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CHAIN CONDITIONS ON FUZZY POSITIVE IMPLICATIVE FILTERS OF BL-ALGEBRAS

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Abstract

In this paper, we discuss chain conditions of fuzzy positive implicative filters of BL-algebras. Specially, by using the notions of maximal and normal fuzzy positive implicative filters, we show that under certain conditions a fuzzy positive implicative filter is two-valued and takes the values 0 and 1.

Keywords: BL-algebra, Positive implicative filter, Normal fuzzy positive implicative filter, Maximal fuzzy positive implicative filter.

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1. Introduction

BL-algebras are the algebraic structures for Hájek's basic logic [4]. The main example of a BL-algebra is the interval [0,1] endowed with the structure induced by a continuous t-norm. MV-algebras [2], introduced by Chang in 1958, are one of the best known classes of BL-algebras. In [17], Mundici proved that MV-algebras are categorically equivalent to the bounded commutative BCK-algebras introduced by Iséki and Tanaka in [11, 12]. Further, Iorgulescu [10] showed that a BL-algebra is a particular case of a reversed left BCK-algebra. In order to research the logical system whose propositional value is given in a lattice, Xu [25] proposed the concept of lattice implication algebras. In [23], Wang proved that lattice implication algebras are categorically equivalent to MV-algebras. For more details of these algebras, we refer the reader to [7, 18, 20-22].

Up to now, these algebras have been widely studied. In particular, emphasis seems to have been put on the theory of ideals and filters. In [11], Iséki proposed the notion of implicative ideals in BCK-algebras, and obtained some results. Subsequently, Hoo and Sessa [9] proposed the notion of Boolean ideals in MV-algebras and proved that implicative ideals and Boolean ideals are equivalent in MV-algebras. Since the notion of ideal was

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missing in BL-algebras, Turunen [21] generalized these ideals to BL-algebras, proposed the notions of implicative filters and Boolean filters (Boolean deductive systems), and proved that implicative filters are equivalent to Boolean filters in BL-algebras. Boolean filters are important filters, because the quotient algebras induced by Boolean filters are Boolean algebras, and a BL-algebra is bipartite if and only if it has a proper Boolean filter.

In 1991, Xi [24] applied the concept of fuzzy sets [28] to BCK-algebras and proposed the notion of fuzzy implicative ideals. Afterwards, Hoo [8] proved that fuzzy implicative and fuzzy Boolean ideals are equivalent in MV-algebras. Also, Xu and Qin [26, 27] proposed the notions of positive implicative filters and fuzzy positive implicative filters (Xu called them implicative filters and fuzzy implicative filters) in lattice implication algebras. Jun *et al.* derived several characterizations of fuzzy positive implicative filters of lattice implication algebras [13, 14, 19].

In this paper, we discuss chain conditions on fuzzy positive implicative filters in BL-algebras. Specially, by using the notions of maximal and normal fuzzy positive implicative filters, we show that under certain conditions a fuzzy positive implicative filter is two-valued and it takes the values 0 and 1.

2. Preliminaries

In this section, we recall certain definitions and results needed for our purpose.

A *BL-algebra* [4] is a structure $(\mathcal{L}, \wedge, \vee, \odot, \to, 0, 1)$ such that $(\mathcal{L}, \wedge, \vee, 0, 1)$ is a bounded lattice, $(\mathcal{L}, \odot, 1)$ is an abelian monoid, i.e. $\odot$ is commutative and associative and the following conditions hold for all $x, y, z \in \mathcal{L}$:

(B1) $x \odot 1 = x$,
(B2) $x \odot y \leq z$ if and only if $x \leq y \to z$,
(B3) $x \wedge y = x \odot (x \to y)$,
(B4) $(x \to y) \vee (y \to x) = 1$.

Let $\mathcal{L}$ be a BL-algebra. A subset $\mathcal{F}$ of $\mathcal{L}$ is called a *positive implicative filter* if it satisfies the following conditions for all $x, y, z \in \mathcal{L}$:

(F1) $1 \in \mathcal{F}$,
(F2) $x \to (y \to z) \in \mathcal{F}$ and $x \to y \in \mathcal{F}$ imply that $x \to z \in \mathcal{F}$.

