

COMMON FIXED POINT THEOREMS FOR TWO SELF-MAPPINGS OF A b-METRIC SPACE UNDER AN IMPLICIT RELATION

Mohamed Akkouchi*

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Abstract

In the setting of a b-metric space (see Stefan Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis **1**, 5–11, 1993 and *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Mat. Fis. Univ. Modena **46** (2), 263–276, 1998), we establish a general common fixed point theorem for two weakly compatible selfmappings satisfying the (E.A) condition (see M. Aamri and D. El Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, Math. Anal. Appl. **270**, 181-188, 2002) under a contractive condition using a class of implicit relations.

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1. Introduction

Let S and T be two selfmappings of a metric space (X, d) . In [9], Jungck defined S and T to be *compatible* if $\lim_{n \rightarrow \infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t,$$

for some $t \in X$.

The concept of compatibility was used by many authors to prove existence theorems in common fixed point theory. The study of common fixed points of noncompatible mappings is also important. Work in this way has been initiated by Pant [13, 15, 16].

Aamri and Moutawakil [1] have generalized the concept of noncompatible mapping. See also, for example, [7, 11, 21] for related results.

*Faculté des Sciences-Semlalia, Département de Mathématiques, Université Cadi Ayyad, Av. Prince My Abdellah, BP. 2390, Marrakech, Maroc (Morocco). E-mail: akkouchimo@yahoo.fr

1.1. Definition. [1] Let S and T be two selfmappings of a metric space (X, d) . We say that T and S satisfy *property (E.A)* if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sx_n = t$ for some $t \in X$.

1.2. Remark. It is clear that two selfmappings of a metric space (X, d) will be non-compatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$, for some $t \in X$, but $\lim_{n \rightarrow \infty} d(STx_n, TSx_n)$ is either nonzero or non-existent. Therefore, two noncompatible selfmappings of a metric space (X, d) satisfy property (E.A).

1.3. Definition. [10] Two self mappings S and T of a metric space (X, d) are said to be *weakly compatible* if $Tu = Su$, for $u \in X$ implies $STu = TSu$.

1.4. Remark. Two compatible mappings are weakly compatible.

We point out that in 1994, Pant [12] introduced the notion of *pointwise R-weakly commuting mappings*. It is proved in [14] that the notion of pointwise R-weakly commuting is equivalent to commutativity at coincidence points.

Popa [18] introduced a class of implicit functions to prove new common fixed point theorems. To describe the implicit functions of Popa [18], let Ψ be the family of real lower semi-continuous functions $F(t_1, t_2, \dots, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ satisfying the following conditions:

- (F1) F is non-increasing in the variables t_5 and t_6 ,
- (F2) There exists $h \in (0, 1)$ such that for every $u, v \geq 0$ with
 - (F2)_a $F(u, v, v, u, u + v, 0) \leq 0$, or
 - (F2)_b $F(u, v, u, v, 0, u + v) \leq 0$
 we have $u \leq hv$, and
- (F3) $F(u, u, 0, 0, u, u) > 0, \forall u > 0$.

Examples of such functions can be found in the papers [18] and [7].

The method of implicit relations has been extensively used in metric fixed point theory. By this method, many common fixed point theorems were unified and generalized. Now, in metric fixed point theory, we can find a large number of papers which are using several kinds of implicit relations. The method is powerful and effective in the study of common fixed points.

In 2008, Imdad and Ali [7] used the class Ψ and established the following result.

1.5. Theorem. [7] *Let T and I be selfmappings of a metric space (X, d) such that*

- (i) T and I satisfy the (E.A) property,
- (ii) $F(d(Tx, Ty), d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), d(Ix, Ty), d(Iy, Tx)) \leq 0$, for each $x, y \in X$ where $F \in \Psi$,
- (iii) $I(X)$ is a complete subspace of X .

Then

- (a) *The pair (T, I) has a point of coincidence,*
- (b) *The pair (T, I) has a common fixed point provided it is weakly compatible.*

In their paper [2], J. Ali and M. Imdad have established some general common fixed point theorems by using a class of implicit relations with weaker conditions than those of the class Ψ .

