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A SHORT NOTE ON THE ROLE OF GRILLS IN NEARNESS FRAMES

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Abstract

Grills and clusters have featured quite significantly in the theory of nearness spaces (see for example H. Herrlich (*Topological structures*, Topological Structures 1. Math. Centre Tracts, 59–122, 1974) and G. Preuss, (*Theory of topological structures: An approach to categorical topology* (D. Reidel Pub. Co., 1987)). In this paper we consider the role they play in nearness frames. Some of the results we establish here are that, in a separated Boolean nearness frame, a near grill is contained in a unique cluster, and that, in quotient-fine nearness frames with spatial completion, near subsets are contained in near grills.

Keywords: Nearness frame, Prime filter, Near subsets, Grill, Cluster.

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1. Introduction

Grills were introduced in 1947 by G. Choquet [5] in connection with filters, where it became apparent that they are duals of filters. In 1973, W.J. Thron [12] carried out an extensive study of proximity structures with grills playing a central role. Some of the preliminary results appearing there are that grills are unions of the ultrafilters they contain, and that ultrafilters are grills. In the pointfree context, that grills are unions of the prime filters they contain becomes immediate.

In [11], G. Preuss carries out a study of grill-determined prenearness spaces, i.e. those prenearness spaces in which each near collection of subsets is contained in a near grill. In this paper we show that, in quotient-fine nearness frames with spatial completion, near subsets are contained in near grills.

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2. Preliminaries

Recall that a *frame* is a complete lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge x \mid x \in S\}$$

holds for all $a \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by 1 and 0 respectively. The morphisms between frames, known as *frame homomorphisms*, are those which preserve finite meets, including the top element, and arbitrary joins, including the bottom element. A homomorphism is called *dense* if the only element it maps to the bottom element is the bottom element. Associated with a homomorphism $h : L \to M$ is its *right adjoint* $h_* : M \to L$ given by

$$h_*(a) = \bigvee \{x \in L \mid h(x) \leq a\}.$$

For a general theory of frames we refer the reader to [9]. The frame of open subsets of a topological space X is denoted by $\mathfrak{O}X$.

An element a of L is *rather below* an element b, written $a \prec b$, if there is an element s such that $a \wedge s = 0$ and $s \vee b = 1$. L is then said to be *regular* if $a = \bigvee \{x \in L \mid x \prec a\}$ for each $a \in L$.

The *pseudocomplement* of an element a is the element $a^* = \bigvee \{x \in L \mid x \wedge a = 0\}$. We say a is a *regular element* in a frame L if $a^{**} = a$. A *Boolean frame* is one where each element is regular.

A *point* of a frame L is an element p such that $p \neq 1$ and $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$. The points of any regular frame are precisely those elements which are maximal below the top. A frame *has enough points* if every element is a meet of points above it. Every compact regular frame has enough points. Frames that have enough points are also said to be *spatial*.

By a *filter* of a frame L, we mean a proper upset of L which is closed under finite meets. An *ideal* is defined dually. A filter is *prime* if it contains at least one element of any two elements whose join it contains.

We say that $A \subseteq L$ is a *cover* of a frame L if $\bigvee A = 1$. The set of all covers of L is denoted by $\operatorname{Cov} L$. For covers A and B of L, A is said to *refine* B, written $A \leq B$, if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. For any $A \in \operatorname{Cov} L$ and $x \in L$, the element Ax of L is defined by $Ax = \bigvee \{a \in A \mid a \wedge x \neq 0\}$.

For any $\mathfrak{N} \subseteq \operatorname{Cov} L$, the relation $\lhd_{\mathfrak{N}}$ (or simply $\lhd$) on L is defined by

$$x \lhd y \text{ if } Cx \leq y \text{ for some } C \in \mathfrak{N},$$

and $\mathfrak{N}$ is said to be *admissible* if $a = \bigvee \{x \in L \mid x \lhd a\}$ for each $a \in L$. A *nearness* on L is an admissible filter $\mathfrak{N}$ in $(\operatorname{Cov} L, \leq)$. A *nearness frame* is a pair $(L, \mathfrak{N})$ where $\mathfrak{N}$ is a nearness on L. A frame has a nearness if and only if it is regular. Therefore all frames considered here are assumed to be regular. For general reference to the theory of nearness frames see [1, 2] and [3].

