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APPROXIMATING FIXED POINTS OF IMPLICIT ALMOST CONTRACTIONS

Vasile Berinde*

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Abstract

In this paper we obtain constructive fixed point theorems for self operators in a general class of almost contractions defined by an implicit relation. Our results unify, extend, generalize, enrich and complement a multitude of related fixed point theorems from the literature.

Keywords: Metric space, Fixed point, Almost contraction, Implicit relation.

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1. Introduction

The study of fixed and common fixed points of mappings satisfying a certain metrical contractive condition has attracted many researchers and stimulated an impressive research work during the last five decades, see for example [34] and the very recent monograph [35].

Among these (common) fixed point theorems, only a few are important from a practical point of view, that is provide a *constructive* method for finding the fixed points or the common fixed points of the mappings involved, and only rarely do they offer information on the error estimate (or rate of convergence) of the iterative method used.

But, as (common) fixed point theorems offer an essential tool for solving nonlinear functional equations, it is important not only to know that the (common) fixed point exists (and, possibly, is unique), but also to be able to construct that (common) fixed point.

In a series of recent papers, see [7]–[15], the author focused on such (common) fixed point theorems, which were called *constructive* (common) fixed point theorems, see [12]. All these results have been obtained by considering operators that satisfy an *explicit* contractive type condition.

On the other hand, several classical fixed point theorems and common fixed point theorems have been recently unified by considering general contractive conditions expressed

*Department of Mathematics and Computer Science, North University of Baia Mare, Victoriei 76, 430072 Baia Mare Romania.
E-mail: vberinde@ubm.ro

by an implicit condition. This approach has been initiated in the seminal papers of Popa [23, 24]. Following Popa's approach, a consistent literature (that cannot be completely cited here) on fixed point, common fixed point and coincidence point theorems, for both single valued and multi-valued mappings, in various ambient spaces, has been developed, see [3]–[5], [23]–[29] and references therein, for a very selective list of references on this topic.

Starting from this background, the aim of this paper is to obtain some constructive fixed point theorems for almost contractions satisfying an implicit relation.

The fixed point theorems we obtain in this way are extremely general. They unify, extend, generalize, enrich and complement a multitude of related results from recent literature: [7]–[19], [23], [30]–[39], and most of the references therein.

2. Explicit almost contractions

Let (X, d) be a complete metric space and $T : X \to X$ a map. The classical contraction mapping principle, which is one of the most useful results in fixed point theory, says that, if T satisfies the inequality

$$d(Tx, Ty) \leq a\, d(x, y), \quad \text{for all } x, y \in X, \tag{2.1}$$

where $0 \leq a < 1$ is constant, then T has a unique fixed point $\overline{x}$ in X and for any $x_0 \in X$ the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n\,, \quad n = 0, 1, 2, \ldots \tag{2.2}$$

converges to $\overline{x}$.

Despite its important features and large field of applications, the contraction mapping principle suffers from one serious drawback - the contractive condition (2.1) forces T be continuous on X.

It was then natural to ask whether or not there exist weaker contractive conditions which do not imply the continuity of T. This was answered in the affirmative by R. Kannan [19] in 1968, who proved a fixed point theorem which extends the contraction mapping principle to mappings that need not be continuous on X (but are continuous at their fixed point, see [33]), by considering instead of (2.1) the next condition: there exists a constant b, $0 \leq 2b < 1$, such that

$$d(Tx, Ty) \leq b\big[d(x, Tx) + d(y, Ty)\big], \quad \text{for all } x, y \in X. \tag{2.3}$$

Following Kannan's theorem, a lot of papers were devoted to obtaining fixed point or common fixed points theorems for various classes of contractive type conditions that do not require the continuity of T, see for example, [34, 35, 11] and the references therein.

One of them, actually a sort of dual of Kannan fixed point theorem, due to Chatterjea [17], is based on a condition similar to (2.3): there exists a constant c, $0 \leq 2c < 1$, such that

$$d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big], \quad \text{for all } x, y \in X. \tag{2.4}$$

For a presentation and comparison of such kind of contractive conditions and fixed point theorems, see [31, 32] and [11].

