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REMARKS ON WEAK NEIGHBORHOOD SPACES

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Abstract

We introduce and study the notions of weak $w(\psi, \phi)$-continuity, strong $w(\psi, \phi)$-continuity, almost $w(\psi, \phi)$-continuity, $w(\psi, \phi)$-open function, weakly $w(\psi, \phi)$-open function and almost $w(\psi, \phi)$-open function. In particular, we investigate the relationships among several types of $w(\psi, \phi)$-continuous function and $w(\psi, \phi)$-open function.

Keywords: Weakly $w(\psi, \phi)$-continuous function, Strongly $w(\psi, \phi)$-continuous function, Almost $w(\psi, \phi)$-continuous function, $w(\psi, \phi)$-open function, Weakly $w(\psi, \phi)$-open function, Almost $w(\psi, \phi)$-open function.

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1. Introduction

In [1], Császár introduced the notions of generalized neighborhood system and generalized topological space. In [3], notions of open function were investigated on generalized neighborhood systems and generalized topological spaces. In [4], the author introduced weak neighborhood systems defined by using the notion of weak neighborhood. These are generalized systems of the open neighborhood system obtained in any topological space. A weak neighborhood system induces a weak neighborhood space (briefly WNS), which is independent of a neighborhood space [2]. Also the author introduced and characterized the notions of $w(\psi, \phi)$-continuity, associated interior and closure operators on WNS's. In this paper, we introduce the notions of weak $w(\psi, \phi)$-continuity, strong $w(\psi, \phi)$-continuity, almost $w(\psi, \phi)$-continuity, $w(\psi, \phi)$-open function, weakly $w(\psi, \phi)$-open function and almost $w(\psi, \phi)$-open function. We obtain some characterizations for several types of $w(\psi, \phi)$-continuous function and $w(\psi, \phi)$-open function. In particular, we investigate the relationships among several types of $w(\psi, \phi)$-continuous function and $w(\psi, \phi)$-open function.

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2. Preliminaries

Let X be a nonempty set, $P(X)$ the power set of X and $\psi : X \to P(P(X))$ satisfy $x \in V$ for $V \in \psi(x)$. Then $V \in \psi(x)$ is called a *generalized neighborhood* [1] of $x \in X$, and ψ is called a *generalized neighborhood system* (briefly GNS) on X. Let g be a collection of subsets of X. Then g is called a *generalized topology* [1] on X iff $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $\bigcup_{i \in I} G_i \in g$.

Let $\psi : X \to P(P(X))$. Then ψ is called a *weak neighborhood system* [4] on X if it satisfies the following axioms:

(a) For $x \in X$, $\psi(x) \neq \emptyset$.
(b) For $x \in X$ and $V \in \psi(x)$, $x \in V$.
(c) For $U, V \in \psi(x)$, $V \cap U \in \psi(x)$.

Then $V \in \psi(x)$ is called a *weak neighborhood* of $x \in X$, and the pair (X, ψ) is called a *weak neighborhood space* (briefly WNS) on X. Set $\psi(X) = \{V : V \in \psi(x) \text{ for all } x \in X\}$.

For $A \subseteq X$, the interior and closure of A on ψ (denoted by $\iota_\psi(A)$, $\gamma_\psi(A)$, respectively) are defined as follows:

$\iota_\psi(A) = \{x \in A : \text{ there exists } V \in \psi(x) \text{ such that } V \subseteq A\}$;
$\gamma_\psi(A) = \{x \in X : V \cap A \neq \emptyset \text{ for all } V \in \psi(x)\}$.