A *fuzzy set* in $\mathcal{L}$ is a mapping $\mu : \mathcal{L} \longrightarrow [0, 1]$. Also, for $t \in [0, 1]$, the set $\mu_t = \{x \in \mathcal{L} \mid \mu(x) \geq t\}$ is called a *level subset* of μ. For convenience, for any $x, y \in [0, 1]$, we denote $\max\{x, y\}$ and $\min\{x, y\}$ by $x \vee y$ and $x \wedge y$, respectively.

A fuzzy set μ in $\mathcal{L}$ is called a *fuzzy positive implicative filter* of $\mathcal{L}$, if for all $x, y, z \in \mathcal{L}$, it satisfies the following conditions:

(F3) $\mu(1) \geq \mu(x)$,
(F4) $\mu(x \to z) \geq \mu(x \to (y \to z)) \wedge \mu(x \to y)$.

Let $\mathcal{L} = \{0, a, b, 1\}$ be a chain with Cayley tables as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Define operations $\wedge$ and $\vee$ on $\mathcal{L}$ as min and max, respectively. Then $(\mathcal{L}, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra. Define a fuzzy set μ in $\mathcal{L}$ by $\mu(1) = t_2$ and $\mu(b) = \mu(a) = \mu(0) = t_1$, where $0 \leq t_1 < t_2 \leq 1$. It is easy to verify that μ is a fuzzy positive implicative filter of $\mathcal{L}$.

2.1. Theorem. [16] *Let μ be a fuzzy set of $\mathcal{L}$. Then μ is a fuzzy positive implicative filter of $\mathcal{L}$ if and only if for all $t \in [0,1]$, μ_t is either empty or a positive implicative filter of $\mathcal{L}$.* □

2.2. Corollary. [16] *Let $\mathcal{L}$ be a BL-algebra. Then, $\mathcal{F}$ is a positive implicative filter of $\mathcal{L}$ if and only if $\chi_{\mathcal{F}}$ is a fuzzy positive implicative filter of $\mathcal{L}$, where $\chi_{\mathcal{F}}$ is the characteristic function of $\mathcal{F}$.* □

3. Fuzzy positive implicative filters

In what follows, $\mathcal{L}$ is a BL-algebra unless otherwise specified.

3.1. Lemma. *Let μ be a fuzzy positive implicative filter of $\mathcal{L}$ and $x \in \mathcal{L}$. Then $\mu(x) = \alpha$ if and only if $x \in \mu_\alpha$ and $x \notin \mu_\gamma$ for all $\gamma > \alpha$.*

Proof. Straightforward. □

3.2. Theorem. *Let $\{\mathcal{F}_\alpha \mid \alpha \in \Lambda \subseteq [0,1]\}$ be a collection of positive implicative filters of $\mathcal{L}$ such that $\mathcal{L} = \bigcup_{\alpha \in \Lambda} \mathcal{F}_\alpha$, and for every $\alpha, \beta \in \Lambda$, $\alpha < \beta$ if and only if $\mathcal{F}_\beta \subset \mathcal{F}_\alpha$. Then the fuzzy set μ of $\mathcal{L}$, defined by $\mu(x) = \sup\{\alpha \in \Lambda \mid x \in \mathcal{F}_\alpha\}$, is a fuzzy positive implicative filter of $\mathcal{L}$.*

Proof. By Theorem 2.1, it is enough to show that for every $\alpha \in [0,1]$, the non-empty set μ_α is a positive implicative filter of $\mathcal{L}$. For this, we consider two cases:

(i) $\alpha = \sup\{\delta \in \Lambda \mid \delta < \alpha\}$, $\quad (ii) \alpha \neq \sup\{\delta \in \Lambda \mid \delta < \alpha\}$.

In the first case

$$x \in \mu_\alpha \iff \forall \delta < \alpha,\ x \in \mathcal{F}_\delta \iff x \in \bigcap_{\delta < \alpha} \mathcal{F}_\delta.$$

So $\mu_\alpha = \bigcap_{\delta < \alpha} \mathcal{F}_\delta$, and hence μ_α is a positive implicative filter of $\mathcal{L}$.

