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ON THE MEAN RESIDUAL LIFETIME AT SYSTEM LEVEL IN TWO-COMPONENT PARALLEL SYSTEMS FOR THE FGM DISTRIBUTION

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Abstract

In this paper, we consider the mean residual lifetime of two-component parallel systems in the case of possibly dependent components. We provide some results and examine the behavior of the mean residual life function at the system level for the bivariate Farlie-Gumbel-Morgenstern (FGM) distribution. Applications of these results to evaluate the relationship between the mean residual life and that of the dependence of components are also given.

Keywords: Mean residual life function, Parallel system, FGM distribution, Copula.

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1. Introduction

System or component reliability is defined as the probability that the system or component is able to work during a given time period. One of the usual assumptions in reliability theory is that the failure of one component does not influence the remaining ones. Generally, this assumption is not satisfied in some situations in which a component failure can effect the remaining components. For example, the failure of an engine may increase the stress on the remaining engine in two-engine planes. Failure modes such as electronic discharges or the performance of a person's paired organs are more likely to affect the neighboring component at that time. Therefore some dependence among the system components must be taken into account.

The reliability properties of the system are affected by the dependence structure of the component lifetimes as well as by the design of the system. If high reliability is

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required for a system, the components should be designed in a parallel structure. A parallel system, consisting of n components, functions if and only if at least one of its n components function. If the random vector $X_1, X_2, \ldots, X_n$ gives the ordered lifetimes of the components, then the lifetime of the parallel system is represented by the last order statistic $X_{n:n}$. When we assume that $X_1, X_2, \ldots, X_n$ are exchangeable, the survival function of the parallel system at time $t > 0$ is given by

$$S_{n:n}(t) = P(X_{n:n} > t) = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} P(X_{1:j} > t), \tag{1.1}$$

for which $X_{1:j} = \min(X_1, \ldots, X_j)$, $1 \le j \le n$, (see David and Nagaraja [5]).

For some results on systems taking into account the dependence among the lifetimes of the components, see e.g. Bairamov and Parsi [4], Gurler [8], Navarro, Ruiz and Sandoval [15], Navarro and Rychlik [16], Navarro, Spizzichino and Balakrishnan [18] and additional results can be found in Li and Zhang [11], Navarro and Lai [14], Navarro and Spizzichino [17], Zhao and Balakrishnan [22] for the ordering properties of systems with dependent components.

If the conditional random variable $(X_{n:n} - t | X_{n:n} > t)$ is defined as the residual lifetime of a parallel system, the usual mean residual life (MRL) function of the system $\Phi_{n:n}(t)$ is

$$\begin{aligned} \Phi_{n:n}(t) &= E(X_{n:n} - t | X_{n:n} > t) \\ &= \frac{1}{S_{n:n}(t)} \int_0^\infty S_{n:n}(x + t)dx, \text{ for } x,\ t > 0, \end{aligned} \tag{1.2}$$

where $S_{n:n}$ denotes the survival function of $X_{n:n}$. Recently, numerous papers have appeared on the MRL of systems. See, for examples of such results and other references Asadi and Bairamov [1], Bairamov et al. [2], Poursaeed [20] and Sadegh [21].

For some parallel systems consisting of n components, the operator may consider some maintenance or replacement procedure when all the components are working. The conditional random variable $(X_{n:n} - t | X_{1:n} > t)$ represents the residual lifetime at the nth component level of the system under the condition that all components are working.

Let X_1 and X_2 denote the ordered lifetimes of two components in a parallel system with marginal distribution and survival functions, F_i, $\overline{F}_i$, $i = 1, 2$, respectively. Then the survival function of $(X_{2:2} - t)$ given that $X_{1:2} > t$ at time $t > 0$ is

$$\begin{aligned} S_{1:2}(x|t) &= P(X_{2:2} - t > x | X_{1:2} > t) \\ &= \frac{2\overline{F}(x + t, t) - \overline{F}(x + t, x + t)}{\overline{F}(t, t)}. \end{aligned} \tag{1.3}$$

Bairamov *et al.* [2] have given a definition of the MRL function for the parallel system consisting of n independent and identical components and have obtained some of their properties at system level. If we assume that the lifetimes of the components are exchangeable in a two-component parallel system, then we can define the MRL function as given below.

