Hacettepe Journal of Mathematics and Statistics
Volume 40 (6) (2011), 829 – 837

SOME CONVEXITY PROPERTIES FOR TWO NEW P-VALENT INTEGRAL OPERATORS

Erhan Deniz*†, Murat Çağlar* and Halit Orhan*

Received 11 : 11 : 2010 : Accepted 09 : 05 : 2011

Abstract

In this paper, we define two new general p-valent integral operators in the unit disc $\mathbb{U}$, and obtain the convexity properties of these integral operators of p-valent functions on some classes of β-uniformly p-valent starlike and β-uniformly p-valent convex functions of complex order. As special cases, the convexity properties of the operators $\int_0^z \left(\frac{f(t)}{t}\right)^{\mu} dt$ and $\int_0^z (g'(t))^{\mu}\, dt$ are given.

Keywords: Analytic functions, Integral operators, β-uniformly p-valent starlike and β-uniformly p-valent convex functions, Complex order.

2000 AMS Classification: Primary 30 C 80. Secondary 30 C 45.

1. Introduction and preliminaries

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \ (p \in \mathbb{N} = \{1, 2, \dots, \}), \tag{1.1}$$

which are analytic in the open disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f \in \mathcal{S}_p^*(\gamma, \alpha)$ is $p-$valently starlike of complex order γ ($\gamma \in \mathbb{C} - \{0\}$) and type α ($0 \leq \alpha < p$), that is, $f \in \mathcal{S}_p^*(\gamma, \alpha)$, if it satisfies the following inequality;

$$\Re\left\{p + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - p\right)\right\} > \alpha, \ (z \in \mathbb{U}). \tag{1.2}$$

*Department of Mathematics, Faculty of Science, Atatürk University, TR-25240 Erzurum, Turkey. E-mail: (E. Deniz) edeniz@atauni.edu.tr (M. Çağlar) mcaglar@atauni.edu.tr (H. Orhan) horhan@atauni.edu.tr

†Corresponding Author.

Furthermore, a function $f \in \mathcal{C}_p(\gamma,\alpha)$ is $p-$valently convex of complex order γ ($\gamma \in \mathbb{C}-\{0\}$) and type α ($0 \leq \alpha < p$), that is, $f \in \mathcal{C}_p(\gamma,\alpha)$ if it satisfies the following inequality;

$$\Re\left\{p+\frac{1}{\gamma}\left(1+\frac{zf''(z)}{f'(z)}-p\right)\right\} > \alpha,\ (z \in \mathbb{U})\,. \tag{1.3}$$

In particular cases, for $p=1$ in the classes $\mathcal{S}_p^*(\gamma,\alpha)$ and $\mathcal{C}_p(\gamma,\alpha)$, we obtain the classes $\mathcal{S}^*(\gamma,\alpha)$ and $\mathcal{C}(\gamma,\alpha)$ of starlike functions of complex order γ ($\gamma \in \mathbb{C}-\{0\}$) and type α ($0 \leq \alpha < p$), and convex functions of complex order γ ($\gamma \in \mathbb{C}-\{0\}$) and type α ($0 \leq \alpha < p$), respectively, which were introduced and studied by Frasin [12].

Also, for $\alpha = 0$ in the classes $\mathcal{S}_p^*(\gamma,\alpha)$ and $\mathcal{C}_p(\gamma,\alpha)$, we obtain the classes $\mathcal{S}_p^*(\gamma)$ and $\mathcal{C}_p(\gamma)$, which are called p-valently starlike of complex order γ ($\gamma \in \mathbb{C}-\{0\}$), and p-valently convex of complex order γ ($\gamma \in \mathbb{C}-\{0\}$), respectively.

Setting $p=1$ and $\alpha=0$, we obtain the classes $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\gamma)$. The class $\mathcal{S}^*(\gamma)$ of starlike functions of complex order γ ($\gamma \in \mathbb{C}-\{0\}$) was defined by Nasr and Aouf (see [18]), while the class $\mathcal{C}(\gamma)$ of convex functions of complex order γ ($\gamma \in \mathbb{C}-\{0\}$) was considered earlier by Wiatrowski (see [25]). Note that $\mathcal{S}_p^*(1,\alpha) = \mathcal{S}_p^*(\alpha)$ and $\mathcal{C}_p(1,\alpha) = \mathcal{C}_p(\alpha)$ are, respectively, the classes of p-valently starlike and p-valently convex functions of order α ($0 \leq \alpha < p$) in $\mathbb{U}$. Also, we note that $\mathcal{S}_1^*(\alpha) = \mathcal{S}^*(\alpha)$ and $\mathcal{C}_1(\alpha) = \mathcal{C}(\alpha)$ are, respectively, the usual classes of starlike and convex functions of order α ($0 \leq \alpha < 1$) in $\mathbb{U}$. In special cases, $\mathcal{S}_1^*(0) = \mathcal{S}^*$ and $\mathcal{C}_1 = \mathcal{C}$ are, respectively, the familiar classes of starlike and convex functions in $\mathbb{U}$.