2.1. Theorem. [4] *Let (X, ψ) be a WNS and $A, B \subseteq X$. Then the following hold.*

(a) $\iota_\psi(A) \subseteq A$ *and* $A \subseteq \gamma_\psi(A)$.
(b) $\iota_\psi(A \cap B) = \iota_\psi(A) \cap \iota_\psi(B)$ *and* $\gamma_\psi(A \cup B) = \gamma_\psi(A) \cup \gamma_\psi(B)$.
(c) $\iota_\psi(X) = X$ *and* $\gamma_\psi(\emptyset) = \emptyset$.
(d) $\gamma_\psi(A) = X \setminus \iota_\psi(X \setminus A)$ *and* $\iota_\psi(A) = X \setminus \gamma_\psi(X \setminus A)$. □

2.2. Definition. [4] Let (X, ψ) and (Y, ϕ) be two WNS's. Then $f : (X, \psi) \to (Y, \phi)$ is said to be $w(\psi, \phi)$*-continuous* if for $x \in X$ and $V \in \phi(f(x))$, there is $U \in \psi(x)$ such that $f(U) \subseteq V$.

2.3. Theorem. [4] *Let $f : (X, \psi) \to (Y, \phi)$ be a function between two WNS's. Then the following statements are equivalent:*

(a) *f is $w(\psi, \phi)$-continuous.*
(b) $f(\gamma_\psi(A)) \subseteq \gamma_\phi(f(A))$ *for* $A \subseteq X$.
(c) $\gamma_\psi(f^{-1}(B)) \subseteq f^{-1}(\gamma_\phi(B))$ *for* $B \subseteq Y$.
(d) $f^{-1}(\iota_\phi(B)) \subseteq \iota_\psi(f^{-1}(B))$ *for* $B \subseteq Y$. □

3. Results on $w(\psi, \phi)$-continuous functions and $w(\psi, \phi)$-open functions

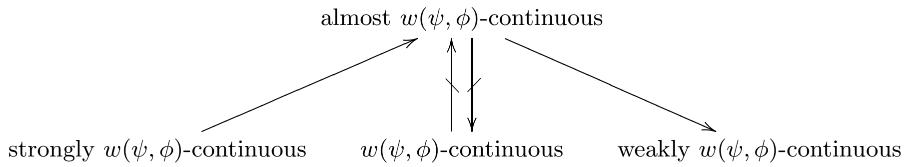
First, we introduce the notions of weak $w(\psi, \phi)$-continuity, strong $w(\psi, \phi)$-continuity and almost $w(\psi, \phi)$-continuity on WNS's. We investigate characterizations for such functions. Secondly, we introduce notions of $w(\psi, \phi)$-open function, weakly $w(\psi, \phi)$-open function and almost $w(\psi, \phi)$-open function, and then investigate properties for such $w(\psi, \phi)$-open functions. Finally, we introduce the notion of $w(\psi, \phi)$-closed function on WNS's.

3.1. Definition. Let (X, ψ) and (Y, ϕ) be two WNS's and let $f : (X, \psi) \to (Y, \phi)$ be a function. Then

(1) f is said to be *weakly* $w(\psi, \phi)$*-continuous* if for each $x \in X$ and each $V \in \phi(f(x))$, there is $U \in \psi(x)$ such that $f(U) \subseteq \gamma_\phi(V)$.
(2) f is said to be *strongly* $w(\psi, \phi)$*-continuous* if for each $x \in X$ and each $V \in \phi(f(x))$, there is $U \in \psi(x)$ such that $f(U) \subseteq \iota_\phi(V)$.

(3) f is said to be *almost* $w(\psi,\phi)$-*continuous* if for each $x \in X$ and each $V \in \phi(f(x))$, there is $U \in \psi(x)$ such that $f(U) \subseteq \iota_\phi(\gamma_\phi(V))$.

The following diagram is obtained, but the converses are not true in general as shown in the subsequent examples:



3.2. Example. Let $X = \{a, b, c\}$. Consider two weak neighborhood systems ψ, ϕ defined as follows:

$$\psi(a) = \{\{a\}\},\ \psi(b) = \{X\},\ \psi(c) = \{\{c\}\},$$
$$\phi(a) = \{\{a, b\}\},\ \phi(b) = \{\{a, b\}\},\ \phi(c) = \{X\}.$$

Let $f : (X, \psi) \to (X, \phi)$ be a function defined by $f(a) = b,\ f(b) = a,\ f(c) = c$. Then f is both weakly $w(\psi,\phi)$-continuous and almost $w(\psi,\phi)$-continuous, but it is not $w(\psi,\phi)$-continuous. Clearly, we know that f is almost $w(\psi,\phi)$-continuous but it is not strongly $w(\psi,\phi)$-continuous.