On the other hand, in 1972, Zamfirescu [39] obtained a very interesting fixed point theorem which gathers together all three contractive conditions mentioned above, i.e., condition (2.1) of Banach, condition (2.3) of Kannan and condition (2.4) of Chatterjea, in a rather unexpected way: if T is such that, for any $x, y \in X$, at least one of the conditions (2.1), (2.3) and (2.4) holds, then T has a unique fixed point. Note that considering conditions (2.1), (2.3) and (2.4) all together is not trivial since, as shown later by Rhoades [31], the contractive conditions (2.1), (2.3) and (2.4) are pairwise independent.

The Zamfirescu fixed point theorem has been further extended by the author to almost contractions [9] (see also [6, 7, 8, 12, 13, 18] and [37]), a class of contractive type mappings which exhibit a totally different feature: any almost contraction does have generally more than one fixed point, see Example 1 in [9] and Example 3.2 in [13].

We give here the full statements of the main results in [9] in view of their extension to fixed point theorems for implicit almost contractions.

2.1. Theorem. [9, Theorem 2.1] *Let (X, d) be a complete metric space and $T : X \to X$ an almost contraction, that is, a mapping for which there exist a constant $\delta \in [0, 1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx), \quad \text{for all } x, y \in X. \tag{2.5}$$

Then

1) $\operatorname{Fix}(T) = \{x \in X : Tx = x\} \neq \emptyset$;
2) *For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (2.2) converges to some $x^* \in \operatorname{Fix}(T)$;*
3) *The following estimate holds*

$$d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta}\, d(x_n, x_{n-1}), \; n = 0, 1, 2, \ldots; \; i = 1, 2, \ldots. \tag{2.6}$$ □

2.2. Theorem. [9, Theorem 2.2] *Let (X, d) be a complete metric space and $T : X \to X$ an almost contraction for which there exist $\theta \in (0, 1)$ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta \cdot d(x, y) + L_1 \cdot d(x, Tx), \quad \text{for all } x, y \in X. \tag{2.7}$$

Then

1) *T has a unique fixed point, i.e.* $\operatorname{Fix}(T) = \{x^*\}$;
2) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (2.2) converges to x^*, for any $x_0 \in X$;*
3) *The error estimate (2.6) holds;*
4) *The rate of convergence of the Picard iteration is given by*

$$d(x_n, x^*) \leq \theta\, d(x_{n-1}, x^*), \; n = 1, 2, \ldots. \tag{2.8}$$ □

Note that in the particular case $L := 0$ and $L_1 := 0$, both Theorem 2.1 and Theorem 2.2 reduce to the well known contraction mapping principle in its complete form, see [10].

It is therefore the main aim of this paper to extend Theorems 2.1 and 2.2 to almost contractions defined by implicit relations. In this way we obtain very general fixed point theorems that unify, extend, generalize, enrich and complement a multitude of related fixed point theorems from the literature: [7]-[19], [20]-[23], [30]-[39], and most of the references therein.

3. Fixed point theorems for mappings satisfying an implicit relation

A simple and natural way to unify and prove in a simple manner several metrical fixed point theorems is to consider an implicit contraction type condition instead of the usual explicit contractive conditions. Popa [23] and [24], initiated this direction of research which has produced so far a consistent literature (that cannot be completely cited here) on fixed point, common fixed point and coincidence point theorems, for both single valued and multi-valued mappings, in various ambient spaces, see [4] and [5], for a partial list of references.

Let $\mathcal{F}$ be the set of all continuous real functions $F : {\mathbb{R}_+}^6 \to \mathbb{R}_+$, for which we consider the following conditions:

(F_{1a}) F is non-increasing in the fifth variable and

$$F(u, v, v, u, u + v, 0) \le 0 \text{ for } u, v \ge 0 \implies \exists\, h \in [0, 1) \text{ such that } u \le hv;$$

(F_{1b}) F is non-increasing in the fourth variable and

$$F(u, v, 0, u + v, u, v) \le 0 \text{ for } u, v \ge 0 \implies \exists\, h \in [0, 1) \text{ such that } u \le hv;$$

(F_{1c}) F is non-increasing in the third variable and

$$F(u, v, u + v, 0, v, u) \le 0 \text{ for } u, v \ge 0 \implies \exists\, h \in [0, 1) \text{ such that } u \le hv;$$

(F_2) $F(u, u, 0, 0, u, u) > 0$, for all $u > 0$.