One frequently abuses notation and simply refers to L as a nearness frame. In such cases one writes $\mathfrak{N}L$ for the nearness.

Given a nearness frame L, the covers in $\mathfrak{N}L$ are called *uniform covers* of L. If L is a nearness frame and $C \in \operatorname{Cov} L$, then $\check{C}$ is the cover defined by

$$\check{C} = \{x \in L \mid x \lhd c \text{ for some } c \in C\}.$$

Let L and M be nearness frames. A homomorphism $h\colon L \to M$ between the respective underlying frames is said to be:

(1) *Uniform* if $h[C] \in \mathfrak{N}M$ for each $C \in \mathfrak{N}L$.

(2) A *surjection* or *quotient map* if it is onto and $\mathfrak{N}M = \{h[C] \mid C \in \mathfrak{N}L\}$. In this case we shall refer to the nearness frame M as a quotient of L.
(3) A *strict surjection* if it is a dense surjection and the uniform covers $h_*[C], C \in \mathfrak{N}M$, generate $\mathfrak{N}L$.

The category of nearness frames and uniform homomorphisms is denoted by **NFrm**.

A nearness frame L is said to be:

(1) *Boolean* if the underlying frame L is Boolean.
(2) *Strong* if for every uniform cover C, $\check{C}$ is also a uniform cover.
(3) *Fine* if $\mathfrak{N}L = \mathrm{Cov}\, L$.
(4) *Quotient-fine* if it is the quotient of a fine nearness frame.
(5) *Complete* if any strict surjection $M \to L$ is an isomorphism.

A completion of a nearness frame L is any strict surjection $M \to L$ with complete M. Fine nearness frames are complete. Any nearness frame L has a completion $\gamma_L : CL \to L$, which is unique up to isomorphism.

In a nearness frame L, a *near subset* $A \subseteq L$ has the property that every uniform cover of L has an element which meets every element of A. It is shown in [6] that A is a near subset of L if and only if the set of pseudocomplements $A^* = \{a^* \mid a \in A\}$ is not a uniform cover of L. We say that $C \subseteq L$ is a *cluster* if it is a maximal near subset.

Given a nearness frame L, and $A \subseteq L$, write

$$\sec A = \{x \in L \mid x \wedge a \neq 0,\ \forall\, a \in A \setminus \{0\}\}.$$

Then A is said to be *semi-Cauchy* if $\sec A$ is a near subset of L.

A nearness frame L is said to be *separated* if whenever a subset $A \subseteq L$ is both near and semi-Cauchy, then the set $\{s \in L \mid A \cup \{s\} \text{ is near}\}$ is near. Strong nearness frames are separated, as shown in [6].

A *grill* $G \subseteq L$ is an upset such that $0 \notin G$ and if $a \vee b \in G$, then $a \in G$ or $b \in G$. Thus, a grill is a complement of an ideal, and a prime filter is a grill. Further, grills are known to be dual to filters, and it is shown in [8] that if G is a uniform grill, then $\sec G \subseteq G$.

3. Grills, clusters, and near subsets

In our discussion we take special interest in establishing the interconnections between the notions *grill, cluster* and *near* subset in the context of nearness frames.

In the theory of nearness spaces, uniformly continuous maps can be characterized by near subsets namely: a function $f : X \longrightarrow Y$ between nearness spaces is uniformly continuous iff for every near subcollection $\mathcal{A} \subseteq \mathcal{P}X$, the collection $\{f[A] \mid A \in \mathcal{A}\}$ is near in Y. We begin by showing that the property of being a near subset characterizes uniform frame homomorphisms only in certain instances as specified by the following result.

3.1. Proposition. *If M is a strong nearness frame, L an arbitrary nearness frame, and $h : M \longrightarrow L$ a dense onto frame homomorphism, then h is uniform iff h_* preserves near subsets.*

Proof. Suppose the hypothesis holds, with h being a uniform frame homomorphism. Let A be a near subset of L. We show that $h_*[A]$ is a near subset of M. If $C \in \mathfrak{N}M$, then $h[C] \in \mathfrak{N}L$. So there exists $c \in C$ such that $h(c) \wedge a \neq 0$ for each $a \in A$, since A is near. This implies $c \wedge h_*(a) \neq 0$ since h is onto. [To see this, suppose $c \wedge h_*(a) = 0$. Then $0 = h(c \wedge h_*(a)) = h(c) \wedge hh_*(a) = h(c) \wedge a$, giving a contradiction]. Therefore $h_*[A]$ is near.