3.3. Example. Let $X = \{a, b, c\}$ and consider two weak neighborhood systems ψ, ϕ defined as follows:

$$\psi(a) = \psi(b) = \{\{a, b\}\},\ \psi(c) = \{\{c\}\},$$
$$\phi(a) = \{\{a, b\}\},\ \phi(b) = \{X\},\ \phi(c) = \{\{c\}\}.$$

Then the identity function $f : (X, \psi) \to (X, \phi)$ is $w(\psi,\phi)$-continuous, but it is neither almost $w(\psi,\phi)$-continuous nor strongly $w(\psi,\phi)$-continuous. It is obvious that f is weakly $w(\psi,\phi)$-continuous but it is not almost $w(\psi,\phi)$-continuous.

3.4. Theorem. *Let $f : (X, \psi) \to (Y, \phi)$ be a function between WNS's. Then the following are equivalent:*

(a) *f is weakly $w(\psi,\phi)$-continuous.*
(b) *$f^{-1}(\iota_\phi(B)) \subseteq \iota_\psi(f^{-1}(\gamma_\phi(B)))$ for every subset B of Y.*
(c) *$\gamma_\psi(f^{-1}(\iota_\phi(B))) \subseteq f^{-1}(\gamma_\phi(B))$ for every subset B of Y.*

Proof. (a) $\Longrightarrow$ (b) Let $x \in f^{-1}(\iota_\phi(B))$. Then there exists $V \in \phi(f(x))$ such that $V \subseteq B$. By hypothesis, there exists $U \in \psi(x)$ such that $f(U) \subseteq \gamma_\phi(V) \subseteq \gamma_\phi(B)$. Since $U \subseteq f^{-1}(\gamma_\phi(V)) \subseteq f^{-1}(\gamma_\phi(B))$ for $U \in \psi(x)$, $x \in \iota_\psi(f^{-1}(\gamma_\phi(B)))$. Hence $f^{-1}(\iota_\phi(B)) \subseteq \iota_\psi(f^{-1}(\gamma_\phi(B)))$.

(b) $\Longrightarrow$ (a) Let $x \in X$ and $V \in \phi(f(x))$. Then $f(x) \in \iota_\phi(V)$ and by (b), $x \in f^{-1}(\iota_\phi(V)) \subseteq \iota_\psi(f^{-1}(\gamma_\phi(V)))$. Thus there exists a subset $U \in \psi(x)$ such that $U \subseteq f^{-1}(\gamma_\phi(V))$ and so $f(U) \subseteq \gamma_\phi(V)$. Hence f is weakly (ψ,ϕ)-continuous.

(b) $\Longleftrightarrow$ (c) Follows from Theorem 2.1. □

3.5. Theorem. *Let $f : (X, \psi) \to (Y, \phi)$ be a function between WNS's. Then the following are equivalent:*

(a) *f is strongly $w(\psi,\phi)$-continuous.*
(b) *$f^{-1}(\iota_\phi(B)) \subseteq \iota_\psi(f^{-1}(\iota_\phi(B)))$ for every subset B of Y.*
(c) *$\gamma_\psi(f^{-1}(\gamma_\phi(B))) \subseteq f^{-1}(\gamma_\phi(B))$ for every subset B of Y.*