The following functions correspond to well known fixed point theorems and satisfy most of the conditions (F_{1a})-(F_2) above.

3.1. Example. The function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2,$$

where $a \in [0, 1)$, satisfies (F_2) and (F_{1a})-(F_{1c}), with $h = a$.

3.2. Example. Let $b \in \left[0, \frac{1}{2}\right)$. Then the function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - b(t_3 + t_4),$$

satisfies (F_2) and (F_{1a})-(F_{1c}), with $h = \dfrac{b}{1-b} < 1$.

3.3. Example. Let $c \in \left[0, \frac{1}{2}\right)$. Then the function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - c(t_5 + t_6),$$

satisfies (F_2) and (F_{1a})-(F_{1c}), with $h = \dfrac{c}{1-c} < 1$.

3.4. Example. The function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \max\left\{t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2}\right\},$$

where $a \in [0, 1)$, satisfies (F_2) and (F_{1a})-(F_{1c}), with $h = a$.

3.5. Example. The function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6),$$

where $a, b, c \in [0, 1)$ and $a + 2b + 2c < 1$, satisfies (F_2) and (F_{1a})-(F_{1c}), with $h = \dfrac{a + b + c}{1 - b - c} < 1$.

3.6. Example. The function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \max\left\{t_2, \frac{t_3 + t_4}{2}, t_5, t_6\right\},$$

where $a \in [0, 1)$, satisfies (F_2) and (F_{1b}), (F_{1c}), with $h = a$ and (F_{1a}), with $h = \dfrac{a}{1-a} < 1$, if $a < 1/2$.

3.7. Example. The function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - Lt_3,$$

where $a \in [0, 1)$ and $L \ge 0$, satisfies (F_2) and (F_{1b}), with $h = a$, but, in general, does not satisfy (F_{1a}) and (F_{1c}).

3.8. Example. The function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - Lt_6,$$

where $a \in [0, 1)$ and $L \geq 0$, satisfies (F_{1a}), with $h = a$, but, in general, does not satisfy (F_{1b}), (F_{1c}) and (F_2).

The following theorem, which is an enriched version of Popa [23, Theorem 3] that unifies the most important metrical fixed point theorems for contractive mappings in Rhoades' classification [31], has been obtained in [16].

3.9. Theorem. *Let (X, d) be a complete metric space, $T : X \to X$ a self mapping for which there exists $F \in \mathcal{F}$ such that for all $x, y \in X$,*

$$F\left(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right) \leq 0. \tag{3.1}$$

If F satisfies (F_{1a}) and (F_2) then:

(p1) *T has a unique fixed point $\overline{x}$ in X;*

(p2) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by*

$$x_{n+1} = Tx_n, \; n = 0, 1, 2, \ldots \tag{3.2}$$

converges to $\overline{x}$, for any $x_0 \in X$.

(p3) *The following estimate holds:*

$$d(x_{n+i-1}, \overline{x}) \leq \frac{h^i}{1-h}\, d(x_n, x_{n-1}), \; n = 0, 1, 2, \ldots; \; i = 1, 2, \ldots, \tag{3.3}$$

where h is the constant appearing in (F_{1a}).

(p4) *If, additionally, F satisfies (F_{1c}), then the rate of convergence of the Picard iteration is given by:*

$$d(x_{n+1}, \overline{x}) \leq hd(x_n, \overline{x}), \; n = 0, 1, 2, \ldots. \tag{3.4}$$

3.10. Remark. (a) If F is the function in Example 3.1, then by Theorem 3.9 we obtain the well known Banach contraction mapping principle, in its complete form, see [10, Theorem B].

(b) If F is the function in Example 3.2, then by Theorem 3.9 we obtain [10, Theorem 1], that extends the well known Kannan fixed point theorem [19].

(c) If F is the function in Example 3.3, then by Theorem 3.9 we obtain a fixed point theorem that extends Chatterjea's fixed point theorem [17].

(d) If F is the function in Example 3.4, then by Theorem 3.9 we obtain [10, Theorem 2], that extends the well known Zamfirescu fixed point theorem [39].

