Hacettepe Journal of Mathematics and Statistics
Volume 41 (1) (2012), 103 – 117

AN APPROXIMATE PROXIMAL POINT ALGORITHM FOR NONLINEAR COMPLEMENTARITY PROBLEMS

Abdellah Bnouhachem[*§†], Muhammad Aslam Noor[‡], Mohamed Khalfaoui[§] and Sheng Zhaohan[¶]

Received 20 : 01 : 2011 : Accepted 08 : 08 : 2011

Abstract

In this paper, we suggest and analyze a new method for solving nonlinear complementarity problems (NCP) where the underlying function F is co-coercive. The theme of this paper is twofold. First, we consider the logarithmic-quadratic proximal (LQP) method which was introduced by Auslender, Teboulle and Ben-Tiba (*A logarithmic-quadratic proximal method for variational inequalities*, Comput. Optim. Appl. **12**, 31–40, 1999). Next, we propose a new modified LQP method by using a new direction with a new step size α_k. We show that the method is globally convergent. Some preliminary computational results are given to illustrate the efficiency of the proposed method.

Keywords: Nonlinear complementarity problems, Co-coercive operator, Logarithmic-quadratic proximal method.

2000 AMS Classification: 49 J 40, 65 N 30.

[*]Ibn Zohr University, ENSA, BP 1136, Agadir, Morocco.
E-mail (A. Bnouhachem) babedallah@yahoo.com
[†]Corresponding Author.
[‡]Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan. E-mail: noormaslam@hotmail.com
[§]Ecole Supérieure de Technologie de Salé, Mohamed 5 University, Agdal-Rabat, Morocco.
[¶]School of Management Science and Engineering, Nanjing University, Nanjing, 210093, P. R. China.

1. Introduction

The nonlinear complementarity problem (NCP) is to determine a vector $x \in \mathbb{R}^n$ such that

(1.1) $$x \geq 0, \quad F(x) \geq 0 \text{ and } x^T F(x) = 0,$$

where F is a nonlinear mapping from $\mathbb{R}^n$ into itself. Complementarity problems were introduced by Lemke [19] and Cottle Dantzig [11] in the early 1960's. These problems are being used as a powerful tool to study a wide class of problems with applications in industry, engineering, optimization, mathematical and physical sciences in a unified framework. It has been shown that the linear and nonlinear problems in operations can be formulated as complementarity problems, which can be solved more effectively. There are several methods for solving complementarity problems, which can be divided into two categories namely direct and indirect (iterative) methods. Direct methods are those based on the process of pivoting, which are mainly due to Lemke [19] and Cottle and Dantzig [11]. The practicality of direct methods is restricted mainly due to the problem size limitations in computer implementations. Also, these methods cannot be extended for nonlinear complementarity problems. These facts and reasons have stimulated much investigation of alternative approaches for solving nonlinear complementarity problems. In this paper, we are only concerned with the iterative approach to proximal point methods. These iterative methods have emerged in the last decades as a powerful technique for solving nonlinear complementarity problems effectively. These methods are user friendly and can be implemented easily. It is well-known that complementarity problems can be formulated as a variational inclusion involving the sum of two monotone operators. This equivalent formulation has played an important part in suggesting and developing proximal point algorithms for solving complementarity problems. In this paper, we use this equivalent formulation in conjunction with suitable quadratic function to suggest a new proximal point method, which uses a new direction with a new suitable step size. We show that this new proximal point method is globally convergent under suitable conditions. Some preliminary computational results are given to illustrate the efficiency and implementation of this new method. Comparison with other methods shows that this new proximal point methods outperform the other methods.

2. Preliminaries

In this section, we summarize some preliminary results which are useful in the following analysis. First, we give some basic properties of the projection mapping.

2.1. Lemma. *Let $P_{\mathbb{R}^n_+}(\cdot)$ denote the projection of $\mathbb{R}^n$ onto $\mathbb{R}^n_+$. Then, we have the following inequalities.*

(2.1) $$(v - P_{\mathbb{R}^n_+}(v))^T (u - P_{\mathbb{R}^n_+}(v)) \leq 0, \ \forall\, u \in \mathbb{R}^n_+, \ \forall\, v \in \mathbb{R}^n;$$

(2.2) $$\|P_{\mathbb{R}^n_+}(v) - P_{\mathbb{R}^n_+}(u)\| \leq \|v - u\|, \ \forall\, u, v \in \mathbb{R}^n;$$

(2.3) $$\|P_{\mathbb{R}^n_+}(v) - u\|^2 \leq \|v - u\|^2 - \|v - P_{\mathbb{R}^n_+}(v)\|^2, \ \forall\, v \in \mathbb{R}^n, \ u \in \mathbb{R}^n_+. \qquad \square$$

2.2. Definition. The operator $F : \mathbb{R}^n_+ \to \mathbb{R}^n$ is said to be *co-coercive* if there exists a constant $c > 0$ such that

$$\langle F(x) - F(y), x - y \rangle \geq c\|F(x) - F(y)\|^2, \ \forall\, x, y \in \mathbb{R}^n_+.$$

The following lemma is similar to [3, Lemma 2]. Hence the proof will be omitted.

2.3. Lemma. [17, 33] *For given $x^k > 0$ and $q \in \mathbb{R}^n$, let x be the positive solution of the following equation:*

$$q + x - (1-\mu)x^k - \mu X_k^2 x^{-1} = 0. \tag{2.4}$$

Then for any $y \geq 0$ we have

$$\langle y - x, q\rangle \geq \tfrac{1+\mu}{2}\left(\|x-y\|^2 - \|x^k - y\|^2\right) + \tfrac{1-\mu}{2}\|x^k - x\|^2. \tag{2.5}$$ □

Throughout this paper we assume that F is co-coercive with modulus $c > 0$ and that the solution set of (1.1), denoted by Ω^*, is nonempty.

3. The proposed method

In this section, we suggest and analyze the new modified LQP method for solving nonlinear complementarity problems 1.1. At the k-th iteration, the LQP method finds the exact solution for the following system of equations:

$$\beta_k F(x) + x - (1-\mu)x^k - \mu X_k^2 x^{-1} = 0. \tag{3.1}$$

We now present a new LQP method for solving NCP. For given $x^k > 0$ and $\beta_k > 0$, each iteration of the proposed method consists of two steps, the first step offers a predictor $\tilde{x}^k$ and the second step produces the new iterate x^{k+1}.