The aim of this paper is to investigate a possible extension of [7, Theorem 1.1] due to M. Imdad and J. Ali to the case of b-metric spaces (introduced by S. Czerwik [4] and [5]), by using a suitable class of implicit relations.

The main result of this paper is Theorem 4.1, in which we establish the existence of a unique common fixed point for a weakly compatible pair of selfmappings of a b-metric space. The paper contains four sections.

2. Implicit relations

Let $s \geq 1$ be fixed and \mathcal{F}_s the set of all real lower semicontinuous functions $F(t_1, \dots, t_1) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ satisfying the following conditions:

- (P₁)(s) F is nondecreasing in the variable t_1 and nonincreasing in the variable t_6 ,
- (P₂)(s) $F(\frac{1}{s}t, 0, 0, t, st, 0) > 0$, for all $t > 0$, and
- (P₃) $F(t, t, 0, 0, t, t) > 0$, for all $t > 0$.

In particular the class \mathcal{F}_1 is the set of all real lower semicontinuous functions $F(t_1, \dots, t_1) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ satisfying the following conditions:

- (P₁) F is nondecreasing in the variable t_1 and nonincreasing in the variable t_6 ,
- (P₂) $F(t, 0, 0, t, t, 0) > 0$, for all $t > 0$, and
- (P₃) $F(t, t, 0, 0, t, t) > 0$, for all $t > 0$.

2.1. Examples. Let s be a given number in the set $[1, \infty)$.

Example 1. $F(t_1, \dots, t_6) := t_1 - q \max\{t_2, \dots, t_6\}$, where $q < \frac{1}{s^2}$.

- (P₁) : Clear.
- (P₂) : $F(\frac{1}{s}t, 0, 0, t, st, 0) = \frac{t}{s}(1 - qs^2) > 0$, for all $t > 0$.
- (P₃) : $F(t, t, 0, 0, t, t) = t(1 - q) > 0$, for all $t > 0$.

Example 2. $F(t_1, \dots, t_6) := t_1 - qs^m \max\{t_2, \dots, t_6\}$, where m is any nonnegative integer and $q < \frac{1}{s^{m+2}}$.

- (P₁) : Clear.
- (P₂) : $F(\frac{1}{s}t, 0, 0, t, st, 0) = \frac{t}{s}(1 - qs^{m+2}) > 0$, for all $t > 0$.
- (P₃) : $F(t, t, 0, 0, t, t) = t(1 - qs^m) > 0$, for all $t > 0$.

Example 3. $F(t_1, \dots, t_6) := t_1^2 - at_2t_3 - bt_4t_5 - ct_5t_6$, where $a \geq 0, b < \frac{1}{s^3}$ and $C < 1$.

- (P₁) : Clear.
- (P₂) : $F(\frac{1}{s}t, 0, 0, t, st, 0) = \frac{t^2}{s^2}(1 - bs^3) > 0$, for all $t > 0$.
- (P₃) : $F(t, t, 0, 0, t, t) = t^2(1 - c) > 0$, for all $t > 0$.

Example 4. $F(t_1, \dots, t_6) := t_1^3 - at_1^2t_2 - bt_1t_4t_5 - ct_3t_5t_6$, where $a < 1$ and $b < \frac{1}{s^3}$.

- (P₁) : Clear.
- (P₂) : $F(\frac{1}{s}t, 0, 0, t, st, 0) = \frac{t^3}{s^3}(1 - bs^3) > 0$, for all $t > 0$.
- (P₃) : $F(t, t, 0, 0, t, t) = t^3(1 - a) > 0$, for all $t > 0$.

3. Preliminaries

3.1. A general result on symmetric spaces. Let X be a nonempty set. A *symmetric* on X is a non-negative real function on $X \times X$ such that

- (i) $d(x, y) = 0$ if and only if $x = y$,
- (ii) $d(x, y) = d(y, x)$, for all $x, y \in X$.

Some fixed point theorems in symmetric spaces for occasionally weakly compatible mappings are proved in [11].