(e) If F is the function in Example 3.5, then by Theorem 3.9 we obtain a fixed point theorem that extends the Reich fixed point theorem [30].

For other important particular cases of Theorem 3.9 see [7, 16] and references therein.

The first main result of this paper extends Theorem 3.9 in such a way as to also include some known fixed point theorems for explicit almost contractions.

3.11. Theorem. *Let (X, d) be a complete metric space, $T : X \to X$ a self mapping for which there exists $F \in \mathcal{F}$, satisfying (F_{1a}), such that for all $x, y \in X$,*

$$F\left(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right) \leq 0. \tag{3.5}$$

Then:

(p1) $\mathrm{Fix}\,(T) \neq \emptyset$;

(p2) *For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point $\overline{x}$ of T.*

(p3) *The following estimate holds:*

$$d(x_{n+i-1},\overline{x}) \le \frac{h^i}{1-h}\, d(x_n,x_{n-1}),\ \ n=0,1,2,\dots;\ \ i=1,2,\dots, \tag{3.6}$$

where h is the constant appearing in (F_{1a}).

(p4) *If, additionally, F satisfies (F_{1c}), then the rate of convergence of the Picard iteration is given by:*

$$d(x_{n+1},\overline{x}) \le h d(x_n,\overline{x}),\ \ n=0,1,2,\dots. \tag{3.7}$$

Proof. (p1) Let x_0 be an arbitrary point in X and $x_{n+1}=Tx_n$, $n=0,1,\dots$, be the Picard iteration. If we take $x:=x_{n-1}$ and $y:=x_n$ in (3.5) and denote $u:=d(x_n,x_{n+1})$, $v:=d(x_{n-1},x_n)$ we get

$$F\left(u,v,v,u,d(x_{n-1},x_{n+1}),0\right) \le 0.$$

By the triangle inequality, $d(x_{n-1},x_{n+1}) \le d(x_{n-1},x_n)+d(x_n,x_{n+1})=u+v$ and, since F is non-increasing in the fifth variable, we have

$$F\left(u,v,v,u,u+v,0\right) \le F\left(u,v,v,u,d(x_{n-1},x_{n+1}),0\right) \le 0$$

and hence, in view of assumption (F_{1a}), there exists $h\in[0,1)$ such that $u\le hv$, that is,

$$d(x_n,x_{n+1}) \le h d(x_{n-1},x_n) \tag{3.8}$$

which, in a straightforward way, leads to the conclusion that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

Since (X,d) is complete, there exists a $\overline{x}$ in X such that

$$\lim_{n\to\infty} x_n = \overline{x}. \tag{3.9}$$

By taking $x:=x_n$ and $y:=\overline{x}$ in (3.5) we get

$$F\left(d(Tx_n,T\overline{x}),d(x_n,\overline{x}),d(x_n,Tx_n),d(\overline{x},T\overline{x}),d(x_n,T\overline{x}),d(\overline{x},Tx_n)\right) \le 0. \tag{3.10}$$

As F is continuous, by letting $n\to\infty$ in (3.10) we obtain

$$F\left(d(\overline{x},T\overline{x}),0,0,d(\overline{x},T\overline{x}),d(\overline{x},T\overline{x}),0)\right) \le 0,$$

which by assumption (F_{1a}) yields $d(\overline{x},T\overline{x})\le 0$, that is, $\overline{x}=T\overline{x}$.

(p2) This follows by the proof of (p1).

(p3) This follows by a double inductive process by means of (3.8).

(p4) By taking $x:=x_n$ and $y:=\overline{x}$ in (3.5) we get

$$F\left(d(Tx_n,\overline{x}),d(x_n,\overline{x}),d(x_n,Tx_n),d(\overline{x},\overline{x}),d(x_n,\overline{x}),d(\overline{x},Tx_n)\right) \le 0,$$

that is,

$$F\left(d(x_{n+1},\overline{x}),d(x_n,\overline{x}),d(x_n,x_{n+1}),0,d(x_n,\overline{x}),d(\overline{x},x_{n+1})\right) \le 0. \tag{3.11}$$