A function $f \in \beta$-$\mathcal{US}_p(\alpha)$ is β-uniformly p-valently starlike of order α ($-1 \leq \alpha < p$), that is, $f \in \beta$-$\mathcal{US}_p(\alpha)$ if it is satisfies the following inequality;

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta\left|\frac{zf'(z)}{f(z)}-p\right| + \alpha,\ (\beta \geq 0,\ z \in \mathbb{U})\,. \tag{1.4}$$

Furthermore, a function $f \in \beta$-$\mathcal{UC}_p(\alpha)$ is β-uniformly p-valently convex of order α ($-1 \leq \alpha < p$), that is, $f \in \beta$-$\mathcal{UC}_p(\alpha)$ if it satisfies the following inequality;

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \beta\left|1+\frac{zf''(z)}{f'(z)}-p\right| + \alpha,\ (\beta \geq 0,\ z \in \mathbb{U})\,. \tag{1.5}$$

These classes generalize various other classes which are worthy of mention here. For example for $p=1$, the classes β-$\mathcal{US}(\alpha)$ and β-$\mathcal{UC}(\alpha)$ introduced by Bharti, Parvatham and Swaminathan (see [2]). Also, the class β-$\mathcal{UC}_1(0) = \beta$-$\mathcal{UCV}$ is the known class of *β-uniformly convex* functions [15]. Using an Alexander type relation, we can obtain the class β-$\mathcal{US}_p(\alpha)$ in the following way:

$$f \in \beta\text{-}\mathcal{UC}_p(\alpha) \iff \frac{zf'}{p} \in \beta\text{-}\mathcal{US}_p(\alpha).$$

The class 1-$\mathcal{UC}_1(0) = \mathcal{UCV}$ of uniformly convex functions was defined by Goodman [14], while the class 1-$\mathcal{US}_1(0) = \mathcal{SP}$ was considered by Rønning [24].

For $f \in \mathcal{A}_p$ given by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \tag{1.6}$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f*g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g*f)(z),\ (z \in \mathbb{U})\,. \tag{1.7}$$

The n-th order Ruscheweyh derivative $R^n : \mathcal{A}_p \to \mathcal{A}_p$ is defined by

$$R^n f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z),\ (n > -p)\,. \tag{1.8}$$

In terms of the binomial coefficients, we can rewrite (1.8) as follows:

$$R^n f(z) = z^p + \sum_{k=p+1}^{\infty} \binom{n+k-1}{k-p} a_k z^k,\ (n > -p)\,. \tag{1.9}$$

In particular, when $n = \lambda \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, it is easily observed from (1.8) and (1.9) that

$$R^\lambda f(z) = \frac{z^p \left(z^{\lambda-p} f(z)\right)^{(\lambda)}}{\lambda!},\ (\lambda \in \mathbb{N}_0,\ p \in \mathbb{N})\,. \tag{1.10}$$

The symbol R^n is called the Ruscheweyh derivative of nth order defined by Goel and Sohi [13].

By using the operator R^λ $(\lambda \in \mathbb{N}_0)$ defined by (1.10), we introduce the new classes β-$\mathcal{US}_p(\lambda, \gamma, \alpha)$ and β-$\mathcal{UC}_p(\lambda, \gamma, \alpha)$ as follows:

1.1. Definition. Let $-1 \le \alpha < p$, $\beta \ge 0$ and $\gamma \in \mathbb{C} - \{0\}$. A function $f \in \mathcal{A}_p$ is in the class β-$\mathcal{US}_p(\lambda, \gamma, \alpha)$ if and only if for all $z \in \mathbb{U}$,

$$\Re\left\{p + \frac{1}{\gamma}\left(\frac{z\left(R^\lambda f(z)\right)'}{R^\lambda f(z)} - p\right)\right\} > \beta\left|\frac{1}{\gamma}\left(\frac{z\left(R^\lambda f(z)\right)'}{R^\lambda f(z)} - p\right)\right| + \alpha. \tag{1.11}$$

1.2. Definition. Let $-1 \le \alpha < p$, $\beta \ge 0$ and $\gamma \in \mathbb{C} - \{0\}$. A function $f \in \mathcal{A}_p$ is in the class β-$\mathcal{UC}_p(\lambda, \gamma, \alpha)$ if and only if for all $z \in \mathbb{U}$

$$\Re\left\{p + \frac{1}{\gamma}\left(\frac{z\left(R^\lambda f(z)\right)''}{\left(R^\lambda f(z)\right)'} + 1 - p\right)\right\} > \beta\left|\frac{1}{\gamma}\left(\frac{z\left(R^\lambda f(z)\right)''}{\left(R^\lambda f(z)\right)'} + 1 - p\right)\right| + \alpha. \tag{1.12}$$