In the second case, we prove that $\mu_\alpha = \bigcup_{\delta \geq \alpha} \mathcal{F}_\delta$. If $x \in \bigcup_{\delta \geq \alpha} \mathcal{F}_\delta$, then $x \in \mathcal{F}_\delta$ for some $\delta \geq \alpha$. Thus $\mu(x) \geq \delta \geq \alpha$, which means $x \in \mu_\alpha$. Hence $\bigcup_{\delta \geq \alpha} \mathcal{F}_\delta \subseteq \mu_\alpha$. Also, if $x \notin \bigcup_{\delta \geq \alpha} \mathcal{F}_\delta$, then $x \notin \mathcal{F}_\delta$ for all $\delta \geq \alpha$. Since $\alpha \neq \sup\{\delta \in \Lambda \mid \delta < \alpha\}$, there exists $\varepsilon > 0$ such that $(\alpha - \varepsilon, \alpha) \cap \Lambda = \phi$. Hence $x \notin \mathcal{F}_\delta$ for all $\delta > \alpha - \varepsilon$, which means that if $x \in \mathcal{F}_\delta$ then $\delta \leq \alpha - \varepsilon$. Thus $\mu(x) \leq \alpha - \varepsilon < \alpha$, and so $x \notin \mu_\alpha$. Therefore $\mu_\alpha = \bigcup_{\delta \geq \alpha} \mathcal{F}_\delta$.

We know that $\bigcup_{\delta \geq \alpha} \mathcal{F}_\delta$ is a positive implicative filter of $\mathcal{L}$, which completes the proof. □

3.3. Corollary. *If μ is a fuzzy positive implicative filter of $\mathcal{L}$, then*

$$\mu(x) = \sup\{t \in [0,1] \mid x \in \mu_t\},$$

for every $x \in \mathcal{L}$.

Proof. Immediate consequence of Theorem 3.2. □

3.4. Theorem. *For any chain $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n = \mathcal{L}$ of positive implicative filters of $\mathcal{L}$, there exists a fuzzy positive implicative filter μ of $\mathcal{L}$ such that the level subsets of μ coincide with the chain.*

Proof. Let $\{\alpha_k \mid k = 0, 1, \ldots, n\}$ be a finite decreasing sequence in $[0, 1]$. Let μ be the fuzzy set of $\mathcal{L}$, defined by $\mu(\mathcal{F}_0) = \alpha_0$ and $\mu(\mathcal{F}_k \setminus \mathcal{F}_{k-1}) = \alpha_k$ for $0 < k \leq n$. Clearly $1 \in \mathcal{F}_0$ and if $x \to (y \to z)$, $x \to y \in \mathcal{F}_k \setminus \mathcal{F}_{k-1}$, then $x \to z \in \mathcal{F}_k$ and $\mu(1) = \alpha_0 \geq \mu(x)$, $\mu(x \to z) \geq \alpha_k = \mu(x \to (y \to z)) \wedge \mu(x \to y)$.

For $i > j$, if $x \to (y \to z) \in \mathcal{F}_i \setminus \mathcal{F}_{i-1}$ and $x \to y \in F_j \setminus \mathcal{F}_{j-1}$, then $\mu(x \to (y \to z)) = \alpha_i = \mu(x \to y)$ and $x \to z \in \mathcal{F}_i$. Thus

$$\mu(x \to z) \geq \alpha_i = \mu(x \to (y \to z)) \wedge \mu(x \to y).$$

Consequently, μ is a fuzzy positive implicative filter of $\mathcal{L}$.

Note that $\mathrm{Im}\mu = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$. It follows that the level subsets of μ are given by the chain of positive implicative filters $\mu_{\alpha_0} \subseteq \mu_{\alpha_1} \subseteq \cdots \subseteq \mu_{\alpha_n} = \mathcal{L}$. Clearly $\mu_{\alpha_0} = \mathcal{F}_0$. We prove that $\mu_{\alpha_k} = \mathcal{F}_k$ for $0 < k \leq n$. Obviously, $\mathcal{F}_k \subseteq \mu_{\alpha_k}$. If $x \in \mu_{\alpha_k}$, then $\mu(x) \geq \alpha_k$ and so $x \notin \mathcal{F}_i$ for $i > k$. Hence $\mu(x) \in \{\alpha_0, \alpha_1, \ldots, \alpha_k\}$, which implies that $x \in \mathcal{F}_i$ for some $i \leq k$. Since $\mathcal{F}_i \subseteq \mathcal{F}_k$, it follows that $x \in \mathcal{F}_k$. Therefore $\mu_{\alpha_k} = \mathcal{F}_k$ for every $0 < k \leq n$. □

In the next theorems, we discuss conditions on a BL-algebra so that every descending chain of positive implicative filters terminates after a finite number of steps.