Let $u:=d(x_{n+1},\overline{x})$, $v:=d(x_n,\overline{x})$. Then, by the triangle inequality we have $d(x_n,x_{n+1})\le d(x_n,\overline{x})+d(x_{n+1},\overline{x})=u+v$ and hence, in view of assumption (F_{1c}), by (3.11) we obtain

$$F\left(u,v,u+v,0,v,u)\right) \le F\left(u,v,d(x_n,x_{n+1}),0,v,u\right) \le 0,$$

which again by (F_{1c}) implies the existence of an $h\in[0,1)$ such that $u\le hv$, which is exactly the desired estimate (3.7). $\square$

3.12. Remark. (a) If F in Theorem 3.11 also satisfies (F_2), then by Theorem 3.11 we obtain Theorem 3.9.

(b) If F is the function in Example 3.7, then by Theorem 3.11 (but not by Theorem 3.9) we obtain Theorem 2.1, i.e. [9, Theorem 2.1].

(c) If F is the function in Example 3.8, then by Theorem 3.11 (or by Theorem 3.9) we obtain Theorem 2.2, i.e. [9, Theorem 2.2].

3.13. Remark. From the unifying error estimates (3.6), and inspired by [38], we get both the *a priori* estimate

$$d(x_n, \overline{x}) \leq \frac{h^n}{1-h}\, d(x_0, x_1),\ \ n = 0, 1, 2, \dots$$

and the *a posteriori* estimate

$$d(x_n, \overline{x}) \leq \frac{h}{1-h}\, d(x_n, x_{n-1}),\ \ n = 1, 2, \dots,$$

which are extremely important in applications, especially when approximating the solutions of nonlinear equations.

One can also obtain an existence and uniqueness fixed point theorem, corresponding to Theorem 3.11.

3.14. Theorem. *Let (X, d) be a complete metric space, $T : X \to X$ a self mapping for which there exists $F \in \mathcal{F}$, satisfying (F_{1a}), such that for all $x, y \in X$,* (3.5) *holds, and there exists $G \in \mathcal{F}$, satisfying (F_2), such that for all $x, y \in X$,*

$$G\left(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right) \leq 0. \tag{3.12}$$

Then:

- (p1) *T has a unique fixed point $\overline{x}$ in X;*
- (p2) *For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ converges to $\overline{x}$.*
- (p3) *The error estimate* (3.6) *holds;*
- (p4) *If, additionally, F or G satisfies (F_{1c}), then the rate of convergence of the Picard iteration is given by:*

$$d(x_{n+1}, \overline{x}) \leq h d(x_n, \overline{x}),\ \ n = 0, 1, 2, \dots. \tag{3.13}$$

Proof. The existence of the fixed point as well as the estimates (3.6) and (3.13) follow as in the proof of Theorem 3.11.

In order to prove the uniqueness of $\overline{x}$, assume the contrary, i.e., there exists $\overline{y} \in \mathrm{Fix}\,(T)$, $\overline{x} \neq \overline{y}$. Then by taking $x := \overline{x}$ and $y := \overline{y}$ in (3.12) and by setting $\delta := d(\overline{x}, \overline{y}) > 0$ we get

$$G(\delta, \delta, 0, 0, \delta, \delta) \leq 0,$$

which contradicts (F_2). This proves that T has a unique fixed point. □

3.15. Remark. (a) If F is the function in Example 3.8 and G is the function in Example 3.7, then by Theorem 3.14 we obtain Theorem 2.2, i.e. [9, Theorem 2.2].

(b) If $F \equiv G$ is the function in Example 3.17, then by Theorem 3.14 one obtains the main result (Theorem 2.3) in [6].

(c) If F is the function in Example 3.19, then by Theorem 3.14 one obtains the second uniqueness result (Theorem 2.4) in [13].

Theorem 3.14 can now be significantly extended by considering two metrics on the set X, similarly to [8, Theorem 5].