We note that by specializing the parameters λ, p, γ, β and α in the classes β-$\mathcal{US}_p(\lambda, \gamma, \alpha)$ and β-$\mathcal{UC}_p(\lambda, \gamma, \alpha)$, these classes reduces to several well-known subclasses of analytic functions. For example, for $p = 1$ and $\lambda = 0$ the classes β-$\mathcal{US}_p(\lambda, \gamma, \alpha)$ and β-$\mathcal{UC}_p(\lambda, \gamma, \alpha)$ reduces to the classes β-$\mathcal{US}(\gamma, \alpha)$ and β-$\mathcal{UC}(\gamma, \alpha)$, respectively. The reader can find more information about these classes in Deniz, Orhan and Sokol [10], Orhan, Deniz and Raducanu [19] and Oros [20].

1.3. Definition. Let $l = (l_1, l_2, \ldots, l_m) \in \mathbb{N}_0^m$, $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}_+^m$ for all $i = \overline{1, m}$, $m \in \mathbb{N}$. We define the following general integral operators

$$\mathcal{I}_{p,m}^{l,\mu}\left(f_1, f_2, \ldots, f_m\right) : \mathcal{A}_p^m \to \mathcal{A}_p$$

$$\mathcal{I}_{p,m}^{l,\mu}\left(f_1, f_2, \ldots, f_m\right) = \mathcal{F}_{p,m,l,\mu}(z),$$

$$\mathcal{F}_{p,m,l,\mu}(z) = \int_0^z p t^{p-1} \prod_{i=1}^{m} \left(\frac{R^{l_i} f_i(t)}{t^p}\right)^{\mu_i} dt \tag{1.13}$$

and

$$\mathcal{J}_{p,m}^{l,\mu}\left(g_1, g_2, \ldots, g_m\right) : \mathcal{A}_p^m \to \mathcal{A}_p,$$

$$\mathcal{J}_{p,m}^{l,\mu}\left(g_1, g_2, \ldots, g_m\right) = \mathcal{G}_{p,m,l,\mu}(z),$$

$$\mathcal{G}_{p,m,l,\mu}(z) = \int_0^z p t^{p-1} \prod_{i=1}^{m} \left(\frac{\left(R^{l_i} g_i(t)\right)'}{p t^{p-1}}\right)^{\mu_i} dt, \tag{1.14}$$

where $f_i, g_i \in \mathcal{A}_p$ for all $i = \overline{1, m}$ and R^l is defined by (1.10).

1.4. Remark. We note that if $l_1 = l_2 = \cdots = l_m = 0$ for all $i = \overline{1,m}$, then the integral operator $\mathcal{F}_{p,m,l,\mu}(z)$ reduces to the operator $F_p(z)$, which was studied by Frasin (see [11]). Upon setting $p = 1$ in the operator (1.13), we can obtain the integral operator $\mathbb{F}_m(z)$ which was studied by Oros and Oros (see [21]). For $p = 1$ and $l_1 = l_2 = \cdots = l_m = 0$ in (1.13), the integral operator $\mathcal{F}_{p,m,l,\mu}(z)$ reduces to the operator $F_m(z)$ which was studied by Breaz and Breaz (see [6]). Observe that when $p = m = 1$, $l_1 = 0$ and $\mu_1 = \mu$, we obtain the integral operator $I_\mu(f)(z)$ which was studied by Pescar and Owa (see [22]), for $\mu_1 = \mu \in [0,1]$ a special case of the operator $I_\mu(f)(z)$ was studied by Miller, Mocanu and Reade (see [17]). For $p = m = 1$, $l_1 = 0$ and $\mu_1 = 1$ in (1.13), we have the Alexander integral operator $I(f)(z)$ in [1].

1.5. Remark. For $l_1 = l_2 = \cdots = l_m = 0$ in (1.14) the integral operator $\mathcal{G}_{p,mn,l,\mu}(z)$ reduces to the operator $G_p(z)$ which was studied by Frasin (see [11]). For $p = 1$ and $l_1 = l_2 = \cdots = l_m = 0$ in (1.14), the integral operator $\mathcal{G}_{p,m,l,\mu}(z)$ reduces to the operator $G_{\mu_1,\mu_2,\ldots,\mu_m}(z)$ which was studied by Breaz, Owa and Breaz (see [8]). If $p = m = 1$, $l_1 = 0$ and $\mu_1 = \mu$, we obtain the integral operator $G(z)$ which was introduced and studied by Pfaltzgraff (see [23]) and Kim and Merkes (see [16]).

In this paper, we consider the integral operators $\mathcal{F}_{p,m,l,\mu}(z)$ and $\mathcal{G}_{p,m,l,\mu}(z)$ defined by (1.13) and (1.14), respectively, and study their properties on the classes β-$\mathcal{US}_p(\lambda,\gamma,\alpha)$ and β-$\mathcal{UC}_p(\lambda,\gamma,\alpha)$. As special cases, the order of convexity of the operators $\int_0^z \left(\frac{f(t)}{t}\right)^\mu dt$ and $\int_0^z (g'(t))^\mu\, dt$ are given.