Prediction step: Find an approximate solution $\tilde{x}^k$ of (3.1), called the predictor, such that

$$0 \approx \beta_k F(\tilde{x}^k) + \tilde{x}^k - (1-\mu)x^k - \mu X_k^2 (\tilde{x}^k)^{-1} = \xi^k := \beta_k(F(\tilde{x}^k) - F(x^k)), \tag{3.2}$$

and ξ^k which satisfies

$$\|\xi^k\| \leq \eta\|x^k - \tilde{x}^k\|,\ 0 < \eta < 1. \tag{3.3}$$

Correction step: For $0 < \rho < 1$, the new iterate $x^{k+1}(\alpha_k)$ is defined by

$$x^{k+1}(\alpha_k) = \rho x^k + (1-\rho)P_{\mathbb{R}^n_+}\left[x^k - \alpha_k d(x^k, \beta_k)\right], \tag{3.4}$$

where

$$d(x^k, \beta_k) := (x^k - \tilde{x}^k) + \frac{\beta_k}{1+\mu}F(\tilde{x}^k) \tag{3.5}$$

and α_k is the step length that will be specified later.

3.1. Remark. Note that that the direction $d(x^k, \beta_k)$ is different from that used in [17, 33].

3.2. Remark. (3.3) implies that

$$|(x^k - \tilde{x}^k)^T \xi^k| \leq \eta\|x^k - \tilde{x}^k\|^2,\ 0 < \eta < 1. \tag{3.6}$$

3.3. Remark. Equation (3.2) can be written as

$$\beta_k F(x^k) + \tilde{x}^k - (1-\mu)x^k - \mu X_k^2 (\tilde{x}^k)^{-1} = 0, \tag{3.7}$$

and the solution of (3.7) can be obtained componentwise by

$$\tilde{x}_j^k = \frac{(1-\mu)x_j^k - \beta_k F_j(x^k) + \sqrt{[(1-\mu)x_j^k - \beta_k F_j(x^k)]^2 + 4\mu(x_j^k)^2}}{2}. \tag{3.8}$$

Moreover for any $x^k > 0$ we have always $\tilde{x}^k > 0$.

How to choose values of α_k to ensure that $x^{k+1}(\alpha_k)$ is closer to the solution set than x^k? For this purpose, we define

$$\Theta(\alpha_k) = \|x^k - x^*\|^2 - \|x^{k+1}(\alpha_k) - x^*\|^2. \tag{3.9}$$

3.4. Theorem. *Let $x^* \in \Omega^*$, $x^{k+1}(\alpha_k)$ be defined by (3.4). Then we have*

$$\Theta(\alpha_k) \geq (1-\rho)\{2\alpha_k(x^k - \tilde{x}^k)^T D(x^k,\beta_k) - \frac{2\alpha_k\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2 - \alpha_k^2\|D(x^k,\beta_k)\|^2 + \|x^k - x_*^k - \alpha_k D(x^k,\beta_k)\|^2 + 2\alpha_k(x_*^k - x^*)^T(x^k - \tilde{x}^k)\},$$

where

$$x_*^k := P_{\mathbb{R}_+^n}\left[x^k - \alpha_k d(x^k,\beta_k)\right] \text{ and } D(x^k,\beta_k) := (x^k - \tilde{x}^k) + \frac{1}{1+\mu}\xi^k. \tag{3.10}$$

Proof. By setting $q = \beta_k F(\tilde{x}^k) - \xi^k$ in (2.4) and $y = x_*^k := P_{\mathbb{R}_+^n}\left[x^k - \alpha_k d(x^k,\beta_k)\right]$ in (2.5), it follows that

$$(x_*^k - \tilde{x}^k)^T\left(\frac{1}{1+\mu}(\xi^k - \beta_k F(\tilde{x}^k))\right) \leq \frac{1}{2}\left(\|x^k - x_*^k\|^2 - \|\tilde{x}^k - x_*^k\|^2\right) - \frac{1-\mu}{2(1+\mu)}\|x^k - \tilde{x}^k\|^2. \tag{3.11}$$

Using the following identity

$$(x_*^k - \tilde{x}^k)^T(x^k - \tilde{x}^k) = \frac{1}{2}\left(\|\tilde{x}^k - x_*^k\|^2 - \|x^k - x_*^k\|^2\right) + \frac{1}{2}\|x^k - \tilde{x}^k\|^2, \tag{3.12}$$

adding (3.11) and (3.12) we then obtain

$$(x_*^k - \tilde{x}^k)^T\left\{(x^k - \tilde{x}^k) + \frac{1}{1+\mu}\left(\xi^k - \beta_k F(\tilde{x}^k)\right)\right\} \leq \frac{\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2,$$

which implies

$$2\alpha_k(x_*^k - \tilde{x}^k)^T\left\{(x^k - \tilde{x}^k) - \frac{\beta_k}{1+\mu}F(x^k)\right\} - \frac{2\alpha_k\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2 \leq 0$$

and

$$2\alpha_k(x_*^k - x^k + x^k - \tilde{x}^k)^T\left\{(x^k - \tilde{x}^k) - \frac{\beta_k}{1+\mu}F(x^k)\right\} - \frac{2\alpha_k\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2 \leq 0. \tag{3.13}$$

Since $x^* \in \Omega^* \subset \mathbb{R}_+^n$ and $x_*^k = P_{\mathbb{R}_+^n}\left[x^k - \alpha_k d(x^k,\beta_k)\right]$, it follows from (2.3) that

$$\|x_*^k - x^*\|^2 \leq \|x^k - \alpha_k d(x^k,\beta_k) - x^*\|^2 - \|x^k - \alpha_k d(x^k,\beta_k) - x_*^k\|^2. \tag{3.14}$$

From (3.4), we get

$$\begin{aligned}\|x^{k+1}(\alpha_k) - x^*\|^2 &= \|\rho(x^k - x^*) + (1-\rho)(x_*^k - x^*)\|^2 \\ &= \rho^2\|x^k - x^*\|^2 + (1-\rho)^2\|x_*^k - x^*\|^2 \\ &\quad + 2\rho(1-\rho)(x^k - x^*)^T(x_*^k - x^*).\end{aligned}$$

Using the following identity

$$2(a+b)^T b = \|a+b\|^2 - \|a\|^2 + \|b\|^2$$

for $a = x^k - x_*^k$, $b = x_*^k - x^*$ and (3.14), we obtain

$$\begin{aligned}\|x^{k+1}(\alpha_k) - x^*\|^2 &= \rho^2\|x^k - x^*\|^2 + (1-\rho)^2\|x_*^k - x^*\|^2 + \rho(1-\rho)\{\|x^k - x^*\|^2 \\ &\quad - \|x^k - x_*^k\|^2 + \|x_*^k - x^*\|^2\} \\ &= \rho\|x^k - x^*\|^2 + (1-\rho)\|x_*^k - x^*\|^2 - \rho(1-\rho)\|x^k - x_*^k\|^2 \\ &\leq \rho\|x^k - x^*\|^2 + (1-\rho)\|x^k - \alpha_k d(x^k,\beta_k) - x^*\|^2 \\ &\quad - (1-\rho)\|x^k - \alpha_k d(x^k,\beta_k) - x_*^k\|^2 - \rho(1-\rho)\|x^k - x_*^k\|^2.\end{aligned} \tag{3.15}$$