1.1. Definition. If $S_{1:2}(x|t)$ represents the conditional survival function of a two-component parallel system, then the MRL at the system level can be defined as

$$\Phi_{1:2}(t) = \int_0^\infty S_{1:2}(x|t)dx, \text{ for } t > 0. \tag{1.4}$$

The MRL at the system level in (1.4) can help an engineer or an operator to improve the design of the system and it may allow the taking of some precautions. Hence, Definition 1.1 is more beneficial than the MRL defined in (1.2) for some situations. In this paper, we focus on the MRL function at the system level for two-component parallel systems where the lifetimes of the components follow the FGM distribution. In the next section, we present the MRL for a parallel system consisting of two exponential components and give some results on how to change the MRL of the system with the chosen dependence parameter. Also, we provide the MRL for the FGM distribution with arbitrary marginals for the given structure.

2. MRL of two-component parallel systems with FGM distribution

The FGM family is a well-known family of bivariate dependent variables. It provides a way of modeling the dependence between variables. The FGM family of distributions was first introduced by Morgenstern [12], then Gumbel [7] investigated its structure and Farlie [6] generalized the FGM family of distributions. Johnson and Kotz [10] studied the multivariate case. Huang and Kotz [9] and Bairamov and Kotz [3] extended the bivariate FGM distribution and provided some new properties.

2.1. Definition. If $\overline{F}_{1,2}(x_1,x_2)$ is the bivariate FGM survival function of the random vector (X_1, X_2), then

$$\overline{F}_{1,2}(x_1,x_2)=\overline{F}_1(x_1)\overline{F}_2(x_2)\left[1+\alpha(1-\overline{F}_1(x_1))(1-\overline{F}_2(x_2))\right], \tag{2.1}$$

where $\overline{F}_i(x_i)=1-F_i(x_i)$, $i=1,2$ and $|\alpha|<1$ is the dependence parameter.

Let X_1 and X_2 be the lifetimes of the components having the bivariate FGM distribution with a common marginal distribution function $F(x)$ and survival function $\overline{F}(x)$. If we denote the MRL function at the system level for a two-component parallel system as $\Phi_{1:2}(t)$, then from (1.4), we have

$$\begin{aligned}\Phi_{1:2}(t)=&\frac{1}{\overline{F}^2(t)\left[1+\alpha(1-\overline{F}(t))^2\right]}\\ &\times\int_0^\infty\Big(2\overline{F}(x+t)\overline{F}(t)\left[1+\alpha(1-\overline{F}(x+t))(1-\overline{F}(t))\right]\\ &-\overline{F}^2(x+t)\left[1+\alpha(1-\overline{F}(x+t))^2\right]\Big)\,dx.\end{aligned} \tag{2.2}$$

2.1. FGM distribution with exponential marginals. The following proposition is given for the MRL at the system level of exponentially distributed components having bivariate FGM distribution with the parameter α.

2.2. Proposition. *Let X_1 and X_2 have bivariate FGM distribution with parameter α and let $\Phi_{1:2}(t)$ be the MRL function of a parallel system consisting of two exponential components having common survival function $\overline{F}(x)$ with parameter $\lambda>0$. Then, for $t>0$,*

$$\Phi_{1:2}(t)=\frac{18(1+\alpha)-28\alpha\overline{F}(t)+9\alpha\overline{F}^2(t)}{12\lambda(1+\alpha-2\alpha\overline{F}(t)+\alpha\overline{F}^2(t))}. \tag{2.3}$$

Proof. The proof can be obtained easily from (2.2). □

The shape of the MRL in (2.3) will depend on the dependence parameter α. Let $t^* > 0$ be a change point obtained by solving $\Phi'_{1:2}(t) = 0$. Then t^* can be obtained as,

$$t^* = -\frac{1}{\lambda} \ln \left(\frac{9(1+\alpha) - \sqrt{81 + 82\alpha + \alpha^2}}{10\alpha} \right). \quad (2.4)$$

From (2.4), the following two cases can be considered. For $\alpha < 0$, $t < t^*$, $\Phi'_{1:2}(t) < 0$, MRL is decreasing; when $t > t^*$, $\Phi'_{1:2}(t) > 0$, MRL is increasing. For the second case $\alpha > 0$, $t < t^*$, $\Phi'_{1:2}(t) > 0$, MRL is increasing; when $t > t^*$, $\Phi'_{1:2}(t) < 0$, MRL is decreasing.

For large values of t, $\Phi_{1:2}(t)$ in (2.3) is constant for negative and positive values of α, that is

$$\lim_{t \to \infty} \Phi_{1:2}(t) = \frac{3}{2\lambda}.$$

This is the MRL value in the case of independence, i.e. $\alpha = 0$, (see Navarro and Hernandez [13]). These results can also be seen from Figure 1 and Figure 2.