2. Sufficient conditions on the integral operator $\mathcal{F}_{p,m,l,\mu}(z)$

First, in this section we prove a sufficient condition for the integral operator $\mathcal{F}_{p,m,l,\mu}(z)$ to be p-valently convex.

2.1. Theorem. *Let $l = (l_1, l_2, \ldots, l_m) \in \mathbb{N}_0^m$, $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}_+^m$, $-1 \le \alpha_i < p$, $\beta_i \ge 0$, $\gamma \in \mathbb{C} - \{0\}$ and $f_i \in \beta_i - \mathcal{US}_p(l_i,\gamma,\alpha_i)$ for all $i = \overline{1,m}$. Moreover, suppose that these numbers satisfy the following inequality*

$$0 \le p + \sum_{i=1}^{m} \mu_i (\alpha_i - p) < p. \tag{2.1}$$

Then the integral operator $\mathcal{F}_{p,m,l,\mu}(z)$ defined by (1.13) is p-valently convex of complex order γ ($\gamma \in \mathbb{C} - \{0\}$) and type $p + \sum_{i=1}^{m} \mu_i (\alpha_i - p)$.

Proof. From the definition (1.13), we observe that $\mathcal{F}_{p,m,l,\mu}(z) \in \mathcal{A}_p$. On the other hand, it is easy to see that

$$\mathcal{F}'_{p,m,l,\mu}(z) = pz^{p-1} \prod_{i=1}^{m} \left(\frac{R^{l_i} f_i(z)}{z^p}\right)^{\mu_i}. \tag{2.2}$$

Now we differentiate (2.2) logarithmically and multiply by z to obtain

$$\frac{z\mathcal{F}''_{p,m,l,\mu}(z)}{\mathcal{F}'_{p,m,l,\mu}(z)} + 1 - p = \sum_{i=1}^{m} \mu_i \left(\frac{z\left(R^{l_i} f_i\right)'(z)}{\left(R^{l_i} f_i\right)(z)} - p\right). \tag{2.3}$$

Then multiplying the relation (2.3) with $\frac{1}{\gamma}$,

$$\frac{1}{\gamma}\left(\frac{z\mathcal{F}''_{p,m,l,\mu}(z)}{\mathcal{F}'_{p,m,l,\mu}(z)} + 1 - p\right) = \sum_{i=1}^{m} \mu_i \frac{1}{\gamma} \left(\frac{z\left(R^{l_i} f_i\right)'(z)}{\left(R^{l_i} f_i\right)(z)} - p\right). \tag{2.4}$$

The relation (2.4) is equivalent to

$$(2.5)\quad p+\frac{1}{\gamma}\left(\frac{z\mathcal{F}''_{p,m,l,\mu}(z)}{\mathcal{F}'_{p,m,l,\mu}(z)}+1-p\right)=p+\sum_{i=1}^{m}\mu_i\left(p+\frac{1}{\gamma}\left(\frac{z\left(R^{l_i}f_i\right)'(z)}{\left(R^{l_i}f_i\right)(z)}-p\right)\right)-p\sum_{i=1}^{m}\mu_i.$$

Lastly, we calculate the real part of both sides of (2.5) and obtain

$$(2.6)\quad \begin{aligned}&\Re\left\{p+\frac{1}{\gamma}\left(\frac{z\mathcal{F}''_{p,m,l,\mu}(z)}{\mathcal{F}'_{p,m,l,\mu}(z)}+1-p\right)\right\}\\&\qquad=\sum_{i=1}^{m}\mu_i\Re\left\{p+\frac{1}{\gamma}\left(\frac{z\left(R^{l_i}f_i\right)'(z)}{\left(R^{l_i}f_i\right)(z)}-p\right)\right\}-p\sum_{i=1}^{m}\mu_i+p.\end{aligned}$$

Since $f_i\in\beta_i-\mathcal{US}_p(l_i,\gamma,\alpha_i)$ for all $i=\overline{1,m}$, from (1.11) and (2.6), we have

$$(2.7)\quad \begin{aligned}&\Re\left\{p+\frac{1}{\gamma}\left(\frac{z\mathcal{F}''_{p,m,l,\mu}(z)}{\mathcal{F}'_{p,m,l,\mu}(z)}+1-p\right)\right\}\\&\qquad>\sum_{i=1}^{m}\frac{\mu_i\beta_i}{|\gamma|}\left|\frac{z\left(R^{l_i}f_i\right)'(z)}{\left(R^{l_i}f_i\right)(z)}-p\right|+p+\sum_{i=1}^{m}\mu_i\left(\alpha_i-p\right).\end{aligned}$$