Let X be a nonempty set. let \mathcal{A} be a set of selfmappings of X . We note that $\text{Fix}(\mathcal{A})$ denotes the set of common fixed point of \mathcal{A} and $\text{Coin}(\mathcal{A})$ the set of coincidence points of \mathcal{A} .

Let $S, T : X \rightarrow X$ be two selfmappings of X . A point $p \in X$ is said to be a *point of coincidence* of S and T if there exists a point $u \in X$ such that $p = Su = Tu$.

The following lemma was proved by V. Popa in [20].

3.1. Lemma. [20] Let X be a nonempty set with a symmetric d , and f, g, S and T selfmappings of X such that

$$(3.1) \quad F(d(fx, gy), d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Sy), d(gy, Sx)) \leq 0$$

for all $x, y \in X$, where F satisfies property (P3). If there are $x, y \in X$ such that $fx = Sx$ and $gy = Ty$, then f and S have a unique point of coincidence $u = fx = Sx$, and g and T have a unique point of coincidence $v = gy = Ty$. \square

3.2. b-metric spaces. The concept of a b-metric space was introduced by S. Czerwik (see [4] and [5]). We recall from [5] the following definition.

3.2. Definition. [4] Let X be a (nonempty) set and $s \geq 1$ a given real number. A function $d : X \times X \rightarrow \mathbb{R}_+$ (nonnegative real numbers) is called a *b-metric* provided that, for all $x, y, z \in X$,

- (bm-1) $d(x, y) = 0$ iff $x = y$,
- (bm-2) $d(x, y) = d(y, x)$,
- (bm-3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a *b-metric space with parameter s* .

We remark that a metric space is evidently a b-metric space. However, S. Czerwik (see [4],[5]) has shown that a b-metric on X need not be a metric on X .

Let d be a b-metric with parameter s on a set X . As in the metric case, the b-metric d induces a topology. The space X will be equipped with this topology associated to d . In particular a sequence $\{x_n\}$ converges to a point $x \in X$ if $\lim_{n \rightarrow \infty} d(x_n, x) = 0$. Almost all the concepts and results obtained for metric spaces can be extended to the case of b-metric spaces. For a large number of results concerning b-metric spaces, the reader is invited to consult the papers [4] and [5].

4. Common fixed point theorems in b-metric spaces

4.1. Theorem. Let (X, d) be a b-metric space with parameter s . Let S and T be self-mappings of X such that:

- (i) T and S satisfy the (E.A) property,
- (ii) $F(d(Tx, Ty), d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)) \leq 0$ for each $x, y \in X$, where $F \in \mathcal{F}_s$,
- (iii) $S(X)$ is a closed subspace of X .

Then

- (a) The pair (T, S) has a point of coincidence,
- (b) For all $x, y \in \text{Coin}(\{S, T\})$, we have $Sx = Sy = Tx = Ty$,
- (c) The pair (T, S) has a unique common fixed point provided it is weakly compatible.

Proof. Since T and S satisfy the property (E.A), there exists in X a sequence $\{x_n\}$ satisfying $\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sx_n = t$, for some $t \in X$.

Since $S(X)$ is closed, there exists a point $a \in X$ such that $t = \lim_{n \rightarrow \infty} Sx_n = Sa$. Also, we have $t = \lim_{n \rightarrow \infty} Tx_n = Sa$. To get a contradiction, suppose that $Sa \neq Ta$. Then by using (ii) for $x = x_n$ and $y = a$, we obtain that

$$F(d(Tx_n, Ta), d(Sx_n, Sa), d(Sx_n, Tx_n), d(Sa, Ta), d(Sx_n, Ta), d(Sa, Tx_n)) \leq 0.$$

Since $d(Sa, Ta) - sd(Sa, Tx_n) \leq sd(Tx_n, Ta)$, and F is nondecreasing in the first variable, then we get

$$F\left(\frac{1}{s}d(Ta, Sa) - d(Sa, Tx_n), d(Sx_n, Sa), d(Sx_n, Tx_n), d(Sa, Ta), d(Sx_n, Ta), d(Sa, Tx_n)\right) \leq 0.$$

