+1)\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(b)\left|L(1,\chi)\right|^2 \\
&= \frac{(p-1)(p-2)(p-3)}{6} + O\left(\sqrt{p}\sum_{b=1}^{p-1}\left|\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(b)\left|L(1,\chi)\right|^2\right|\right) \\
&= \frac{(p-1)(p-2)(p-3)}{6} + O\left(p^{\frac{3}{2}+\epsilon}\right).
\end{aligned}
$$

This completes the proof of Theorem 1.2. □

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