3.5. Theorem. *If every fuzzy positive implicative filter of $\mathcal{L}$ has a finite image, then every descending chain of positive implicative filters of $\mathcal{L}$ terminates after a finite number of steps.*

Proof. Suppose that there exists a strictly descending chain $\mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots$ of positive implicative filters of $\mathcal{L}$ which does not terminate after a finite number of steps. We prove that μ defined by $\mu(x) = \dfrac{n}{n+1}$ if $x \in \mathcal{F}_n \setminus \mathcal{F}_{n+1}$ (for $n = 0, 1, 2, \ldots$) and $\mu(x) = 1$ if $x \in \bigcap_{n=0}^{\infty} \mathcal{F}_n$, where $\mathcal{F}_0 = \mathcal{L}$, is a fuzzy positive implicative filter of $\mathcal{L}$ with an infinite image. Since $1 \in \bigcap_{n=0}^{\infty} \mathcal{F}_n$, so $\mu(1) = 1 \geq \mu(x)$ for all $x \in \mathcal{L}$. Let $x, y, z \in \mathcal{L}$. Assume that $x \to (y \to z) \in \mathcal{F}_n \setminus \mathcal{F}_{n+1}$, and $x \to y \in \mathcal{F}_k \setminus \mathcal{F}_{k+1}$ for some n and k. Without loss of generality, we can assume that $n \leq k$. Then, obviously $x \to z$, $x \to y \in \mathcal{F}_n$ and

$$\mu(x \to z) \geq \frac{n}{n+1} = \mu(x \to (y \to z)) \wedge \mu(x \to y).$$

If $x \to y$, $x \to (y \to z) \in \bigcap_{n=0}^{\infty} \mathcal{F}_n$, then $x \to z \in \bigcap_{n=0}^{\infty} \mathcal{F}_n$. Thus

$$\mu(x \to z) = 1 = \mu(x \to y) \wedge \mu(x \to (y \to z)).$$

If $x \to y \notin \bigcap_{n=0}^{\infty} \mathcal{F}_n$ and $x \to (y \to z) \in \bigcap_{n=0}^{\infty} \mathcal{F}_n$, then there exists $k \in \mathbb{N}$ such that $x \to y \in \mathcal{F}_k \setminus \mathcal{F}_{k+1}$. So $x \to z \in \mathcal{F}_k$ and

$$\mu(x \to z) \geq \frac{k}{k+1} = \mu(x \to y) \wedge \mu(x \to (y \to z)).$$

Finally suppose that $x \to y \in \bigcap_{n=0}^{\infty} \mathcal{F}_n$ and $x \to (y \to z) \notin \bigcap_{n=0}^{\infty} \mathcal{F}_n$. Then $x \to (y \to z) \in \mathcal{F}_r \setminus \mathcal{F}_{r+1}$ for some $r \in \mathbb{N}$. Hence $x \to z \in \mathcal{F}_r$, which implies that

$$\mu(x \to z) \geq \frac{r}{r+1} = \mu(x \to y) \wedge \mu(x \to (y \to z)).$$

Therefore, μ is a fuzzy positive implicative filter of $\mathcal{L}$ with an infinite image. This is a contradiction. □

3.6. Theorem. *Let every descending chain of positive implicative filters of $\mathcal{L}$ terminates after a finite number of steps. If μ is a fuzzy positive implicative filter of $\mathcal{L}$ such that a sequence of elements of* Imμ *is strictly increasing, then μ has a finite number of different values.*

Proof. Suppose that Imμ is not finite. Let $0 \leq \alpha_1 < \alpha_2 < \cdots \leq 1$ be a strictly increasing sequence of elements of Imμ. Then every μ_{α_t} is a positive implicative filter of $\mathcal{L}$. For $x \in \mu_{\alpha_t}$ we have $\mu(x) \geq \alpha_t > \alpha_{t-1}$, which implies that $x \in \mu_{\alpha_{t-1}}$. Hence $\mu_{\alpha_t} \subseteq \mu_{\alpha_{t-1}}$. But for $\alpha_{t-1} \in$ Imμ, there exists $x_{t-1} \in \mathcal{L}$ such that $\mu(x_{t-1}) = \alpha_{t-1}$. This gives $x_{t-1} \in \mu_{\alpha_{t-1}}$ and $x_{t-1} \notin \mu_{\alpha_t}$. Thus $\mu_{\alpha_t} \subsetneqq \mu_{\alpha_{t-1}}$, and so we obtain a strictly descending chain $\mu_{\alpha_1} \supset \mu_{\alpha_2} \supset \mu_{\alpha_3} \supset \cdots$ of positive implicative filters of $\mathcal{L}$ which is not terminating. This is a contradiction, which completes the proof. □

In the next theorem, we prove an equivalent statement for BL-algebras with an ascending chain condition of positive implicative filters.