Since d is a b-metric with parameter s , then we have

$$d(Sx_n, Ta) \leq s[d(Sx_n, Sa) + d(Sa, Ta)].$$

Since F is nonincreasing in the fifth variable then we get

$$F\left(\frac{1}{s}d(Ta, Sa) - d(Sa, Tx_n), d(Sx_n, Sa), d(Sx_n, Tx_n), d(Sa, Ta), s[d(Sx_n, Sa) + d(Sa, Ta)], d(Sa, Tx_n)\right) \leq 0.$$

It is easy to show that $\lim_{n \rightarrow \infty} d(Sx_n, Tx_n) = 0$, so by letting n tend to infinity and using the continuity of F , we get:

$$F\left(\frac{1}{s}d(Ta, Sa), 0, 0, d(Sa, Ta), sd(Sa, Ta), 0\right) \leq 0,$$

a contradiction of (P1)(s). Hence, $Sa = Ta$. That is a is a coincidence point of the pair $\{S, T\}$. We set $z = Sa = Ta$. So, z is a point of coincidence of the pair $\{S, T\}$.

Suppose that $x, y \in \text{Coin}(\{S, T\})$. As in the proof of Lemma 3.1 (see [20]), one can prove that $Sx = Sy$.

Suppose that S and T are weakly compatible. Then S and T commute at the point $z = Sa = Ta$. Next, we show that z is a common fixed point of T and S . We have

$$Tz = T Sa = S T a = Sz.$$

By (ii) for $x = a$ and $y = z$ we have successively:

$$F(d(Ta, Tz), d(Sa, Sz), d(Sa, Ta), d(Tz, Sz), d(Sa, Tz), d(Sz, Ta)) \leq 0, \\ F(d(z, Tz), d(z, Tz), 0, 0, d(z, Tz), d(z, Tz)) \leq 0,$$

a contradiction of (P3) if $d(z, Tz) \neq 0$. Hence, $Tz = z$ and $Sz = Tz = z$. Therefore z is a common fixed point of S and T .

Suppose that $Su = Tu = u$ and $Sv = Tv = v$ for $u \neq v$. Then, by (ii) we have successively:

$$F(d(Tu, Tv), d(Su, Sv), d(Su, Tu), d(Sv, Tv), d(Su, Tv), d(Sv, Tu)) \leq 0, \\ F(d(u, v), d(u, v), 0, 0, d(u, v), d(u, v)) \leq 0,$$

a contradiction to (P3) if $d(u, v) \neq 0$. Hence, $u = v$. This completes the proof. □

As a consequence, we have

4.2. Corollary. *Let $s \geq 1$ and let d be a b-metric space on a set X with parameter s . Let S and T two noncompatible and weakly compatible selfmappings of X such that:*

- (1) $F(d(Tx, Ty), d(Sx, Sy), d(Tx, Sx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)) \leq 0$, for each $(x, y) \in X^2$ and $F \in \mathcal{F}_s$.

If $S(X)$ is a closed subspace of X , then S and T have a unique common fixed point.

4.3. Remark. Let (X, d) be a b-metric space with parameter $s \geq 1$. Suppose that d is continuous on the topological space X endowed with the topology induced by d . Then the lines of the proof of Theorem 4.1 show that one needs only to use the class \mathcal{F}_1 to define the contractive condition.

More precisely, we have the following theorem.

4.4. Theorem. *Let (X, d) be a b-metric space with parameter s . We suppose that d is continuous. Let S and T be two selfmappings of X such that:*

- (1) S and T satisfy property (E.A),
- (2) $F(d(Tx, Ty), d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)) \leq 0$, for each $(x, y) \in X^2$, where $F \in \mathcal{F}_1$.
- (3) $S(X)$ is a closed subspace of X .

Then

- (a) The pair (T, S) has a point of coincidence,
- (b) The pair (T, S) has a unique common fixed point provided it is weakly compatible.

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