Conversely, suppose h_* preserves near subsets. We show that h is uniform. Let $C \in \mathfrak{N}M$, and suppose on the contrary $h[C] \notin \mathfrak{N}L$. Since M is strong, we have that

$$\check{C} = \{x \in M \mid \exists c \in C,\ x \lhd c\} \in \mathfrak{N}M.$$

Now for any $x \in \check{C}$, $x^{**} \leq c$ for some $c \in C$ (since $x \lhd c$). Thus, in line with our supposition, $h[\check{C}^{**}] \notin \mathfrak{N}L$. Since h is dense onto, $h[\check{C}^{**}] = h[\check{C}^*]^* \notin \mathfrak{N}L$ and hence $h[\check{C}^*]$ is near. So, by the hypothesis $h_*h[\check{C}^*]$ is near.

Since $\check{C} \in \mathfrak{N}M$, we can find $x \in \check{C}$ which meets with every element of $h_*h[\check{C}^*]$. But $h_*(h(x)^*)$ is an element of $h_*h[\check{C}^*]$ and

$$h(x \wedge h_*(h(x)^*)) = h(x \wedge h_*h(x^*)) = h(x) \wedge h(x^*) = 0,$$

which implies $x \wedge h_*(h(x)^*) = 0$ by denseness. So we have a contradiction. Hence the desired result holds. □

3.2. Proposition. *Every cluster is a grill.*

Proof. Let L be a nearness frame and $C \subseteq L$ a cluster. Now, since C is near, we have $0 \notin C$. Suppose $a \in C$ or $b \in C$. Then $C \cup \{a \vee b\}$ is near, and consequently $a \vee b \in C$, since C is a maximal near subset.

On the other hand, suppose $a \vee b \in C$ with $a \notin C$ and $b \notin C$. Then since C is a cluster, $C \cup \{a\}$ and $C \cup \{b\}$ are not near and so both $D = C^* \cup \{a^*\}$ and $E = C^* \cup \{b^*\}$ are uniform covers of L. But $D \wedge E \leq C^* \cup \{a^* \wedge b^*\}$ since $a^* \wedge b^* = (a \vee b)^* \in C^*$. So $C^* \in \mathfrak{N}L$, implying that C is not near. This is a contradiction. Hence the result holds. □

The following characterization of grills is immediate, when one invokes the dual version of Stone's Separation Lemma [7, Theorem 15].

3.3. Lemma. *A nonempty subset G of a frame L is a grill if and only if it is the union of the prime filters it contains.* □

3.4. Proposition. *In any nearness frame L, every near subset is contained in a grill.*

Proof. Let A be a near subset of L and C a uniform cover of L. Choose $c \in C$ such that $c \wedge a \neq 0$ for all $a \in A$. This implies $a \not\leq c^*$ for each $a \in A$. [Note that if $a \leq c^*$, then $c \wedge a \leq c \wedge c^* = 0$, so that $c \wedge a = 0$ which gives a contradiction]. So we have $A \subseteq L \setminus \downarrow c^*$. Now $\downarrow c^*$ is an ideal, so that its complement $L \setminus \downarrow c^*$ is a grill. □

Our next result identifies separated nearness frames among the arbitrary ones.

3.5. Proposition. *If L is a nearness frame in which every near subset is contained in a unique cluster, then L is separated.*

Proof. Given the hypothesis, let $A \subseteq L$ be near and semi-Cauchy, and let C be the unique cluster with $A \subseteq C$. Put $S = \{s \in L \mid A \cup \{s\} \text{ is near}\}$. For each $s \in S$, let C_s be the unique cluster such that $A \cup \{s\} \subseteq C_s$. Then $A \subseteq C_s$ for each $s \in S$. So $S \subseteq C$. Since C is near, we have that S is near, and therefore L is separated. □

3.6. Remark. We note that if L is a nearness frame, $A \subseteq L$ is near and

$$C = \{c \in L \mid A \cup \{c\} \text{ is near}\}$$

is near, then C is the unique cluster containing A. To see this let $B \supseteq A$ be near. Then for each $b \in B$, we have $A \cup \{b\} \subseteq B$ and so $A \cup \{b\}$ is near. In that case $b \in C$, so that $B \subseteq C$.