Proof. Similar to the proof of Theorem 3.4. □

3.6. Theorem. *Let $f : (X, \psi) \to (Y, \phi)$ be a function between WNS's. Then the following are equivalent:*

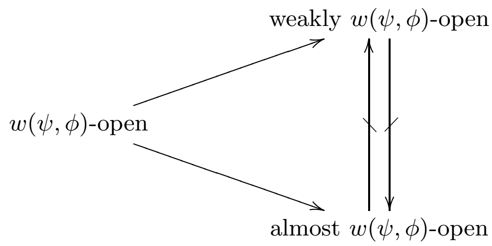
(a) *f is almost $w(\psi, \phi)$-continuous.*
(b) *$f^{-1}(\iota_\phi(B)) \subseteq \iota_\psi(f^{-1}(\iota_\phi(\gamma_\phi(B))))$ for every subset B of Y.*
(c) *$\gamma_\psi(f^{-1}(\gamma_\phi(\iota_\phi(B)))) \subseteq f^{-1}(\gamma_\phi(B))$ for every subset B of Y.*

Proof. Similar to the proof of Theorem 3.4. □

3.7. Definition. Let (X, ψ) and (Y, ϕ) be two WNS's. Then $f : X \to Y$ is said to be

(a) *$w(\psi, \phi)$-open* if for each $x \in X$ and $U \in \psi(x)$, there exists an element $V \in \phi(f(x))$ such that $V \subseteq f(U)$;
(b) *weakly $w(\psi, \phi)$-open* if for each $x \in X$ and $U \in \psi(x)$, there exists an element $V \in \phi(f(x))$ such that $V \subseteq f(\gamma_\psi(U))$;
(c) *almost $w(\psi, \phi)$-open* if for each $x \in X$ and $U \in \psi(x)$, there exists an element $V \in \phi(f(x))$ such that $V \subseteq \gamma_\psi(f(U))$.

3.8. Remark. We have the following diagram from the above definitions.



In the above diagram, the converses are not true in general as shown in the following examples.

3.9. Example. Let $X = \{a, b\}$. Consider a weak neighborhood system ψ defined as follows:

$$\psi(a) = \{\{a\}\}, \psi(b) = \{X\}.$$

Let $f : (X, \psi) \to (X, \psi)$ be a function defined by $f(a) = b$, $f(b) = a$. Then f is both weakly $w(\psi, \phi)$-open and almost $w(\psi, \phi)$-open, but it is not $w(\psi, \phi)$-open.

3.10. Example. (1) In Example 3.2, the function f is obviously almost $w(\psi, \phi)$-open. For $c \in X$, we have $\gamma_\psi(\{c\}) = \{b, c\}$ and $f(\gamma_\psi(\{c\})) = \{a, c\}$. But since $\phi(f(c))$ only contains X, f can not be weakly $w(\psi, \phi)$-open.

(2) Let $X = \{a, b, c\}$. Consider two weak neighborhood systems ψ, ϕ defined as follows:

$$\psi(a) = \{X\},\ \psi(b) = \{X\},\ \psi(c) = \{X\},$$
$$\phi(a) = \{\{a\}\},\ \phi(b) = \{X\},\ \phi(c) = \{\{c\}\}.$$

Note that:

$$\gamma_\phi(\{a\}) = \{a, b\};\ \gamma_\phi(\{b\}) = \{X\};\ \gamma_\phi(\{c\}) = \{b, c\}.$$

Consider a function $f : (X, \psi) \to (X, \phi)$ defined as follows: $f(a) = b$, $f(b) = a$, $f(c) = c$. Then f is weakly $w(\psi, \phi)$-open but not almost $w(\psi, \phi)$-open.

3.11. Theorem. *Let (X, ψ) and (Y, ϕ) be two WNS's and let $f : X \to Y$ be a function. Then the following are equivalent:*

(a) *f is $w(\psi, \phi)$-open.*
(b) *$\iota_\psi(f^{-1}(A)) \subseteq f^{-1}(\iota_\phi(A))$ for $A \subseteq Y$.*

(c) $f^{-1}(\gamma_\phi(A)) \subseteq \gamma_\psi(f^{-1}(A))$ *for* $A \subseteq Y$.
(d) $f(\iota_\psi(B)) \subseteq \iota_\phi(f(B))$ *for* $B \subseteq X$.