Figure 1. MRL curves of the two-component parallel system for FGM distribution with exponential marginals ($\lambda = 1$)

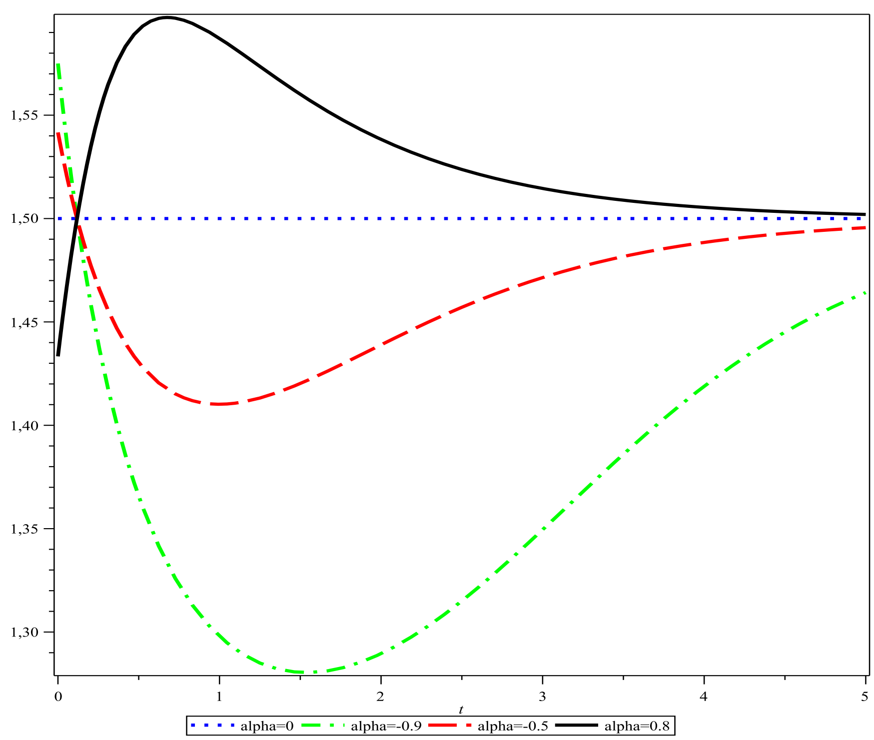
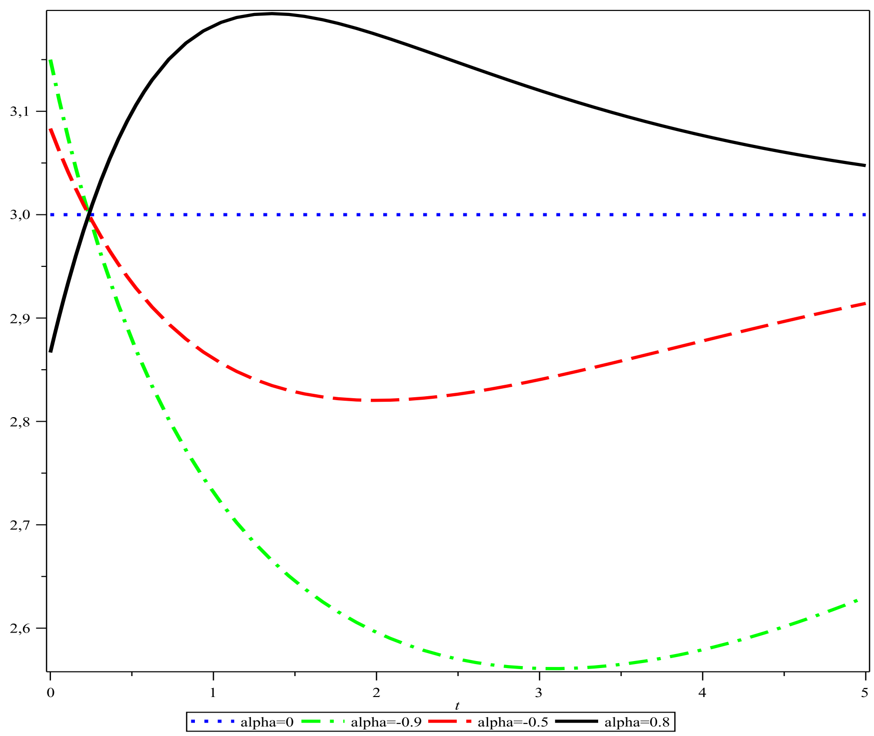


Figure 2. MRL curves of the two-component parallel system for FGM distribution with exponential marginals ($\lambda = 0.5$)



2.2. The bivariate copula case. The multivariate distribution of the component lifetimes plays an important role in the reliability of a system structure. A copula is defined as the multivariate distribution function with uniform marginals in $[0,1]$. From Sklar's theorem, the bivariate survival function is defined as

$$\bar{F}(x_1, x_2) = \tilde{C}(\bar{F}_1(x_1), \bar{F}_2(x_2)), \tag{2.5}$$

where $\tilde{C}$ is the survival copula (see, Nelsen [19]).