Because $\sum_{i=1}^{m}\frac{\mu_i\beta_i}{|\gamma|}\left|\frac{z\left(R^{l_i}f_i\right)'(z)}{\left(R^{l_i}f_i\right)(z)}-p\right|>0$, for all $i=\overline{1,m}$, from (2.7), we obtain

$$\Re\left\{p+\frac{1}{\gamma}\left(\frac{z\mathcal{F}''_{p,m,l,\mu}(z)}{\mathcal{F}'_{p,m,l,\mu}(z)}+1-p\right)\right\}>p+\sum_{i=1}^{m}\mu_i\left(\alpha_i-p\right).$$

Therefore, the operator $\mathcal{F}_{p,m,l,\mu}(z)$ is p-valently convex of complex order γ ($\gamma\in\mathbb{C}-\{0\}$) and type $p+\sum_{i=1}^{m}\mu_i\left(\alpha_i-p\right)$. This evidently completes the proof of Theorem 2.1. □

2.2. Remark.

(1) Letting $\gamma=1$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 2.1, we obtain [11, Theorem 2.1].
(2) Letting $p=1$, $\gamma=1$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 2.1, we obtain [4, Theorem 1].
(3) Letting $p=1$, $\gamma=1$ and $\alpha_i=l_i=0$ for all $i=\overline{1,m}$ in Theorem 2.1, we obtain [7, Theorem 2.5]
(4) Letting $p=1$, $\beta=0$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 2.1, we obtain [3, Theorem 1].
(5) Letting $p=1$, $\beta=0$, $\alpha_i=\mu$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 2.1, we obtain [9, Theorem 1].
(6) Letting $p=1$, $\beta=0$, $\alpha_i=0$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 2.1, we obtain [5, Theorem 1].

Putting $p=m=1$, $l_1=0$, $\mu_1=\mu$, $\alpha_1=\alpha$, $\beta_1=\beta$ and $f_1=f$ in Theorem 2.1, we have

2.3. Corollary. *Let $\mu>0$, $-1\le\alpha<1$, $\beta\ge0$, $\gamma\in\mathbb{C}-\{0\}$ and $f\in\beta$-$\mathcal{US}(\gamma,\alpha)$. If $0\le1+\mu(\alpha-1)<1$, then $\int_0^z\left(\frac{f(t)}{t}\right)^{\mu}dt$ is convex of complex order γ ($\gamma\in\mathbb{C}-\{0\}$) and type $\mu(\alpha-1)+1$ in $\mathbb{U}$.* □

2.4. Theorem. *Let $l=(l_1,l_2,\ldots,l_m)\in\mathbb{N}_0^m$, $\mu=(\mu_1,\mu_2,\ldots,\mu_m)\in\mathbb{R}_+^m$, $-1\le\alpha_i<p$, $\beta_i>0$, $\gamma\in\mathbb{C}-\{0\}$ for all $i=\overline{1,m}$ and*

$$(2.8)\quad \left|\frac{z\left(R^{l_i}f_i\right)'(z)}{\left(R^{l_i}f_i\right)(z)}-p\right|>-\frac{p+\sum_{i=1}^{m}\mu_i\left(\alpha_i-p\right)}{\sum_{i=1}^{m}\frac{\mu_i\beta_i}{|\gamma|}}$$

for all $i=\overline{1,m}$, *then the integral operator* $\mathcal{F}_{p,m,l,\mu}(z)$ *defined by (1.13) is p-valently convex of complex order* γ $(\gamma\in\mathbb{C}-\{0\})$.

Proof. From (2.7) and (2.8) we easily get $\mathcal{F}_{p,m,l,\mu}(z)$ is p-valently convex of complex order γ. □

From Theorem 2.4, we easily get

2.5. Corollary. *Let* $l=(l_1,l_2,\ldots,l_m)\in\mathbb{N}_0^m$, $\mu=(\mu_1,\mu_2,\ldots,\mu_m)\in\mathbb{R}_+^m$, $-1\le\alpha_i<p$, $\beta_i>0$, $\gamma\in\mathbb{C}-\{0\}$ *for all* $i=\overline{1,m}$ *and*

$$\Re\left(\frac{z\left(R^{l_i}f_i\right)'(z)}{\left(R^{l_i}f_i\right)(z)}\right)>p-\frac{p+\sum_{i=1}^m\mu_i\left(\alpha_i-p\right)}{\sum_{i=1}^m\frac{\mu_i\beta_i}{|\gamma|}},$$

that is $R^{l_i}f_i\in\mathcal{S}_p^*(\sigma)$, *where* $\sigma=p-\left(p+\sum_{i=1}^m\mu_i\left(\alpha_i-p\right)\right)\big/\sum_{i=1}^m\frac{\mu_i\beta_i}{|\gamma|}$; $0\le\sigma<p$ *for all* $i=\overline{1,m}$, *then the integral operator* $\mathcal{F}_{p,m,l,\mu}(z)$ *is p-valently convex of complex order* γ $(\gamma\in\mathbb{C}-\{0\})$. □