3.7. Proposition. *If L is a Boolean separated nearness frame, then every near grill in L is contained in a unique cluster.*

Proof. Let $G \subseteq L$ be a near grill. We first show that G is semi-Cauchy, and then use the above remark to draw our conclusion. So we begin by showing that $\sec G$ is near.

Suppose on the contrary that $\sec G$ is not near. Then the set $\{a^* \mid a \in \sec G\} \in \mathfrak{N}L$, and so there exists $b \in \sec G$ such that $b^* \wedge x \neq 0$ for each $x \in G$, since G is near. But for each $x \in G$, we have $x = (x \wedge b) \vee (x \wedge b^*)$, since L is Boolean. Since G is a grill, we should have $x \wedge b \in G$ or $x \wedge b^* \in G$. But we cannot have $x \wedge b \in G$ since $b^* \wedge (x \wedge b) = 0$. So $x \wedge b^* \in G$; which contradicts the fact that $b \in \sec G$, as $b \wedge (x \wedge b^*) = 0$. Therefore $\sec G$ is near, so that G is semi-Cauchy.

Since L is separated, the set $C = \{c \in L \mid G \cup \{c\} \text{ is near}\}$ is near. So, by Remark 3.6, C is the unique cluster containing G. □

Our next result shows that clusters are preserved by dense surjections. We shall need the following result appearing in [6].

3.8. Lemma. *A surjection $h : M \to L$ is dense iff for every near subset A of M, $h[A]$ is a near subset of L.* □

3.9. Proposition. *Let $h : M \longrightarrow L$ be a dense surjection. If $C \subseteq M$ is a cluster, then $h[C]$ is a cluster in L.*

Proof. Suppose $C \subseteq M$ is a cluster. Since C is near, we have, by Lemma 3.8, that $h[C]$ is near. Now suppose $h[C] \subseteq D$ for some $D \subseteq L$ which is near. We show that $D \subseteq h[C]$, which will show that $h[C]$ is a maximal near subset. Let $d \in D$, and choose $b \in M$ such that $h(b) = d$. Let $A \in \mathfrak{N}M$. Then $h[A] \in \mathfrak{N}L$. Since D is near, there exists $a \in A$ such that $h(a) \wedge x \neq 0$ for each $x \in D$. In particular, $0 \neq h(a) \wedge h(b) = h(a \wedge b)$, which implies $a \wedge b \neq 0$.

Consider any element $c \in C$. Since $h[C] \subseteq D$, $h(a) \wedge h(c) \neq 0$, which implies $a \wedge c \neq 0$. Thus, a meets every element of $C \cup \{b\}$, which implies that $C \cup \{b\}$ is near, and therefore $b \in C$ as C is a maximal near subset. Thus, $d = h(b) \in h[C]$. Hence $h[C] = D$, as required. □

If denseness is dropped in the above proposition, then $h[C]$ can fail to be a cluster mainly because $h[C]$ need not be near when C is near, as shown by the following example.

3.10. Example. Let $\mathbf{4} = \{0, a, a^*, 1\}$ be the Boolean algebra of four elements and $\mathbf{2}$ the two-element chain. Regard these frames as fine nearness frames. Let $h : \mathbf{4} \longrightarrow \mathbf{2}$ be the frame homomorphism given by

$$0 \mapsto 0,\ a \mapsto 0,\ a^* \mapsto 1,\ 1 \mapsto 1.$$

Then h is a nondense surjection. The set $C = \{a, 1\}$ is a cluster in $\mathbf{4}$ for which $h[C]$ is not near, and therefore not a cluster.