Proof. (a) $\Longrightarrow$ (b) Let f be (ψ,ϕ)-open and $x \in \iota_\psi(f^{-1}(A))$. Then there is $U \in \psi(x)$ such that $U \subseteq f^{-1}(A)$. Since f is $w(\psi,\phi)$-open, there is $V \in \phi(f(x))$ such that $V \subseteq f(U) \subseteq A$, so that $f(x) \in \iota_\phi(A)$.

(b) $\Longrightarrow$ (a) Let $U \in \psi(x)$ for $x \in X$. Then $x \in \iota_\psi(U) \subseteq \iota_\psi(f^{-1}(f(U)))$. By hypothesis, $x \in f^{-1}(\iota_\phi(f(U)))$, and so $f(x) \in \iota_\phi(f(U))$. From the definition of the interior operator ι_ϕ, there exists an element $V \in \phi(f(x))$ such that $V \subseteq f(U)$.

(b) $\Longleftrightarrow$ (c) Follows from Theorem 2.1.

(b) $\Longrightarrow$ (d) Easily obtained.

(d) $\Longrightarrow$ (a) Let $U \in \psi(x)$ for $x \in X$. Since $x \in \iota_\psi(U)$, we have $f(x) \in f(\iota_\psi(U))$ and so $f(x) \in \iota_\phi(f(U))$ by (d). Thus there exists an element $V \in \phi(f(x))$ such that $V \subseteq f(U)$. □

3.12. Theorem. *Let (X,ψ) and (Y,ϕ) be two WNS's, and let $f : X \to Y$ be a function. Then f is weakly $w(\psi,\phi)$-open if and only if $f(\iota_\psi(B)) \subseteq \iota_\phi(f(\gamma_\psi(B)))$ for $B \subseteq X$.*

Proof. Let f be weakly $w(\psi,\phi)$-open. Then for each $x \in \iota_\psi(B)$, there exists $U \in \psi(x)$ such that $U \subseteq B$. From the weakly $w(\psi,\phi)$-openness of f, there exists an element $V \in \phi(f(x))$ such that $V \subseteq f(\gamma_\psi(U))$. This implies $x \in \iota_\phi(f(\gamma_\psi(B)))$, and hence $f(\iota_\psi(B)) \subseteq \iota_\phi(f(\gamma_\psi(B)))$.

For the converse, let $U \in \psi(x)$ for $x \in X$. Since $x \in \iota_\psi(U)$, by hypothesis, we have $f(x) \in \iota_\phi(f(\gamma_\psi(U)))$. This implies that there exists an element $V \in \phi(f(x))$ such that $V \subseteq f(\gamma_\psi(U))$. Hence f is weakly (ψ,ϕ)-open. □

3.13. Theorem. *Let (X,ψ) and (Y,ϕ) be two WNS's, and let $f : X \to Y$ be a function. Then the following are equivalent:*

(a) *f is almost $w(\psi,\phi)$-open.*
(b) $\iota_\psi(f^{-1}(A)) \subseteq f^{-1}(\iota_\phi(\gamma_\phi(A)))$ *for* $A \subseteq Y$.
(c) $f^{-1}(\gamma_\phi(\iota_\phi(A))) \subseteq \gamma_\psi(f^{-1}(A))$ *for* $A \subseteq Y$.
(d) $f(\iota_\psi(B)) \subseteq \iota_\phi(\gamma_\phi(f(B)))$ *for* $B \subseteq X$.

Proof. (a) $\Longrightarrow$ (b) Let f be almost $w(\psi,\phi)$-open and $A \subseteq Y$. For $x \in \iota_\psi(f^{-1}(A))$, there is $U \in \psi(x)$ such that $U \subseteq f^{-1}(A)$. From the almost $w(\psi,\phi)$-openness of f, there is $V \in \phi(f(x))$ such that $V \subseteq \gamma_\phi(f(U)) \subseteq \gamma_\phi(A)$. This implies $f(x) \in \iota_\phi(A)$.