Putting $p=m=1$, $l_1=0$, $\mu_1=\mu$, $\alpha_1=\alpha$, $\beta_1=\beta$ and $f_1=f$ in Corollary 2.5, we have

2.6. Corollary. *Let* $\mu>0$, $-1\le\alpha<1$, $\beta>0$, $\gamma\in\mathbb{C}-\{0\}$ *and* $f\in\mathcal{S}^*(\rho)$, *where* $\rho=\left[\mu(\beta+(1-\alpha)|\gamma|)-|\gamma|\right]\big/\mu\beta$; $0\le\rho<1$, *then the integral operator* $\int_0^z\left(\frac{f(t)}{t}\right)^\mu dt$ *is convex of complex order* γ $(\gamma\in\mathbb{C}-\{0\})$ *in* $\mathbb{U}$. □

3. Sufficient conditions on the integral operator $\mathcal{G}_{p,m,l,\mu}(z)$

Next, in this section we give a sufficient condition for the integral operator $\mathcal{G}_{p,m,l,\mu}(z)$ to be p-valently convex.

3.1. Theorem. *Let* $l=(l_1,l_2,\ldots,l_m)\in\mathbb{N}_0^m$, $\mu=(\mu_1,\mu_2,\ldots,\mu_m)\in\mathbb{R}_+^m$, $-1\le\alpha_i<p$, $\beta_i\ge0$, $\gamma\in\mathbb{C}-\{0\}$ *and* $f_i\in\beta_i$-$\mathcal{UC}_p(l_i,\gamma,\alpha_i)$ *for all* $i=\overline{1,m}$. *Moreover, suppose that these numbers satisfy the following inequality*

$$0\le p+\sum_{i=1}^m\mu_i\left(\alpha_i-p\right)<p.$$

Then the integral operator $\mathcal{G}_{p,m,l,\mu}(z)$ *defined by (1.14) is p-valently convex of complex order* γ $(\gamma\in\mathbb{C}-\{0\})$*and type* $p+\sum_{i=1}^m\mu_i\left(\alpha_i-p\right)$.

Proof. From the definition (1.14), we observe that $\mathcal{G}_{p,m,l,\mu}(z)\in\mathcal{A}_p$. On the other hand, it is easy to see that

$$\mathcal{G}'_{p,m,l,\mu}(z)=pz^{p-1}\prod_{i=1}^m\left(\frac{\left(R^{l_i}g_i(z)\right)'}{pz^{p-1}}\right)^{\mu_i}. \tag{3.1}$$

Now, we differentiate (3.1) logarithmically to obtain

$$\frac{\mathcal{G}''_{p,m,l,\mu}(z)}{\mathcal{G}'_{p,m,l,\mu}(z)}=\frac{p-1}{z}+\sum_{i=1}^m\mu_i\left(\frac{\left(R^{l_i}g_i\right)''(z)}{\left(R^{l_i}g_i\right)'(z)}-\frac{p-1}{z}\right). \tag{3.2}$$

Then multiplying this relation (3.2) with $\frac{z}{\gamma}$, we obtain

$$\frac{1}{\gamma}\left(\frac{z\mathcal{G}''_{p,m,l,\mu}(z)}{\mathcal{G}'_{p,m,l,\mu}(z)}+1-p\right)=\sum_{i=1}^m\mu_i\frac{1}{\gamma}\left(\frac{z\left(R^{l_i}g_i\right)''(z)}{\left(R^{l_i}g_i\right)'(z)}+1-p\right)$$

or

$$p+\frac{1}{\gamma}\left(\frac{z\mathcal{G}''_{p,m,l,\mu}(z)}{\mathcal{G}'_{p,m,l,\mu}(z)}+1-p\right)=p+\sum_{i=1}^{m}\mu_i\frac{1}{\gamma}\left(\frac{z\left(R^{l_i}g_i\right)''(z)}{\left(R^{l_i}g_i\right)'(z)}+1-p\right). \tag{3.3}$$

Taking the real part of both sides of (3.3), we have

$$\begin{aligned}
&\Re\left\{p+\frac{1}{\gamma}\left(\frac{z\mathcal{G}''_{p,n,l,\mu}(z)}{\mathcal{G}'_{p,n,l,\mu}(z)}+1-p\right)\right\}\\
&\qquad=p+\sum_{i=1}^{m}\mu_i\Re\frac{1}{\gamma}\left(\frac{z\left(R^{l_i}g_i\right)''(z)}{\left(R^{l_i}g_i\right)'(z)}+1-p\right)\\
&\qquad=p-p\sum_{i=1}^{m}\mu_i+\sum_{i=1}^{m}\mu_i\Re\left\{p+\frac{1}{\gamma}\left(\frac{z\left(R^{l_i}g_i\right)''(z)}{\left(R^{l_i}g_i\right)'(z)}+1-p\right)\right\}.
\end{aligned} \tag{3.4}$$