Using the definition of $\Theta(\alpha_k)$ and (3.15), we get

$$\begin{aligned}\Theta(\alpha_k) &\geq (1-\rho^2)\|x^k - x_*^k\|^2 + 2\alpha_k(1-\rho)(x_*^k - x^k)^T d(x^k,\beta_k) \\ &\qquad + 2\alpha_k(1-\rho)(x^k - x^*)^T d(x^k,\beta_k) \\ &\geq (1-\rho)\big\{\|x^k - x_*^k\|^2 + 2\alpha_k(x_*^k - x^k)^T d(x^k,\beta_k) \\ &\qquad + 2\alpha_k(x^k - x^*)^T d(x^k,\beta_k)\big\}. \end{aligned} \tag{3.16}$$

From the definition of $d(x^k,\beta_k)$ (see 3.5), we have

$$\begin{aligned}\Theta(\alpha_k) &\geq (1-\rho)\Big\{\|x^k - x_*^k\|^2 + 2\alpha_k(x_*^k - x^k)^T\Big(x^k - \tilde{x}^k + \frac{\beta_k}{1+\mu}F(\tilde{x}^k)\Big) \\ &\qquad + 2\alpha_k(x^k - x^*)^T\Big(x^k - \tilde{x}^k + \frac{\beta_k}{1+\mu}F(\tilde{x}^k)\Big)\Big\} \\ &= (1-\rho)\Big\{\frac{2\alpha_k\beta_k}{1+\mu}(x^k - x^*)^T F(\tilde{x}^k) + \|x^k - x_*^k\|^2 \\ &\qquad - \frac{2\alpha_k\beta_k}{1+\mu}(x^k - x_*^k)^T F(\tilde{x}^k) + 2\alpha_k(x_*^k - x^*)^T(x^k - \tilde{x}^k)\Big\} \\ &= (1-\rho)\Big\{\frac{2\alpha_k\beta_k}{1+\mu}(x^k - x^*)^T F(\tilde{x}^k) + \|x^k - x_*^k - \alpha_k D(x^k,\beta_k)\|^2 \\ &\qquad + 2\alpha_k(x^k - x_*^k)^T D(x^k,\beta_k) - \alpha_k^2\|D(x^k,\beta_k)\|^2 \\ &\qquad - \frac{2\alpha_k\beta_k}{1+\mu}(x^k - x_*^k)^T F(\tilde{x}^k) + 2\alpha_k(x_*^k - x^*)^T(x^k - \tilde{x}^k)\Big\} \\ &= (1-\rho)\Big\{\frac{2\alpha_k\beta_k}{1+\mu}(x^k - x^*)^T F(\tilde{x}^k) + \|x^k - x_*^k - \alpha_k D(x^k,\beta_k)\|^2 \\ &\qquad - \alpha_k^2\|D(x^k,\beta_k)\|^2 + 2\alpha_k(x^k - x_*^k)^T\Big(x^k - \tilde{x}^k - \frac{\beta_k}{1+\mu}F(x^k)\Big) \\ &\qquad + 2\alpha_k(x_*^k - x^*)^T(x^k - \tilde{x}^k)\Big\}. \end{aligned} \tag{3.17}$$

Since $\tilde{x}^k \in \mathbb{R}^n_{++}$ and x^* is a solution of NCP, using the co-coercivity of F we obtain

$$(\tilde{x}^k - x^*)^T F(x^*) = (\tilde{x}^k)^T F(x^*) \geq 0 \Rightarrow (\tilde{x}^k - x^*)^T F(\tilde{x}^k) \geq 0$$

and consequently

$$(x^k - x^*)^T F(\tilde{x}^k) \geq (x^k - \tilde{x}^k)^T F(\tilde{x}^k). \tag{3.18}$$

Applying (3.18) to the first term in the right side of (3.17) and using $0 < \rho < 1$, we obtain

$$\begin{aligned}\Theta(\alpha_k) &\geq (1-\rho)\Big\{\frac{2\alpha_k\beta_k}{1+\mu}(x^k - \tilde{x}^k)^T F(\tilde{x}^k) + \|x^k - x_*^k - \alpha_k D(x^k,\beta_k)\|^2 \\ &\qquad - \alpha_k^2\|D(x^k,\beta_k)\|^2 + 2\alpha_k(x^k - x_*^k)^T\Big(x^k - \tilde{x}^k - \frac{\beta_k}{1+\mu}F(x^k)\Big) \\ &\qquad + 2\alpha_k(x_*^k - x^*)^T(x^k - \tilde{x}^k)\Big\}. \end{aligned} \tag{3.19}$$

Adding (3.13) (multiplied by $1-\rho$) to (3.19) and using the definition of $D(x^k,\beta_k)$, we obtain

$$\begin{aligned}\Theta(\alpha_k) &\geq (1-\rho)\Big\{2\alpha_k(x^k - \tilde{x}^k)^T D(x^k,\beta_k) - \frac{2\alpha_k\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2 - \alpha_k^2\|D(x^k,\beta_k)\|^2 \\ &\qquad + \|x^k - x_*^k - \alpha_k D(x^k,\beta_k)\|^2 + 2\alpha_k(x_*^k - x^*)^T(x^k - \tilde{x}^k)\Big\}, \end{aligned} \tag{3.20}$$

and the theorem is proved. □

3.5. Lemma. *Let $x^* \in \Omega^*$ be a solution point. For any $x^k > 0$ we have*

$$(x^k - x^*)^T (x^k - \tilde{x}^k) \geq \frac{1}{1+\mu} \left(1 - \frac{\beta_k}{4c}\right) \|x^k - \tilde{x}^k\|^2. \tag{3.21}$$

Proof. Since x^* is a solution, it follows from (1.1) that

$$\frac{\beta_k}{1+\mu} F(x^*)^T (\tilde{x}^k - x^*) \geq 0. \tag{3.22}$$

By setting $q = \beta_k F(\tilde{x}^k) - \xi^k$ in (2.4) and $y = x^*$ in (2.5), it follows from (3.11)–(3.13) that

$$((x^k - x^*) - (x^k - \tilde{x}^k))^T \Big\{(x^k - \tilde{x}^k) - \frac{\beta_k}{1+\mu} F(x^k)\Big\} + \frac{\mu}{1+\mu} \|x^k - \tilde{x}^k\|^2 \geq 0. \tag{3.23}$$