(b) $\Longrightarrow$ (a) Let $U \in \psi(x)$ for $x \in X$. Then $x \in \iota_\psi(U) \subseteq \iota_\psi(f^{-1}f(U))$. By (b), $x \in f^{-1}(\iota_\phi(\gamma_\phi(f(U))))$ and so $f(x) \in \iota_\phi(\gamma_\phi(f(U)))$. From the definition of the interior operator, there exists an element $V \in \phi(f(x))$ such that $V \subseteq \gamma_\phi(f(U))$. Hence f is almost $w(\psi,\phi)$-open.

(b) $\Longleftrightarrow$ (c) Obvious from Theorem 2.1.

(b) $\Longrightarrow$ (d) Easily obtained from (b).

(d) $\Longrightarrow$ (a) Let $U \in \psi(x)$ for $x \in X$. Then by (d), we have $f(x) \in \iota_\phi(\gamma_\phi(f(U))$. Thus there exists an element $V \in \phi(f(x))$ such that $V \subseteq \gamma_\phi(f(U))$. Hence f is almost $w(\psi,\phi)$-open. □

3.14. Definition. Let (X,ψ) and (Y,ϕ) be two WNS's. Then $f : X \to Y$ is said to be $w(\psi,\phi)$*-closed* if $\gamma_\phi(f(B)) \subseteq f(\gamma_\psi(B))$ for $B \subseteq X$.

3.15. Theorem. *Let (X,ψ) and (Y,ϕ) be two WNS's, and let $f : X \to Y$ be a bijective function. Then*

(a) f *is* $w(\psi,\phi)$*-closed;*
(b) $f^{-1}(\gamma_\phi(A)) \subseteq \gamma_\psi(f^{-1}(A))$ *for* $A \subseteq Y$*;*
(c) $\iota_\psi(f^{-1}(A)) \subseteq f^{-1}(\iota_\phi(A))$ *for* $A \subseteq Y$.

Proof. (a) $\Longrightarrow$ (b) Suppose f is $w(\psi,\phi)$-closed. Then for $A \subseteq Y$, $\gamma_\phi(f(f^{-1}(A))) \subseteq f(\gamma_\psi(f^{-1}(A)))$ is satisfied. Since f is surjective, we have $\gamma_\phi(A) \subseteq f(\gamma_\psi(f^{-1}(A)))$. And from injectivity, it follows that $f^{-1}(\gamma_\phi(A)) \subseteq f^{-1}(f(\gamma_\psi(f^{-1}(A)))) = \gamma_\psi(f^{-1}(A))$. Hence (b) is obtained.

(b) $\Longrightarrow$ (a) For $B \subseteq X$, from (b) and injectivity, it follows that

$$f^{-1}(\gamma_\phi(f(B))) \subseteq \gamma_\psi(f^{-1}(f(B))) = \gamma_\psi(B).$$

Finally from surjectivity, we have $\gamma_\phi(f(B)) \subseteq f(\gamma_\psi(B))$.

(b) $\Longleftrightarrow$ (c) Follows from Theorem 2.1. □

4. Decompositions of several types of $w(\psi,\phi)$-continuous function and $w(\psi,\phi)$-open function

In this section, we investigate the relationships among several types of $w(\psi,\phi)$-continuous function and $w(\psi,\phi)$-open function.