Since $g_i\in\beta_i$-$\mathcal{UC}_p(l_i,\gamma,\alpha_i)$ for all $i=\overline{1,m}$, from (1.12) and (3.4), we have

$$\begin{aligned}
&\Re\left\{p+\frac{1}{\gamma}\left(\frac{z\mathcal{G}''_{p,m,l,\mu}(z)}{\mathcal{G}'_{p,m,l,\mu}(z)}+1-p\right)\right\}\\
&\qquad>p-p\sum_{i=1}^{m}\mu_i+\sum_{i=1}^{m}\mu_i\left\{\beta_i\left|\frac{1}{\gamma}\left(\frac{z\left(R^{l_i}g_i\right)''(z)}{\left(R^{l_i}g_i\right)'(z)}+1-p\right)\right|+\alpha_i\right\}\\
&\qquad=p+\sum_{i=1}^{m}\mu_i\left(\alpha_i-p\right)+\sum_{i=1}^{m}\frac{\mu_i\beta_i}{|\gamma|}\left|\frac{z\left(R^{l_i}g_i\right)''(z)}{\left(R^{l_i}g_i\right)'(z)}+1-p\right|\\
&\qquad>p+\sum_{i=1}^{m}\mu_i\left(\alpha_i-p\right).
\end{aligned}$$

Therefore, the operator $\mathcal{G}_{p,m,l,\mu}(z)$ is p-valently convex of complex order γ $(\gamma\in\mathbb{C}-\{0\})$ and type $p+\sum_{i=1}^{m}\mu_i\left(\alpha_i-p\right)$. This evidently completes the proof of Theorem 3.1. □

3.2. Remark.

(1) Letting $\gamma=1$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 3.1, we obtain [11, Theorem 3.1].
(2) Letting $p=1$, $\beta=0$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 3.1, we obtain [3, Theorem 3].
(3) Letting $p=1$, $\beta=0$, $\alpha_i=\mu$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 3.1, we obtain [9, Theorem 3].
(4) Letting $p=1$, $\beta=0$, $\alpha_i=0$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 3.1, we obtain [5, Theorem 2].

Putting $p=m=1$, $l_1=0$, $\mu_1=\mu$, $\alpha_1=\alpha$, $\beta_1=\beta$ and $g_1=g$ in Theorem 3.1, we have

3.3. Corollary. *Let $\mu>0$, $-1\le\alpha<1$, $\beta\ge0$, $\gamma\in\mathbb{C}-\{0\}$ and $g\in\beta$-$\mathcal{UC}(\gamma,\alpha)$. If $0\le1+\mu\left(\alpha-1\right)<1$, then $\int_0^z\left(g'(t)\right)^{\mu}dt$ is convex of complex order γ $(\gamma\in\mathbb{C}-\{0\})$ and type $\mu\left(\alpha-1\right)+1$ in $\mathcal{U}$.* □

3.4. Theorem. *Let $l=(l_1,l_2,\dots,l_m)\in\mathbb{N}_0^m$, $\mu=(\mu_1,\mu_2,\dots,\mu_m)\in\mathbb{R}_+^m$, $-1\le\alpha_i<p$, $\beta_i>0$, $\gamma\in\mathbb{C}-\{0\}$ for all $i=\overline{1,m}$ and*

$$\left|\frac{z\left(R^{l_i}g_i\right)''(z)}{\left(R^{l_i}g_i\right)'(z)}+1-p\right|>-\frac{p+\sum_{i=1}^{m}\mu_i\left(\alpha_i-p\right)}{\sum_{i=1}^{m}\frac{\mu_i\beta_i}{|\gamma|}} \tag{3.5}$$

for all $i=\overline{1,m}$, *then the integral operator* $\mathcal{G}_{p,m,l,\mu}(z)$ *defined by (1.14) is p-valently convex of complex order* γ $(\gamma\in\mathbb{C}-\{0\})$.

Proof. From the proof of Theorem 3.1 and (3.5) we easily get $\mathcal{G}_{p,m,l,\mu}(z)$ is $p-$valently convex of complex order γ. □

From Theorem 3.4, we easily get

3.5. Corollary. *Let* $l=(l_1,l_2,\dots,l_m)\in\mathbb{N}_0^m$, $\mu=(\mu_1,\mu_2,\dots,\mu_m)\in\mathbb{R}_+^m$, $-1\leq\alpha_i<p$, $\beta_i>0$, $\gamma\in\mathbb{C}-\{0\}$ *for all* $i=\overline{1,m}$ *and* $R^{l_i}g_i\in\mathcal{C}_p(\sigma)$, *where* $\sigma=p-\left(p+\sum_{i=1}^m\mu_i\left(\alpha_i-p\right)\right)\diagup\sum_{i=1}^m\frac{\mu_i\beta_i}{|\gamma|}$; $0\leq\sigma<p$ *for all* $i=\overline{1,m}$, *then the integral operator* $\mathcal{G}_{p,m,l,\mu}(z)$ *is p-valently convex of complex order* γ $(\gamma\in\mathbb{C}-\{0\})$. □