A bivariate survival copula is the distribution function of the random vector $(\bar{F}_1(x_1), \bar{F}_2(x_2))$, where $\bar{F}_i(x_i)$, $i = 1, 2$ is the marginal reliability function of X_i. For a two-component parallel system, the survival function of $(X_{2:2} - t | X_{1:2} > t)$ can be represented in terms of the survival copula as

$$S_{\tilde{C}}(x,t) = \frac{2\tilde{C}(\bar{F}_1(x+t), \bar{F}_2(t)) - \tilde{C}(\bar{F}_1(x+t), \bar{F}_2(x+t))}{\tilde{C}(\bar{F}_1(t), \bar{F}_2(t))}. \tag{2.6}$$

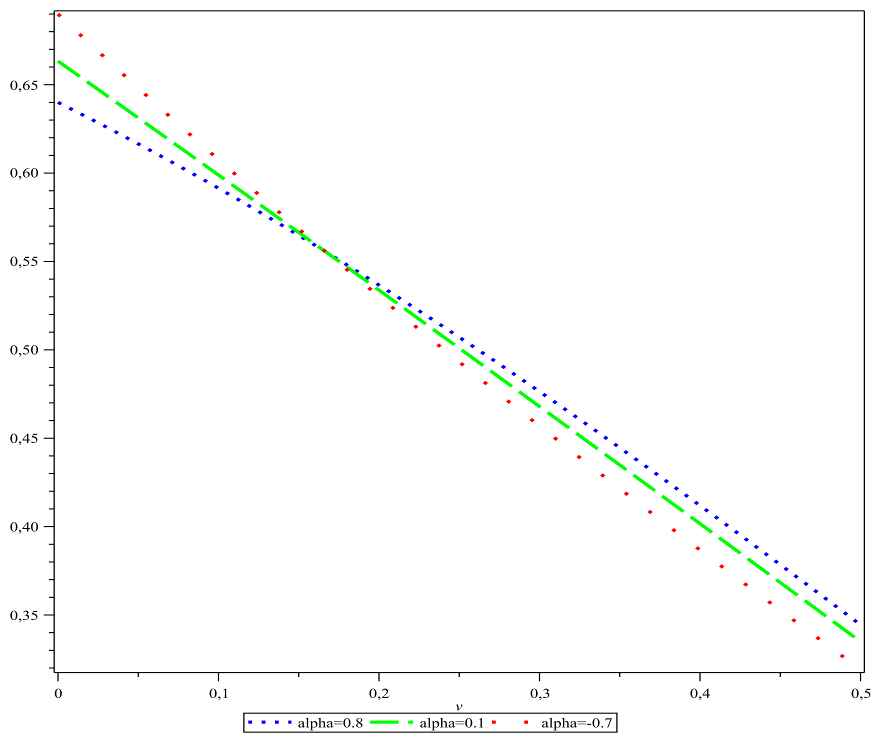
An alternative expression for the MRL function at system level in the case of the FGM distribution can be obtained by a transformation $U_1 = F_1(x)$ and $U_2 = F_2(x)$, where F_i,

$i = 1, 2$ is an arbitrary continuous marginal distribution function. Then, we have

$$\Phi_{1:2}(u_2) = \int_{u_2}^{1} S_{\tilde{C}}(u_1, u_2)\, du_1$$
$$= \frac{\alpha u_2 \left(-14u_2^2 + 7u_2 + 8\right) - 20(u_2 - 1) - \alpha}{30(1 + \alpha u_2^2)}, \text{ for } u_1, u_2 \in [0, 1].$$

Since $\Phi_{1:2}(u_2)$ is continuous on the interval $[0, 1]$ and $\Phi'_{1:2}(u_2)$ exists and is negative, we conclude that $\Phi_{1:2}(u_2)$ is decreasing on $[0, 1]$, at every level of α. This result can also be seen from Figure 3.

Figure 3. MRL curves of the two-component parallel system for FGM copula



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