3.7. Theorem. *Every ascending chain of positive implicative filters of $\mathcal{L}$ terminates after a finite number of steps if and only if for any fuzzy positive implicative filter of $\mathcal{L}$,* Imμ *is a well-ordered subset of* $[0, 1]$.

Proof. Suppose that for a fuzzy positive implicative filter μ, the set of Imμ is not a well-ordered subset of $[0, 1]$. Then there exists a strictly decreasing sequence $\{\alpha_n\}_{n=0}^{\infty}$ such that $\alpha_n = \mu(x_n)$ for some $x_n \in \mathcal{L}$. In this case μ_{α_n} form a strictly ascending chain of positive implicative filters of $\mathcal{L}$ which is not terminating. This is a contradiction. Therefore Imμ is a well-ordered subset of $[0, 1]$.

Conversely, suppose that there exists a strictly ascending chain $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \cdots$ of positive implicative filters of $\mathcal{L}$ which does not terminate after a finite number of steps. Then $\mathcal{F} = \bigcup_{k=1}^{\infty} \mathcal{F}_k$ is a positive implicative filter of $\mathcal{L}$. Define μ on $\mathcal{L}$ by $\mu(x) = \dfrac{1}{k}$ for $x \in \mathcal{F}_k \setminus \mathcal{F}_{k-1}$ and $\mu(x) = 0$ for $x \notin \mathcal{F}$, where $\mathcal{F}_0 = \phi$. Clearly $\mu(1) = 1 \geq \mu(x)$ for all $x \in \mathcal{L}$. Let $x, y, z \in \mathcal{L}$. We consider the following cases:

(1) $x \to (y \to z)$, $x \to y \in \mathcal{F}$. In this case there are m, n such that $x \to (y \to z) \in \mathcal{F}_n \setminus \mathcal{F}_{n-1}$ and $x \to y \in \mathcal{F}_m \setminus \mathcal{F}_{m-1}$. Obviously $x \to z \in \mathcal{F}_k \setminus \mathcal{F}_{k-1} \subset \mathcal{F}_p$, where $k \leq p = m \vee n$. So $\mu(x \to (y \to z)) = \dfrac{1}{n}$, $\mu(x \to y) = \dfrac{1}{m}$ and

$$\mu(x \to z) = \frac{1}{k} \geq \frac{1}{p} = \mu(x \to (y \to z)) \wedge \mu(x \to y).$$

(2) $x \to (y \to z) \notin \mathcal{F}$ and $x \to y \in \mathcal{F}$. In this case $x \to y \in \mathcal{F}_m \setminus \mathcal{F}_{m-1}$ for some natural number m. Hence $\mu(x \to (y \to z)) = 0$ and $\mu(x \to y) = \dfrac{1}{m}$, which imply that

$$\mu(x \to z) \geq 0 = \mu(x \to (y \to z)) \wedge \mu(x \to y).$$

(3) $x \to (y \to z) \in \mathcal{F}$ and $x \to y \notin \mathcal{F}$. In this case $x \to (y \to z) \in \mathcal{F}_n \setminus \mathcal{F}_{n-1}$ for some natural n. Hence $\mu(x \to y) = 0$ and $\mu(x \to (y \to z)) = \dfrac{1}{n}$, which imply that

$$\mu(x \to z) \geq 0 = \mu(x \to (y \to z)) \wedge \mu(x \to y).$$

(4) $x \to (y \to z)$, $x \to y \notin \mathcal{F}$. Obviously,

$$\mu(x \to z) \geq 0 = \mu(x \to (y \to z)) \wedge \mu(x \to y).$$

Therefore, μ is a fuzzy positive implicative filter of $\mathcal{L}$. Since the chain $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \cdots$ is not terminating, μ has a strictly descending sequence of values. This contradicts that the value set of any fuzzy positive implicative filter of $\mathcal{L}$ is well-ordered. This completes the proof. □

4. Maximal fuzzy positive implicative filters of BL-algebras

4.1. Definition. A fuzzy positive implicative filter μ of $\mathcal{L}$ is said to be *normal* if there exists an element $x_0 \in \mathcal{L}$ such that $\mu(x_0) = 1$.