Adding (3.22) and (3.23), we get

$$\Big\{(x^k - \tilde{x}^k) - \frac{\beta_k}{1+\mu}(F(x^k) - F(x^*))\Big\}^T \{(x^k - x^*) - (x^k - \tilde{x}^k)\} + \frac{\mu}{1+\mu} \|x^k - \tilde{x}^k\|^2 \geq 0$$

and consequently

$$\begin{aligned} &\Big\{(x^k - x^*) + \frac{\beta_k}{1+\mu}(F(x^k) - F(x^*))\Big\}^T (x^k - \tilde{x}^k) \\ &\qquad \geq \frac{1}{1+\mu} \|x^k - \tilde{x}^k\|^2 + \frac{\beta_k}{1+\mu} (x^k - x^*)^T (F(x^k) - F(x^*)). \end{aligned} \tag{3.24}$$

Using the co-coercivity of F and by a simple manipulation, it follows from (3.24) that

$$\begin{aligned} (x^k - x^*)^T (x^k - \tilde{x}^k) &\geq \frac{1}{1+\mu} \{\|x^k - \tilde{x}^k\|^2 + \beta_k c \|F(x^k) - F(x^*)\|^2 \\ &\qquad - \beta_k (F(x^k) - F(x^*))^T (x^k - \tilde{x}^k)\} \\ &\geq \frac{1}{1+\mu} \{\|x^k - \tilde{x}^k\|^2 + \|\sqrt{\beta_k c}(F(x^k) - F(x^*)) \\ &\qquad - \frac{1}{2}\sqrt{\frac{\beta_k}{c}}(x^k - \tilde{x}^k)\|^2 - \frac{\beta_k}{4c} \|x^k - \tilde{x}^k\|^2\} \\ &\geq \frac{1}{1+\mu} \left(1 - \frac{\beta_k}{4c}\right) \|x^k - \tilde{x}^k\|^2. \end{aligned}$$

□

Now we consider the last two terms on the right-hand side of (3.20). We have

$$\begin{aligned} &\|x^k - x_*^k - \alpha_k D(x^k, \beta_k)\|^2 + 2\alpha_k (x_*^k - x^*)^T (x^k - \tilde{x}^k) \\ &\quad = \|x^k - x_*^k - \alpha_k D(x^k, \beta_k)\|^2 - 2\alpha_k (x^k - x_*^k - \alpha_k D(x^k, \beta_k))^T (x^k - \tilde{x}^k) \\ &\qquad - 2\alpha_k^2 D(x^k, \beta_k)^T (x^k - \tilde{x}^k) + 2\alpha_k (x^k - x^*)^T (x^k - \tilde{x}^k) \\ &\quad \geq 2\alpha_k (x^k - x^*)^T (x^k - \tilde{x}^k) - 2\alpha_k^2 D(x^k, \beta_k)^T (x^k - \tilde{x}^k) - \alpha_k^2 \|x^k - \tilde{x}^k\|^2 \\ &\quad \geq \left(\frac{2\alpha_k}{1+\mu}(1 - \frac{\beta_k}{4c}) - \alpha_k^2\right) \|x^k - \tilde{x}^k\|^2 - 2\alpha_k^2 D(x^k, \beta_k)^T (x^k - \tilde{x}^k), \end{aligned}$$

where the last inequality follows from (3.21). Then from (3.20), we obtain

$$\begin{aligned} \Theta(\alpha_k) \geq (1-\rho)\Big\{ &2\alpha_k (x^k - \tilde{x}^k)^T D(x^k, \beta_k) - \alpha_k^2 \Big(\|D(x^k, \beta_k)\|^2 \\ &+ 2D(x^k, \beta_k)^T (x^k - \tilde{x}^k)\Big) + \left(\frac{2\alpha_k}{1+\mu}\Big(1 - \mu - \frac{\beta_k}{4c}\Big) - \alpha_k^2\right) \|x^k - \tilde{x}^k\|^2 \Big\}. \end{aligned} \tag{3.25}$$

4. Convergence analysis

In this section, we prove some useful results which will be used in the consequent analysis, and then investigate the strategy of how to chose the new step size α_k. For this purpose we define,

$$\Phi(\alpha_k) := 2\alpha_k(x^k - \tilde{x}^k)^T D(x^k, \beta_k) - \alpha_k^2 \left(\|D(x^k, \beta_k)\|^2 + 2D(x^k, \beta_k)^T (x^k - \tilde{x}^k) \right).$$

Note that $\Phi(\alpha_k)$ is a quadratic function of α_k and it reaches its maximum at

$$\alpha_k^* = \frac{(x^k - \tilde{x}^k)^T D(x^k, \beta_k)}{\|D(x^k, \beta_k)\|^2 + 2D(x^k, \beta_k)^T (x^k - \tilde{x}^k)} \tag{4.1}$$

and

$$\Phi(\alpha_k^*) = \frac{((x^k - \tilde{x}^k)^T D(x^k, \beta_k))^2}{\|D(x^k, \beta_k)\|^2 + 2D(x^k, \beta_k)^T (x^k - \tilde{x}^k)} = \alpha_k^* (x^k - \tilde{x}^k)^T D(x^k, \beta_k). \tag{4.2}$$

In the next theorem we show that α_k^* and $\Phi(\alpha_k^*)$ are lower bounded away from zero, whenever $x^k \neq \tilde{x}^k$. This is one of the keys to proving the global convergence results.

4.1. Theorem. *For given $x^k \in \mathbb{R}^n_{++}$ and $\beta_k > 0$, let $\tilde{x}^k$ and ξ^k satisfy the condition* (3.3). *Then we have the following:*

$$\alpha_k^* \geq \frac{1}{4} \tag{4.3}$$

and

$$\Phi(\alpha_k^*) \geq \frac{1}{4} \left(\frac{1-\eta}{1+\mu} \right) \|x^k - \tilde{x}^k\|^2. \tag{4.4}$$

Proof. It follows from (3.6) and $0 < \mu < 1$ that

$$\begin{aligned}
(x^k - \tilde{x}^k)^T D(x^k, \beta_k) &= \|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T \xi^k \\
&\geq \frac{1}{1+\mu}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T \xi^k \\
&\geq \left(\frac{1-\eta}{1+\mu} \right) \|x^k - \tilde{x}^k\|^2. \qquad (4.5)
\end{aligned}$$