In regular nearness spaces there is a characterization of the subtopological ones that says a nearness space is subtopological if and only if every near collection of subsets is contained in a near grill (see [4] for definitions and the cited result). A close scrutiny of the validating arguments suggests that what makes the characterization valid is that, talking frame-theoretically, the frame $\mathfrak{P}(X)$ of subsets of X is Boolean, and furthermore the near collections are allowed to contain any type of subset and not just the open ones.

We shall show shortly that if a nearness frame is quotient-fine and its completion has enough points, then every near subset is contained in a near grill. A remark is in order here. If the completion of a nearness frame (or a quotient-fine nearness frame for that matter) has enough points, it does not follow that the nearness frame has enough points. For instance let L be a Boolean frame with no atoms. Then L has no points. Consider the Stone-Čech compactification $\sigma \colon \beta L \to L$ of L, and endow L with the nearness $\sigma[\mathrm{Cov}(\beta L)]$. Then L is a quotient-fine nearness frame and its completion has enough points.

3.11. Proposition. *Let L be a quotient-fine nearness frame with a spatial completion. Then any near subset of L is contained in a near grill.*

Proof. Let $h\colon M \to L$ be a completion of L, so that $\mathfrak{N}L = h[\mathrm{Cov}\, M]$. Let $A \subseteq L$ be near. We show that $\bigvee\{h_*(a^*) \mid a \in A\} \neq 1$. If not, then $h_*[A^*] \in \mathrm{Cov}\, M$, and hence $hh_*[A^*]$ is a uniform cover of L, that is, A^* is a uniform cover of L, contradicting the lemma. Since M has enough points, by hypothesis, there is a point $p \in M$ such that

$$\bigvee\{h_*(a^*) \mid a \in A\} \leq p.$$

Now define a subset G of L by

$$G = \{x \in L \mid h_*(x^*) \leq p\}.$$

Then, clearly, $A \subseteq G$. We show that G is near. If not, then there is a cover U of M such that $h[U] = G^*$. But this implies $U \leq h_*[G^*]$, and hence

$$1 = \bigvee U \leq \bigvee\{h_*(x^*) \mid x \in G\} \leq p,$$

which is false. Next, we show that G is a grill. Since $h_*(0^*) = h_*(1) = 1$, $0 \notin G$. Also, since $a \geq b \in G$ implies $h_*(a^*) \leq h_*(b^*) \leq p$, G is an upset. Now suppose $u \vee v \in G$. Then

$$h_*(u^*) \wedge h_*(v^*) = h_*(u^* \wedge v^*) = h_*((u \vee v)^*) \leq p,$$

which implies $h_*(u^*) \leq p$ or $h_*(v^*) \leq p$ since p is a point. Thus $u \in G$ or $v \in G$. Therefore G is a near grill containing A. □

Denote by $\mathrm{FCov}(L)$ the set of all covers of L which have finite subcovers. Then $\mathrm{FCov}(L)$ is a nearness on L. A nearness frame L is said to be *finitely fine* if $\mathfrak{N}L = \mathrm{FCov}(L)$. It is shown in [1] that:

> *For any finitely fine Boolean nearness frame L, the map $\mathfrak{J}L \to L$ from its ideal lattice by taking joins is a completion.*

In view of the last proposition, we have:

3.12. Corollary. *Any near subset of a finitely fine Boolean nearness frame is contained in a maximal near grill.*

Proof. Let L be a Boolean finitely fine nearness frame. Then, being compact, the completion of L is fine; and so L is a quotient-fine nearness frame with a spatial completion. Thus, by the proposition, every near subset is contained in a near grill. So it remains to show maximality. For definiteness, denote the map $\mathfrak{J}L \to L$ by h. Also, let A, p and G be as in the preceding proof. We must show maximality of G. So, let H be a near grill with $G \subseteq H$, and let $a \in H$. Suppose, by way of contradiction, that $a \notin G$. Then $h_*(a^*) \not\leq p$. Since h is dense,

$$h_*(a^*) \wedge h_*(a^{**}) = h_*(0) = 0 \leq p,$$

and so $h_*(a^{**}) \leq p$ as p is a point. This implies $a^* \in G \subseteq H$, so that both a and a^* are in H. But now $\{a, a^*\}$ is a uniform cover each of whose members misses at least one member of H; contradicting the fact that H is near. □

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