Clearly, a fuzzy positive implicative filter μ is normal if and only if $\mu(1) = 1$. Also, any fuzzy positive implicative filter containing a normal fuzzy positive implicative filter is normal too.

4.2. Example. Let $\mathcal{L} = \{0, a, b, 1\}$ be a chain with Cayley tables as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

Define operations $\wedge$ and $\vee$ on $\mathcal{L}$ as min and max, respectively. Then $(\mathcal{L}, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra. Define a fuzzy set μ in $\mathcal{L}$ by $\mu(1) = \mu(b) = \mu(a) = 1$ and $\mu(0) = \frac{1}{2}$. It is easy to verify that μ is a normal fuzzy positive implicative filter of $\mathcal{L}$.

4.3. Theorem. *Let μ be a fuzzy positive implicative filter of $\mathcal{L}$. Then the fuzzy set μ^+, where $\mu^+(x) = \mu(x) + 1 - \mu(1)$ for all $x \in \mathcal{L}$, is a normal fuzzy positive implicative filter of $\mathcal{L}$ containing μ.*

Proof. Clearly $\mu^+(x) \in [0, 1]$ for every $x \in \mathcal{L}$ and $\mu^+(1) = 1$. We prove that μ^+ is a fuzzy positive implicative filter of $\mathcal{L}$. Let $x, y, z \in \mathcal{L}$. We have

$$\begin{aligned}\mu^+(x \rightarrow z) &= \mu(x \rightarrow z) + 1 - \mu(1)\\ &\geq \big(\mu(x \rightarrow (y \rightarrow z)) \wedge \mu(x \rightarrow y)\big) + 1 - \mu(1)\\ &= \big(\mu(x \rightarrow (y \rightarrow z)) + 1 - \mu(1)\big) \wedge (\mu(x \rightarrow y) + 1 - \mu(1))\\ &= \mu^+(x \rightarrow (y \rightarrow z)) \wedge \mu^+(x \rightarrow y),\end{aligned}$$

which proves that μ^+ is a fuzzy positive implicative filter of $\mathcal{L}$. Clearly, $\mu \subseteq \mu^+$, which completes the proof. □

4.4. Corollary. $(\mu^+)^+ = \mu^+$ *for any fuzzy positive implicative filter μ of $\mathcal{L}$. If μ is normal, then $\mu^+ = \mu$.* □

We denote the set of all normal fuzzy positive implicative filters of $\mathcal{L}$ by $N(\mathcal{L})$. Clearly, $N(\mathcal{L})$ is a partially ordered set under fuzzy set inclusion.

4.5. Theorem. *A non-constant maximal element μ of $N(\mathcal{L})$ is two-valued and takes only the values* 0 *and* 1.

Proof. We know that $\mu(1) = 1$. Let $x \in \mathcal{L}$ be such that $\mu(x) \neq 1$. We claim that $\mu(x) = 0$. If not, then there exists $a \in \mathcal{L}$ such that $0 < \mu(a) < 1$. Let ν be the fuzzy set of $\mathcal{L}$ defined by $\nu(x) = \frac{1}{2}\big(\mu(x) + \mu(a)\big)$ for all $x \in \mathcal{L}$. Clearly, ν is well-defined. For all $x, y, z \in \mathcal{L}$ we have

$$\nu(1) = \frac{1}{2}\big(\mu(1) + \mu(a)\big) \geq \frac{1}{2}\big(\mu(x) + \mu(a)\big) = \nu(x),$$

also

$$\begin{aligned}\nu(x \to z) &= \frac{1}{2}\big(\mu(x \to z) + \mu(a)\big)\\ &\geq \frac{1}{2}\big((\mu(x \to (y \to z)) \wedge \mu(x \to y)) + \mu(a)\big)\\ &= \Big[\frac{1}{2}\big(\mu(x \to (y \to z)) + \mu(a)\big)\Big] \wedge \Big[\frac{1}{2}\big(\mu(x \to y)) + \mu(a)\big)\Big]\\ &= \nu(x \to (y \to z)) \wedge \nu(x \to y).\end{aligned}$$

This proves that ν is a fuzzy positive implicative filter of $\mathcal{L}$. Now, by Theorem 4.3, $\nu^+ \in N(\mathcal{L})$. Clearly $\mu \subseteq \nu^+$, and since $\nu^+(a) = \frac{1}{2}(1 + \mu(a)) > \mu(a)$, so μ is a proper subset of ν^+. Obviously $\nu^+(a) < 1 = \nu^+(1)$. Hence ν^+ is non-constant, and μ is not maximal element of $N(\mathcal{L})$. This is a contradiction. Therefore $|\mathrm{Im}\mu| = 2$ and μ takes only the values 0 and 1. □

4.6. Definition. A non-constant fuzzy positive implicative filter μ of $\mathcal{L}$ is called *maximal* if μ^+ is a maximal element of the poset $N(\mathcal{L})$.