3.16. Theorem. *Let X be a nonempty set and d, ρ two metrics on X, such that (X, d) is complete. Let $T : X \to X$ be a self operator for which:*

(i) *There exists $F \in \mathcal{F}$ satisfying (F_{1a}) such that for all $x, y \in X$,*

$$F\left(d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right) \leq 0.$$

(ii) *There exists $G \in \mathcal{F}$ satisfying (F_{1c}) and (F_2) such that for all $x, y \in X$,*

$$G\left(\rho(Tx,Ty), \rho(x,y), \rho(x,Tx), \rho(y,Ty), \rho(x,Ty), \rho(y,Tx)\right) \leq 0.$$

Then

1) *T has a unique fixed point $\overline{x}$;*
2) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$, $x_{n+1} = Tx_n$, $n \geq 0$, converges to $\overline{x}$, for all $x_0 \in X$;*
3) *The a error estimate (3.6) holds;*
4) *The rate of convergence of the Picard iteration is given by*

$$\rho(x_n, x^*) \leq h\rho(x_{n-1}, x^*), \ n \geq 1. \tag{3.14}$$

Proof. The existence of the fixed point as well as the estimates (3.6) and (3.14) follow as in the proof of Theorem 3.11.

In order to prove the uniqueness of $\overline{x}$, assume the contrary, i.e., there exists $\overline{y} \in \mathrm{Fix}\,(T)$, $\overline{x} \neq \overline{y}$. Then by taking $x := \overline{x}$ and $y := \overline{y}$ in (3.12) and letting $\delta := \rho(\overline{x}, \overline{y}) > 0$ we get

$$G(\delta, \delta, 0, 0, \delta, \delta) \leq 0,$$

which contradicts (F_2). This proves that T has a unique fixed point. □

In order to illustrate the generality of Theorem 3.14 and Theorem 3.16 we consider three more examples of functions $F \in \mathcal{F}$.

3.17. Example. The function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - L\min\{t_3, t_4, t_5, t_6\},$$

where $a \in [0, 1)$ and $L \geq 0$, satisfies (F_2) and (F_{1a})-(F_{1c}), with $h = a$.

3.18. Example. The function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a\max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\} - Lt_6,$$

where $a \in [0, 1)$ and $L \geq 0$, satisfies (F_{1a}), with $h = a$, but, in general, does not satisfy (F_{1b}), (F_{1c}) and (F_2). To prove (F_{1a}) let us observe that by $F(u, v, v, u, u+v, 0) \leq 0$ one obtains

$$u - a\max\left\{v, v, u, \frac{u+v}{2}\right\} \leq 0.$$

If one admits that $u > v$, then by the previous inequality one obtains $u - au \leq 0 \iff (1-a)u \leq 0$, a contradiction. Hence $u \leq v$ and thus (F_{1a}) is satisfied with $h = a$.

3.19. Example. The function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a\max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\} - L\min\{t_3, t_4, t_5, t_6\},$$

where $a \in [0, 1)$ and $L \geq 0$, satisfies (F_2) and (F_{1a})-(F_{1c}), with $h = a$.

3.20. Remark. (a) If we set $d \equiv \rho$, by Theorem 3.16 we obtain Theorem 3.14.

(b) If F is the function in Example 3.8 and G is the function in Example 3.7, then by Theorem 3.14 we obtain Theorem 2.2, i.e. [9, Theorem 2.2].

(c) If F is the function given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\} - Lt_6,$$

where $a \in [0, 1)$ and $L \geq 0$, and G is an in Example 3.7, then F satisfies (F_2) and (F_{1a})-(F_{1c}), with $h = a$, and hence by Theorem 3.14 one obtains the first uniqueness result (Theorem 2.3) in [13].

4. Conclusions and further research

All the contractive conditions considered in this paper are defined by *linear* functions $F \in \mathcal{F}$, see Examples 3.1-3.19 and Remark 3.20, but, generally, in Theorems 3.9-3.16, neither F nor G is assumed to be linear. This ensures a great generality to the results obtained in the present paper. Several *nonlinear* contractive conditions associated to such fixed point theorems can be found in most of the papers in the list of references, see for example [3, 5, 23, 24] etc.

We shall consider and study such nonlinear contractive conditions in some forthcoming papers. It is also our aim to obtain fixed point results similar to the ones in [4] by removing the decreasing assumption in (F_{1a})-(F_{1c}).

It is very important to note that the contraction conditions defined by the functions in Examples 3.7-3.19 in this paper have not been considered in any other paper devoted to fixed point theorems for mappings defined by implicit relations.

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