From $0 < \mu < 1, 0 < \eta < 1$ and (3.3), we have

$$\begin{aligned}
(x^k - \tilde{x}^k)^T D(x^k, \beta_k) &= \|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T \xi^k \\
&= \frac{1}{2}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T \xi^k \\
&\qquad + \frac{1}{2}\|x^k - \tilde{x}^k\|^2 \\
&\geq \frac{1}{2}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T \xi^k \\
&\qquad + \frac{1}{2(1+\mu)^2}\|\xi^k\|^2 \\
&= \frac{1}{2}\|D(x^k, \beta_k)\|^2
\end{aligned}$$

and thus

$$\alpha_k^* = \frac{(x^k - \tilde{x}^k)^T D(x^k, \beta_k)}{\|D(x^k, \beta_k)\|^2 + 2D(x^k, \beta_k)^T (x^k - \tilde{x}^k)} \geq \frac{1}{4}.$$

Using (4.2), (4.3) and (4.5) directly we obtain (4.4). □

Let β_k satisfy

$$0 < \beta_l \leq \inf_{k=0}^{\infty} \beta_k \leq \sup_{k=0}^{\infty} \beta_k \leq \beta_u < 4c(1-\mu).$$

We take the new step size as

$$\alpha_k^{**} = \min\left\{ \frac{(1-\mu-\frac{\beta_k}{4c})}{1+\mu}, \alpha_k^* \right\}.$$

For fast convergence, we take a relaxation factor $\gamma \in [1,2)$ and set the step-size α_k in (3.4) as $\alpha_k = \gamma\alpha_k^{**}$.

If $\alpha_k = \gamma\frac{(1-\mu-\frac{\beta_k}{4c})}{1+\mu}$, it follows from (3.25) that

$$\Theta(\alpha_k) \geq \gamma(2-\gamma)(1-\rho)\frac{(1-\mu-\frac{\beta_u}{4c})^2}{(1+\mu)^2}\|x^k - \tilde{x}^k\|^2. \tag{4.6}$$

If $\alpha_k = \gamma\alpha_k^*$, it follows from (3.25) and Theorem 4.1 that

$$\begin{aligned} \Theta(\alpha_k) &\geq \gamma\alpha_k^*(1-\rho)\{2(x^k-\tilde{x}^k)^T D(x^k,\beta_k) \qquad (4.7)\\ &\qquad - \gamma\alpha_k^*\big(\|D(x^k,\beta_k)\|^2 + 2D(x^k,\beta_k)^T(x^k-\tilde{x}^k)\big)\}\\ &\geq \gamma(2-\gamma)(1-\rho)\alpha_k^*(x^k-\tilde{x}^k)^T D(x^k,\beta_k)\\ &\geq \gamma(2-\gamma)(1-\rho)\frac{(1-\eta)}{4(1+\mu)}\|x^k-\tilde{x}^k\|^2. \qquad (4.8) \end{aligned}$$

Then from Theorem 3.4, (4.6) and (4.7), there is a constant

$$\tau := \gamma(2-\gamma)(1-\rho)\min\left\{ \frac{(1-\mu-\frac{\beta_u}{4c})^2}{(1+\mu)^2}, \frac{(1-\eta)}{4(1+\mu)} \right\} > 0$$

such that

$$\|x^{k+1}-x^*\|^2 \leq \|x^k-x^*\|^2 - \tau\|x^k-\tilde{x}^k\|^2 \ \forall x^* \in \Omega^*. \tag{4.9}$$

The following result can be proved by arguments similar to those in [17, 33]. Hence the proof will be omitted.

4.2. Theorem. [5, 17, 33] *If* $\inf_{k=0}^{\infty}\beta_k = \beta_l > 0$, *then the sequence* $\{x^k\}$ *generated by the proposed method converges to some* x^∞ *which is a solution of NCP.*

The detailed algorithm is as follows.

Step 0. Let $\beta_0 = 1, \eta(:= 0.9) < 1, 0 < \rho < 1, \mu = 0.1, \gamma = 1.9, \epsilon = 10^{-7}, k = 0$ and $x^0 > 0$.

Step 1. If $\|\min(x,F(x))\|_\infty \leq \epsilon$, then stop. Otherwise, go to Step 2.

Step 2. (Prediction step)

$s := (1-\mu)x^k - \beta_k F(x^k)$, $\quad \tilde{x}_i^k := \big(s_i + \sqrt{(s_i)^2 + 4\mu(x_i^k)^2}\big)/2$,

$\xi^k := \beta_k\big(F(\tilde{x}^k) - F(x^k)\big)$, $\quad r := \|\xi^k\|/\|x^k - \tilde{x}^k\|$.

while $(r > \eta)$

$\beta_k := \beta_k * 0.8/r$,

$s := (1-\mu)x^k - \beta_k F(x^k)$, $\quad \tilde{x}_i^k := \big(s_i + \sqrt{(s_i)^2 + 4\mu(x_i^k)^2}\big)/2$,

$\xi^k := \beta_k\big(F(\tilde{x}^k) - F(x^k)\big)$, $\quad r := \|\xi^k\|/\|x^k - \tilde{x}^k\|$.

end while

Step 3. (correction step)

$$D(x^k,\beta_k) := (x^k - \tilde{x}^k) + \frac{1}{1+\mu}\xi^k, \qquad d(x^k,\beta_k) := (x^k - \tilde{x}^k) + \frac{\beta_k}{1+\mu}F(\tilde{x}^k),$$

$$\alpha_k^* = \frac{(x^k - \tilde{x}^k)^T D(x^k,\beta_k)}{\|D(x^k,\beta_k)\|^2 + 2D(x^k,\beta_k)^T(x^k - \tilde{x}^k)}, \quad \alpha_k = \gamma \min\left\{\frac{\left(1-\mu-\frac{\beta_k}{4c}\right)}{(1+\mu)}, \alpha_k^*\right\},$$

$$x^{k+1} = \rho x^k + (1-\rho)P_{\mathbb{R}^n_+}\left[x^k - \alpha_k d(x^k,\beta_k)\right],$$

Step 4.

$$\beta_{k+1} = \begin{cases} \frac{\beta_k * 0.7}{r}, & \text{if} \quad r \le 0.3; \\ \beta_k, & \text{otherwise.} \end{cases}$$

Step 5.

$$k := k+1; \text{ go to Step 1.}$$

5. Preliminary computational results

In this section, we consider two examples to illustrate the efficiency and performance of the proposed algorithm.

5.1. Numerical experiments I. We consider the nonlinear complementarity problems:

$$x \ge 0, \; F(x) \ge 0, \; x^T F(x) = 0, \tag{5.1}$$

where

$$F(x) = D(x) + Mx + q,$$

so $D(x)$ and $Mx+q$ are the nonlinear part and linear part of $F(x)$ respectively.