4.7. Theorem. *A maximal fuzzy positive implicative filter μ of $\mathcal{L}$ is normal and takes only the values* 0 *and* 1.

Proof. Let μ be a maximal fuzzy positive implicative filter μ of $\mathcal{L}$. Then μ^+ is a non-constant maximal element of the poset $N(\mathcal{L})$ and by Theorem 4.5, μ^+ takes only the values 0 and 1. Clearly $\mu^+(x) = 1$ if and only if $\mu(x) = \mu(1)$ and $\mu^+(x) = 0$ if and only if $\mu(x) = \mu(1) - 1$. But $\mu \subseteq \mu^+$ (by Theorem 4.3). So $\mu^+(x) = 0$ implies that $\mu(x) = 0$, consequently $\mu(1) = 1$. Therefore μ is normal. □

4.8. Theorem. *If μ is a maximal fuzzy positive implicative filter of $\mathcal{L}$, then μ_1 is a maximal positive implicative filter of $\mathcal{L}$.*

Proof. Let $\mathcal{F}_1 = \mu_1 = \{x \in \mathcal{L} \mid \mu(x) = 1\}$. By Theorem 2.1, $\mathcal{F}_1$ is a positive implicative filter of $\mathcal{L}$. Obviously $\mathcal{F}_1 \neq \mathcal{L}$, because μ is two-valued. Let $\mathcal{F}_2 (\neq \mathcal{L})$ be a positive implicative filter of $\mathcal{L}$ containing $\mathcal{F}_1$. Then $\chi_{\mathcal{F}_1} \subseteq \chi_{\mathcal{F}_2}$ (characteristic functions). But we know that μ is a maximal fuzzy positive implicative filter of $\mathcal{L}$, so $\chi_{\mathcal{F}_1} = \mu = \chi_{\mathcal{F}_2}$ or $\chi_{\mathcal{F}_2}(x) = 1$ for all $x \in \mathcal{L}$. If $\chi_{\mathcal{F}_2}(x) = 1$ for all $x \in \mathcal{L}$, then $\mathcal{F}_2 = \mathcal{L}$, which is a contradiction. So $\mu = \chi_{\mathcal{F}_1} = \chi_{\mathcal{F}_2}$, which implies $\mathcal{F}_1 = \mathcal{F}_2$. Therefore $\mathcal{F}_1$ is a maximal positive implicative filter of $\mathcal{L}$. □

4.9. Definition. A normal fuzzy positive implicative filter μ of $\mathcal{L}$ is called *completely normal* if there exists $x \in \mathcal{L}$ such that $\mu(x) = 0$.

We denote the set of completely normal fuzzy positive implicative filters of $\mathcal{L}$ by $C(\mathcal{L})$. It is obvious that $C(\mathcal{L}) \subseteq N(\mathcal{L})$.

4.10. Theorem. *A non-constant element of $N(\mathcal{L})$ is a maximal element of $C(\mathcal{L})$.*

Proof. Let μ be a non-constant maximal element of $N(\mathcal{L})$. By Theorem 4.5, μ is two-valued and takes only the values 0 and 1. Let $\mu(x_0) = 1$ and $\mu(x_1) = 0$ for some $x_0, x_1 \in \mathcal{L}$. Hence $\mu \in C(\mathcal{L})$. Assume that there exists $\nu \in C(\mathcal{L})$ such that $\mu \subseteq \nu$ in $N(\mathcal{L})$. Since μ is maximal in $N(\mathcal{L})$ and since ν is non-constant, thus $\mu = \nu$. Therefore μ is a maximal element of $C(\mathcal{L})$. □

4.11. Theorem. *Every maximal fuzzy positive implicative filter μ of $\mathcal{L}$ is completely normal.*

Proof. Let μ be a maximal fuzzy positive implicative filter of $\mathcal{L}$. Then by Theorem 4.7 and Corollary 4.4, μ is normal, $\mu = \mu^+$ and μ is two-valued. Since μ is non-constant, it follows that $\mu(1) = 1$ and $\mu(0) = 0$. Therefore μ is completely normal. □