4.1. Theorem. *Let* (X,ψ) *and* (Y,ϕ) *be two WNS's. Then if* $f : (X,\psi) \to (Y,\phi)$ *is* $w(\psi,\phi)$*-continuous and almost* $w(\psi,\phi)$*-open, and if it satisfies* $\iota_\psi\iota_\psi = \iota_\psi$ *for the interior operator* ι_ψ*, then* f *is almost* $w(\psi,\phi)$*-continuous.*

Proof. For $B \subseteq Y$, from Theorem 2.3 (d) and Theorem 3.13 (b), it follows that

$$\begin{aligned} f^{-1}(\iota_\phi(B)) &\subseteq \iota_\psi(f^{-1}(B))) \\ &= \iota_\psi(\iota_\psi(f^{-1}(B))) \\ &= \iota_\psi(f^{-1}(\iota_\phi(\gamma_\phi(B))). \end{aligned}$$

Hence by Theorem 3.6 (b), f is almost $w(\psi,\phi)$-continuous. □

4.2. Corollary. *Let* (X,ψ) *and* (Y,ϕ) *be two WNS's. Then if* $f : (X,\psi) \to (Y,\phi)$ *is* $w(\psi,\phi)$*-continuous and* $w(\psi,\phi)$*-open, and if it satisfies* $\iota_\psi\iota_\psi = \iota_\psi$ *for the interior operator* ι_ψ*, then* f *is almost* $w(\psi,\phi)$*-continuous.*

Proof. Since every $w(\psi,\phi)$-open function is almost $w(\psi,\phi)$-open, the result follows from Theorem 4.1. □

4.3. Theorem. *Let* (X,ψ) *and* (Y,ϕ) *be two WNS's. Then if* $f : (X,\psi) \to (Y,\phi)$ *is* $w(\psi,\phi)$*-continuous and* $w(\psi,\phi)$*-open, and if it satisfies* $\iota_\psi\iota_\psi = \iota_\psi$ *for the interior operator* ι_ψ*, then* f *is strongly* $w(\psi,\phi)$*-continuous.*

Proof. For $B \subseteq Y$, from Theorem 2.3 (d) and Theorem 3.11 (b), it follows that

$$\begin{aligned} f^{-1}(\iota_\phi(B)) &\subseteq \iota_\psi(f^{-1}(B))) \\ &= \iota_\psi(\iota_\psi(f^{-1}(B))) \\ &= \iota_\psi(f^{-1}(\iota_\phi(B))). \end{aligned}$$

Hence by Theorem 3.5 (b), f is strongly $w(\psi,\phi)$-continuous. □

4.4. Theorem. *Let* (X,ψ) *and* (Y,ϕ) *be two WNS's. Then if* $f : (X,\psi) \to (Y,\phi)$ *is weakly* $w(\psi,\phi)$*-continuous and* $w(\psi,\phi)$*-open, and if it satisfies* $\iota_\psi\iota_\psi = \iota_\psi$ *for the interior operator* ι_ψ*, then* f *is almost* $w(\psi,\phi)$*-continuous.*

Proof. For $B \subseteq Y$, from Theorem 3.4 (b) and Theorem 3.11 (b), it follows that

$$\begin{aligned} f^{-1}(\iota_\phi(B)) &\subseteq \iota_\psi(f^{-1}(\gamma_\phi(B))) \\ &= \iota_\psi(\iota_\psi(f^{-1}(\gamma_\phi(B)))) \\ &= \iota_\psi(f^{-1}(\iota_\phi(\gamma_\phi(B)))). \end{aligned}$$

Hence by Theorem 3.6 (b), f is almost $w(\psi, \phi)$-continuous. □

4.5. Theorem. *Let (X, ψ) and (Y, ϕ) be two WNS's. Then if $f : (X, \psi) \to (Y, \phi)$ is weakly $w(\psi, \phi)$-continuous and almost $w(\psi, \phi)$-open, and if it satisfies $\iota_\psi \iota_\psi = \iota_\psi$ and $\gamma_\phi \gamma_\phi = \gamma_\phi$ for the operators ι_ψ, γ_ϕ, then f is almost $w(\psi, \phi)$-continuous.*