We form the linear part in the test problems similarly as in Harker and Pang [16]. The matrix $M = A^TA + B$, where A is an $n \times n$ matrix whose entries are randomly generated in the interval $(-5,+5)$ and a skew-symmetric matrix B is generated in the same way. The vector q is generated from a uniform distribution in the interval $(-500,500)$ or in $(-500,0)$. In $D(x)$, the nonlinear part of $F(x)$, the components are chosen to be $D_j(x) = d_j * \arctan(x_j)$, where d_j is a random variable in $(0,1)$. A similar type of problems was tested in [20] and [31].

In all the tests we take the logarithmic proximal parameter $\mu = 0.01$, $\rho = 0.01$ and $c = 0.9$. All iterations start with $x^0 = (1,\dots,1)^T$ and $\beta_0 = 1$, and are stopped whenever

$$\|\min(x^k, F(x^k))\|_\infty \le 10^{-7}.$$

All the code was written in Matlab, and we compare the proposed method with that in [33]. The number k of iterations, the number l of evaluations of the mapping F and the computation time for the problem (5.1) with different dimensions are given in the following tables.

Table 1. Numerical results for problem (5.1) with $q \in (-500,\ 500)$

n	The proposed method			The method in [33]		
	k	l	CPU(Sec.)	k	l	CPU(Sec.)
200	297	651	0.34	395	822	0.42
300	329	707	1.28	435	902	1.61
400	334	724	1.61	446	929	1.99
500	366	797	3.31	487	1012	4.29
700	364	751	5.71	466	968	7.27
1000	338	746	9.57	454	943	11.79

Table 2. Numerical results for problem (5.1) with $q \in (-500,\ 0)$

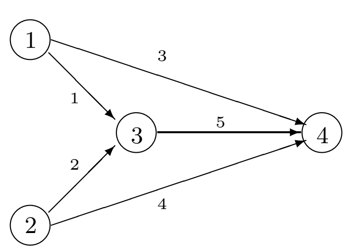
n	The proposed method			The method in [33]		
	k	l	CPU(Sec.)	k	l	CPU(Sec.)
200	581	1242	0.67	781	1618	0.86
300	588	1260	2.25	777	1613	2.89
400	767	1594	3.57	998	2068	4.41
500	835	1759	7.24	1076	2229	9.68
700	701	1499	11.28	923	1909	14.52
1000	814	1716	19.91	1056	2187	25.62

Tables 1 and 2 show that the proposed method is more efficient. The numerical results indicate that the proposed method can save about 21 $\sim$ 25 percent of the number of iterations and about 20 $\sim$ 23 of the amount of computing of the value of the function F.

5.2. Numerical experiments II. In this subsection, we apply the proposed method to traffic equilibrium problems and present corresponding numerical results.

Consider a network $[N, L]$ of nodes N and directed links L, which consists of a finite sequence of connecting links with a certain orientation. Let a, b, etc., denote the links, and let p, q, etc., denote the paths. We let ω denote an origin/destination (O/D) pair of nodes of the network and P_ω the set of all paths connecting the O/D pair ω. Note that the path-arc incidence matrix and the path-O/D pair incidence matrix, denoted by A and B, respectively, are determined by the given network and O/D pairs. To see how to convert a traffic equilibrium problem into a variational inequality, we take into account the simple example depicted in Figure 1.

Figure 1. An illustrative example of a directed network and the O/D pairs



O/D Pairs	Path No. & the link on the path
ω_1: ①→ ④	$p_1 = \{3\}$ $p_2 = \{1, 5\}$
ω_2: ②→ ④	$p_3 = \{4\}$ $p_4 = \{2, 5\}$

For the given example in Figure 1, the path-arc incidence matrix A and the path-O/D pair incidence matrix B have the following forms:

$$A = \begin{array}{c} \text{Link No.} \quad 1\ \ 2\ \ 3\ \ 4\ \ 5 \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}, \qquad B = \begin{array}{c} \text{O/D pair No.} \quad \omega_1\ \ \omega_2 \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \end{array}.$$

Let x_p represent the traffic flow on path p and f_a the link load on link a. Then the arc-flow vector f is given by

$$f = A^T x.$$

Let d_ω denote the amount of traffic between the O/D pair ω, which must satisfy

$$d_\omega = \sum_{p \in P_\omega} x_p.$$

Thus, the O/D pair-traffic amount vector d is given by

$$d = B^T x.$$

Let $t(f) = \{t_a, a \in L\}$ be the vector of link travel costs, which is a function of the link flow. A user traveling on path p incurs a (path) travel cost θ_p. For a given link travel cost vector t, the path travel cost vector θ is given by

$$\theta = At(f) \text{ and thus } \theta(x) = At(A^T x).$$

Associated with every O/D pair ω, there is a travel disutility $\lambda_\omega(d)$. Since both the path costs and the travel disutilities are functions of the flow pattern x, the traffic network equilibrium problem is to seek the path flow pattern x^* such that

$$x^* \geq 0,\ (x - x^*)^T F(x^*) \geq 0,\ \forall x \geq 0 \tag{5.2}$$

where

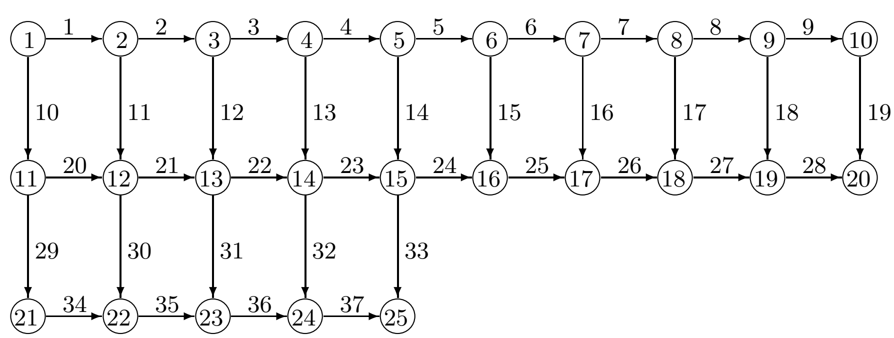
$$F_p(x) = \theta_p(x) - \lambda_\omega(d(x)),\ \forall \omega,\ p \in P_\omega,$$

and thus

$$F(x) = At(A^T x) - B\lambda(B^T x).$$

We apply the proposed method to [23, Example 7.5]), which consists of 25 nodes, 37 links and 6 O/D pairs. The network is depicted in Figure 2.

Figure 2. A directed network with 25 nodes and 37 links



For this example, there are together 55 paths for the 6 given O/D pairs and hence the dimension of the variable x is 55. Therefore, the path-arc incidence matrix A is a 55×37 matrix and the path-O/D pair incidence matrix B a 55×6 matrix. The user cost of traversing link a is given in Table 3. The disutility function is given by

$$\lambda_\omega(d) = -m_\omega d_\omega + q_\omega, \tag{5.3}$$

and the coefficients m_ω and q_ω in the disutility function of different O/D pairs for this example are given in Table 4.