4.12. Theorem. *Let $f : [0,1] \longrightarrow [0,1]$ be a strictly increasing function and μ a fuzzy set of $\mathcal{L}$. Then μ_f, defined by $\mu_f(x) = f(\mu(x))$ for all $x \in \mathcal{L}$, is a fuzzy positive implicative filter of $\mathcal{L}$ if and only if μ is a fuzzy positive implicative filter of $\mathcal{L}$.*

Proof. Let μ_f be a fuzzy positive implicative filter of $\mathcal{L}$. Then

$$f(\mu(1)) = \mu_f(1) \geq \mu_f(x) = f(\mu(x)).$$

This gives $f(\mu(1)) \geq f(\mu(x))$ for all $x \in \mathcal{L}$. Since f is strictly increasing, it implies that $\mu(1) \geq \mu(x)$. Also we have

$$\begin{aligned} f\Big(\mu(x \to z)\Big) &= \mu_f(x \to z) \\ &\geq \mu_f(x \to (y \to z)) \wedge \mu_f(x \to y) \\ &= f\big(\mu(x \to (y \to z))\big) \wedge f\big(\mu(x \to y)\big). \end{aligned}$$

Hence,

$$f\big(\mu(x \to z)\big) \geq f\big(\mu(x \to (y \to z))\big) \wedge f\big(\mu(x \to y)\big)$$

for all $x, y, z \in \mathcal{L}$. Since f is strictly increasing, it implies that

$$\mu(x \to z) \geq \mu(x \to (y \to z)) \wedge \mu(x \to y).$$

Conversely, if μ is a fuzzy positive implicative filter of $\mathcal{L}$, then for all $x, y, z \in \mathcal{L}$ we have

$$\mu_f(1) = f(\mu(1)) \geq f(\mu(x)) = \mu_f(x).$$

This gives $\mu_f(1) \geq \mu_f(x)$. Also we have

$$\begin{aligned} \mu_f(x \to z) &= f\big(\mu(x \to z)\big) \\ &\geq f\big(\mu(x \to (y \to z))\big) \wedge f\big(\mu(x \to y)\big) \\ &= \mu_f(x \to (y \to z)) \wedge \mu_f(x \to y). \end{aligned}$$

Hence,

$$\mu_f(x \to z) \geq \mu_f\big(x \to (y \to z)\big) \wedge \mu_f(x \to y).$$

Therefore μ_f is a fuzzy positive implicative filter of $\mathcal{L}$. □

4.13. Theorem. *Let μ be a fuzzy positive implicative filter of $\mathcal{L}$, $\mu(0) \neq 0$ and let $\widetilde{\mu}$ be the fuzzy set of $\mathcal{L}$ defined by $\widetilde{\mu}(x) = \dfrac{\mu(x)}{\mu(0)}$ for all $x \in \mathcal{L}$. Then $\widetilde{\mu}$ is a normal fuzzy positive implicative filter of $\mathcal{L}$ and $\mu \subseteq \widetilde{\mu}$.*

Proof. Let $x, y, z \in \mathcal{L}$. We have

$$\widetilde{\mu}(1) = \frac{\mu(1)}{\mu(0)} \geq \frac{\mu(x)}{\mu(0)} = \widetilde{\mu}(x).$$

This gives $\widetilde{\mu}(1) \geq \widetilde{\mu}(x)$. Also we have

$$\begin{aligned}\widetilde{\mu}(x \to z) &= \frac{\mu(x \to z)}{\mu(0)} \\ &\geq \frac{\mu(x \to (y \to z)) \wedge \mu(x \to y)}{\mu(0)} \\ &= \frac{\mu(x \to (y \to z))}{\mu(0)} \wedge \frac{\mu(x \to y)}{\mu(0)} \\ &= \widetilde{\mu}(x \to (y \to z)) \wedge \widetilde{\mu}(x \to y).\end{aligned}$$

Hence,

$$\widetilde{\mu}(x \to z) \geq \widetilde{\mu}(x \to (y \to z)) \wedge \widetilde{\mu}(x \to y).$$

Therefore $\widetilde{\mu}$ is a fuzzy positive implicative filter of $\mathcal{L}$. Clearly, $\widetilde{\mu}$ is normal and $\mu \subseteq \widetilde{\mu}$. □

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