Proof. For $B \subseteq Y$, from Theorem 3.4 (b) and Theorem 3.11 (b), it follows that

$$\begin{aligned} f^{-1}(\iota_\phi(B)) &\subseteq \iota_\psi(f^{-1}(\gamma_\phi(B))) \\ &= \iota_\psi(\iota_\psi(f^{-1}(\gamma_\phi(B)))) \\ &\subseteq \iota_\psi(f^{-1}(\iota_\phi(\gamma_\phi(\gamma_\phi(B))))) \\ &= \iota_\psi(f^{-1}(\iota_\phi(\gamma_\phi(B)))). \end{aligned}$$

Hence f is almost $w(\psi, \phi)$-continuous. □

4.6. Theorem. *Let $f : X \to Y$ be a function between the WNS's (X, ψ) and (Y, ϕ). Then if f is almost $w(\psi, \phi)$-open and $w(\psi, \phi)$-closed, then f is weakly $w(\psi, \phi)$-open.*

Proof. Suppose f is almost $w(\psi, \phi)$-open and $w(\psi, \phi)$-closed. For each $x \in X$ and $U \in \psi(x)$, from the almost $w(\psi, \phi)$-openness of f, there exists an element $V \in \phi(f(x))$ such that $V \subseteq \gamma_\phi(f(U))$. From the definition of $w(\psi, \phi)$-closed functions, it follows that

$$V \subseteq \gamma_\phi(f(U)) \subseteq f(\gamma_\psi(B)).$$

Consequently, f is weakly $w(\psi, \phi)$-open. □

4.7. Theorem. *Let $f : X \to Y$ be a function on WNS's (X, ψ) and (Y, ϕ). Then if f is weakly $w(\psi, \phi)$-open and $w(\psi, \phi)$-continuous, then f is almost $w(\psi, \phi)$-open.*

Proof. For each $x \in X$ and $U \in \psi(x)$, from the weakly $w(\psi, \phi)$-openness of f, there exists an element $V \in \phi(f(x))$ such that $V \subseteq f(\gamma_\psi(U))$. Since f is $w(\psi, \phi)$-continuous, from Theorem 2.3, it follows that

$$V \subseteq f(\gamma_\psi(U)) \subseteq \gamma_\phi(f(U)).$$

This implies f is almost $w(\psi, \phi)$-open. □

4.8. Theorem. *Let $f : X \to Y$ be a function between the WNS's (X, ψ) and (Y, ϕ). Then f is $w(\psi, \phi)$-continuous and $w(\psi, \phi)$-closed if and only if $\gamma_\phi(f(B)) = f(\gamma_\psi(B))$ for $B \subseteq X$.*

Proof. Follows from Theorem 2.3 and Definition 3.14. □

4.9. Theorem. *Let $f : X \to Y$ be a function between the WNS's (X, ψ) and (Y, ϕ). Then the following are equivalent:*

(a) *f is $w(\psi, \phi)$-continuous and $w(\psi, \phi)$-open.*
(b) *$f^{-1}(\gamma_\phi(A)) = \gamma_\psi(f^{-1}(A))$ for $A \subseteq Y$.*
(c) *$\iota_\psi(f^{-1}(A)) = f^{-1}(\iota_\phi(A))$ for $A \subseteq Y$.*

Proof. Follows from Theorem 2.3 and Theorem 3.11. □

From Theorem 4.8 and Theorem 4.9, we have the following:

4.10. Corollary. *Let $f : X \to Y$ be a function between the WNS's (X, ψ) and (Y, ϕ). If f is bijective, then the following are equivalent:*

(a) *f is $w(\psi, \phi)$-continuous and $w(\psi, \phi)$-closed.*
(b) $\gamma_\phi(f(B)) = f(\gamma_\psi(B))$ *for* $B \subseteq X$.
(c) $f^{-1}(\gamma_\phi(A)) = \gamma_\psi(f^{-1}(A))$ *for* $A \subseteq Y$.
(d) $\iota_\psi(f^{-1}(A)) = f^{-1}(\iota_\phi(A))$ *for* $A \subseteq Y$. □

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