The test results for problems (5.2) for different ε are reported in Table 5, where k is the number of iterations and l the number of evaluations of the mapping F. The stopping criterion is

$$\frac{\|\min\{x, F(x)\}\|_\infty}{\|\min\{x^0, F(x^0)\}\|_\infty} \le \varepsilon.$$

Table 3. The link traversing cost functions $t_a(f)$

$t_1(f) = 5 \cdot 10^{-5} f_1^4 + 5f_1 + 2f_2 + 500$	$t_{20}(f) = 3 \cdot 10^{-5} f_{20}^4 + 6f_{20} + f_{21} + 300$
$t_2(f) = 3 \cdot 10^{-5} f_2^4 + 4f_2 + 4f_1 + 200$	$t_{21}(f) = 4 \cdot 10^{-5} f_{21}^4 + 4f_{21} + f_{22} + 400$
$t_3(f) = 5 \cdot 10^{-5} f_3^4 + 3f_3 + f_4 + 350$	$t_{22}(f) = 2 \cdot 10^{-5} f_{22}^4 + 6f_{22} + f_{23} + 500$
$t_4(f) = 3 \cdot 10^{-5} f_4^4 + 6f_4 + 3f_5 + 400$	$t_{23}(f) = 3 \cdot 10^{-5} f_{23}^4 + 9f_{23} + 2f_{24} + 350$
$t_5(f) = 6 \cdot 10^{-5} f_5^4 + 6f_5 + 4f_6 + 600$	$t_{24}(f) = 2 \cdot 10^{-5} f_{24}^4 + 8f_{24} + f_{25} + 400$
$t_6(f) = 7f_6 + 3f_7 + 500$	$t_{25}(f) = 3 \cdot 10^{-5} f_{25}^4 + 9f_{25} + 3f_{26} + 450$
$t_7(f) = 8 \cdot 10^{-5} f_7^4 + 8f_7 + 2f_8 + 400$	$t_{26}(f) = 6 \cdot 10^{-5} f_{26}^4 + 7f_{26} + 8f_{27} + 300$
$t_8(f) = 4 \cdot 10^{-5} f_8^4 + 5f_8 + 2f_9 + 650$	$t_{27}(f) = 3 \cdot 10^{-5} f_{27}^4 + 8f_{27} + 3f_{28} + 500$
$t_9(f) = 10^{-5} f_9^4 + 6f_9 + 2f_10 + 700$	$t_{28}(f) = 3 \cdot 10^{-5} f_{28}^4 + 7f_{28} + 650$
$t_{10}(f) = 4f_{10} + f_{12} + 800$	$t_{29}(f) = 3 \cdot 10^{-5} f_{29}^4 + 3f_{29} + f_{30} + 450$
$t_{11}(f) = 7 \cdot 10^{-5} f_{11}^4 + 7f_{11} + 4f_{12} + 650$	$t_{30}(f) = 4 \cdot 10^{-5} f_{30}^4 + 7f_{30} + 2f_{31} + 600$
$t_{12}(f) = 8f_{12} + 2f_{13} + 700$	$t_{31}(f) = 3 \cdot 10^{-5} f_{31}^4 + 8f_{31} + f_{32} + 750$
$t_{13}(f) = 10^{-5} f_{13}^4 + 7f_{13} + 3f_{18} + 600$	$t_{32}(f) = 6 \cdot 10^{-5} f_{32}^4 + 8f_{32} + 3f_{33} + 650$
$t_{14}(f) = 8f_{14} + 3f_{15} + 500$	$t_{33}(f) = 4 \cdot 10^{-5} f_{33}^4 + 9f_{33} + 2f_{31} + 750$
$t_{15}(f) = 3 \cdot 10^{-5} f_{15}^4 + 9f_{15} + 2f_{14} + 200$	$t_{34}(f) = 6 \cdot 10^{-5} f_{34}^4 + 7f_{34} + 3f_{30} + 550$
$t_{16}(f) = 8f_{16} + 5f_{12} + 300$	$t_{35}(f) = 3 \cdot 10^{-5} f_{35}^4 + 8f_{35} + 3f_{32} + 600$
$t_{17}(f) = 3 \cdot 10^{-5} f_{17}^4 + 7f_{17} + 2f_{15} + 450$	$t_{36}(f) = 2 \cdot 10^{-5} f_{36}^4 + 8f_{36} + 4f_{31} + 750$
$t_{18}(f) = 5f_{18} + f_{16} + 300$	$t_{37}(f) = 6 \cdot 10^{-5} f_{37}^4 + 5f_{37} + f_{36} + 350$
$t_{19}(f) = 8f_{19} + 3f_{17} + 600$	

Table 4. The O/D pairs and the parameters in (5.3)

(O,D) Pair ω	(1,20)	(1,25)	(2,20)	(3,25)	(1,24)	(11,25)
m_ω	1	6	10	5	7	9
q_ω	1000	800	2000	6000	8000	7000
$\|P_\omega\|$	10	15	9	6	10	5

Table 5. Numerical results for different values of ε

Different	The proposed method			The method in [33]		
ε	k	l	CPU(Sec.)	k	l	CPU(Sec.)
10^{-5}	198	439	0.016	401	864	0.109
10^{-6}	262	574	0.031	519	1119	0.141
10^{-7}	330	710	0.015	643	1386	0.141
10^{-8}	393	843	0.017	757	1632	0.156
10^{-9}	455	979	0.031	877	1892	0.204

Table 5 shows that the new method is more flexible and efficient in solving traffic equilibrium problem. Moreover, it demonstrates computationally that the new method is more effective than the method presented in [33] in the sense that the new method needs fewer iteration and a smaller number of evaluations of F, which clearly illustrate its efficiency.

6. Concluding remarks and outlook

We would like to mention that the ideas and techniques of this paper can be extended to solve the more general complementarity problem of finding a vector $x \in \mathbb{R}^n$ such that

$$g(x) \geq 0, \ \ F(x) \geq 0 \text{ and } (g(x))^T F(x) = 0.$$

Here g and F are nonlinear from $\mathbb{R}^n$ into itself. This problem was introduced by Noor [24] in 1988. Note that, for $g = I$ this problem is exactly (1.1). It is now well known that a class of odd-order and nonsymmetric obstacle can be studied via the general complementarity problems. See, for example, Noor [25, 27] and the reference therein for applications, numerical methods and other aspects of general complementarity problems. In spite of their importance, very few methods have been developed for solving these problems.

It is an interesting problem to extend the idea and technique of this paper to develop a proximal point method for solving these general complementarity problems and this is another direction for future research activities.

In passing, we would like to remark that this problem can be formulated as an equivalent variational inclusion problem under suitable conditions. This equivalent formulation may stimulate further research in this dynamic field. Interested readers are encouraged to discover new methods and applications of these complementarity problems in different areas of pure and applied sciences.

Acknowledgments.

The authors would like to thank the referee for his/her very constructive comments and suggestions.

This research was supported partly by NSFC Grants Nos: 70731002 and 70571034, "The Second Term '985' Engineering-Innovation Center for Computational Laboratory of Evolutionary Economic Theory and Its Applications" and "Study on the Evolution of Complex Economic System" at "Innovation Center of Economic Transition and Development of Nanjing University" of the Ministry of Education, China.

The research of Prof. M. Aslam Noor is supported by the Visiting Professor Program of King Saud University, Riyadh, Saudi Arabia and Grant No: KSU. VPP. 108.

References

[1] Auslender, A. *Optimization Méthodes Numériques* (Mason, Paris, 1976).

[2] Auslender, A. and Teboule, M. *Interior projection-like methods for monotone variational inequalities*, Math. Prog., Ser. A **104**, 39–68, 2005.

[3] Auslender, A., Teboulle, M. and Ben-Tiba, S. *A logarithmic-quadratic proximal method for variational inequalities*, Comput. Optim. Appl. **12**, 31–40, 1999.

[4] Auslender, A., Teboulle, M. and Ben-Tiba, S. *Interior proximal and multiplier methods based on second order homogenous kernels*, Math. Oper. Res. **24**, 646–668, 1999.

[5] Bnouhachem, A. *An LQP method for pseudomonotone variational inequalities*, J. Global Optim. **36** (3), 351–363, 2006.

[6] Bnouhachem, A. and Noor, M. A. *A new predictor-corrector method for pseudomonotone nonlinear complementarity problems*, Inter. J. Compt. Math. **85**, 1023–1038, 2008.

[7] Bnouhachem, A., Noor, M. A., Khalfaoui, M. and Zhaohan, S. *A new logarithmic-quadratic proximal method for nonlinear complementarity problems*, Appl. Math. Comput. **215**, 695–706, 2009.

[8] Bnouhachem, A. and Noor, M. A. *An interior proximal point algorithm for nonlinear complementarity problems*, Nonlinear Analysis: Hybrid Systems **4** (3), 371–380, 2010.

[9] Burachik, R. S. and Iusem, A. N. *A generalized proximal point algorithm for the variational inequality problem in a Hilbert Space*, SIAM J. Optim. **8**, 197–216, 1998.

[10] Burachik, R. S. and Svaiter, B. F. *A relative error tolerance for a family of generalized proximal point methods*, Math. Oper. Res. **26** (4), 816–831, 2001.

[11] Cottle, R. W. and Dantzig, G. B. *Complemenatriy pivot theory of mathematical programming*, Linear Algbra Appl. **1**, 103–125, 1968.

[12] Eckestein, J. *Approximate iterations in Bregman-function-based proximal algorithms*, Math. Prog. **83**, 113–123, 1998.

[13] Ferris, M. C. and Pang, J. S. *Engineering and economic applications of complementariry problems*, SIAM Review. **39**, 669–713, 1997.

[14] Fischer, A. *Solution of monotone complementarity problems with locally Lipschitzian functions*, Math. Prog. **76**, 513–532, 1997.

[15] Han, D. *A new hybrid generalized proximal point algorithm for variational inequality problems*, J. Global Optim. **26**, 125–140, 2003.

[16] Harker, P. T. and Pang, J. S. *A damped-Newton method for the linear complementarity problem* (Lectures in Applied Mathematics **26**, 1990), 265–284.

[17] He, B. S., Liao, L.-Z. and Yuan, X. M. *A LQP based interior prediction-correction method for nonlinear complementarity problems*, Journal of Computational Mathematics **24** (1), 33–44, 2006.

[18] Kaplan, A. and Tichatschke, R. *On inexact generalized proximal method with a weakned error tolerance criterion*, Optimization **53** (1), 3–17, 2004.

[19] Lemke, C. E. *Bimatrix equilibrium point and mathematical programming*, Management Sci. **11**, 681–689, 1965.

[20] Marcotte, P. and Dussault, J. P. *A Note on a globally convergent Newton method for solving variational inequalities*, Operation Research Letters **6**, 35–42, 1987.

[21] Martinet, B. *Regularization d'inequations variationnelles par approximations successives*, Revue Francaise d'Informatique et de Recherche Operationelle **4**, 154–159, 1970.

[22] Moré, J. J. *Global methods for nonlinear complementarity problems*, Math. Oper. Res. **21**, 589–614, 1996.

[23] Nagurney, A. and Zhang, D. *Projected Dnamical Systems and Variational Inequalities with Applications* (Kluwer Academic Publishers, Boston, Dordrecht, London, 1996).

[24] Noor, M. A. *General variational inequalities*, Appl. Math. Letters **1**, 119–121, 1988.

[25] Noor, M. A. *Some developments in general variational inequalities*, Appl. Math. Comput. **152**, 199–277, 2004.

[26] Noor, M. A. and Bnouhachem, B. *Modified proximal point methods for nonlinear complementarity problems*, J. Comput. Appl. Math. **197**, 395–405, 2006.

[27] Noor, M. A. *Extended general variational inequalities*, Appl. Math. Letters **22**, 182–185, 2009.

[28] Rockafellar, R. T. *Augmented Lagrangians and applications of the proximal point algorithm in convex programming*, Math. Oper. Res. **1**, 97–116, 1976.

[29] Rockafellar, R. T. *Monotone operators and the proximal point algorithm*, SIAM J. Control. Optim. **14**, 877–898, 1976.

[30] Solodov, M. V. and Svaiter, B. F. *An inexact hybrid generalized proximal point algorithms and some new results on the theory of Bregman functions*, Math. Oper. Res. **25**, 214–230, 2000.

[31] Taji, K., Fukushima, M. and Ibaraki, T. *A globally convergent Newton method for solving strongly monotone variational inequalities*, Math. Prog. **58**, 369–383, 1993.

[32] Teboulle, M. *Convergence of proximal-like algorithms*, SIAM J. Optim. **7**, 1069–1083, 1997.

[33] Xu, Y., He, B. S. and Yuan, X. *A hybrid inexact logarithmic-quadratic proximal method for nolinear complementarity problems*, J. Math. Anal. Appl. **322**, 276–287, 2006.

[34] Yan, X., Han, D. and Sun, W. *A self-adaptive projection method with improved step-size for solving variational inequalities*, Computers & Mathematics with Applications **55**, 819–832, 2008.