Putting $p=m=1$, $l_1=0$, $\mu_1=\mu$, $\alpha_1=\alpha$, $\beta_1=\beta$ and $g_1=g$ in Corollary 3.5, we have

3.6. Corollary. *Let* $\mu>0$, $-1\leq\alpha<1$, $\beta>0$, $\gamma\in\mathbb{C}-\{0\}$ *and* $g\in\mathcal{C}(\rho)$, *where* $\rho=\left[\mu(\beta+(1-\alpha)\,|\gamma|)-|\gamma|\right]\diagup\mu\beta$; $0\leq\rho<1$, *then the integral operator* $\int_0^z\left(g'(t)\right)^\mu\,dt$ *is convex of complex order* γ $(\gamma\in\mathbb{C}-\{0\})$ *in* $\mathbb{U}$. □

Acknowledgement

The present investigation was supported by Atatürk University Rectorship under BAP Project (The Scientific and Research Project of Atatürk University) Project No: 2010/28.

References

[1] Alexander, J. W. *Functions which map the interior of the unit circle upon simple regions*, Annals of Mathematics **17** (1), 12–22, 1915.

[2] Bharti, R., Parvatham, R. and Swaminathan, A. *On subclasses of uniformly convex functions and corresponding class of starlike functions*, Tamkang J. Math. **28** (1), 17–32, 1997.

[3] Bulut, S. *A note on the paper of Breaz and Güney*, J. Math. Ineq. **2** (4), 549–553, 2008.

[4] Breaz, D. *A convexity properties for an integral operator on the classes* $S_p(\alpha)$, Gen. Math. **15** (2-3), 177–183, 2007.

[5] Breaz, D., Aouf, M. K. and Breaz, N. *Some properties for integral operators on some analytic functions with complex order*, Acta. Math. Acad. Paedagog. Nyhazi. **25**, 39–43, 2009.

[6] Breaz, D. and Breaz, N. *Two integral operators*, Stud. Univ. Babes-Bolyai Math. **47** (3), 13–19, 2002.

[7] Breaz, D. and Breaz, N. *Some convexity properties for a general integral operator*, J. Ineq. Pure Appl. Math. **7** (5), Art. 177, 2006.

[8] Breaz, D., Owa, S. and Breaz, N. *A new integral univalent operator*, Acta Univ. Apulensis Math. Inform. **16**, 11–16, 2008.

[9] Breaz, D. and Güney, H. Ö. *The integral operator on the classes* $S_\alpha^*(b)$ *and* $C_\alpha(b)$, J. Math. Ineq. **2** (1), 97–100, 2008.

[10] Deniz, E., Orhan, H. and Sokol, J. *Classes of analytic functions defined by a differential operator related to conic domains*, submitted.

[11] Frasin, B. A. *Convexity of integral operators of p-valent functions*, Math. Comput. Model. **51**, 601–605, 2010.

[12] Frasin, B. A. *Family of analytic functions of complex order*, Acta Math. Acad. Paed. Ny. **22**, 179–191, 2006.

[13] Goel, R. M. and Sohi, N. S. *A new criterion for p-valent functions*, Proc. Amer. Math. Soc. **78**, 353–357, 1980.

[14] Goodman, A. W. *On uniformly convex functions*, Ann. Polon. Math. **56**, 87–92, 1991.

[15] Kanas, S. and Wisniowska, A. *Conic regions and k-uniform convexity*, Comput. Appl. Math. **105**, 327–336, 1999.

[16] Kim, Y. J. and Merkes, E. P. *On an integral of powers of a spirallike function*, Kyungpook Math. J. **12**, 249–252, 1972.

[17] Miller, S. S., Mocanu, P. T. and Reade, M. O. *Starlike integral operators*, Pacific J. Math. **79** (1), 157–168, 1978.

[18] Nasr, M. A. and Aouf, M. K. *Starlike function of complex order*, J. Natur. Sci. Math. **25** (1), 1–12, 1985.

[19] Orhan, H., Deniz, E. and Răducanu, D. *The Fekete–Szegö problem for subclasses of analytic functions defined by a differential operator related to conic domains*, Comput. Math. Appl. **59** (1), 283–295, 2010.

[20] Oros, G. I. *New results related to the convexity and starlikeness of the Bernardi integral operator*, Hacet. J. Math. Stat. **38** (2), 137–143, 2009.

[21] Oros, G. I. and Oros, G. *A convexity property for an integral operator* $\mathbb{F}_m$, Stud. Univ. Babes-Bolyai Math. **55** (3), 169–177, 2010.

[22] Pescar, V. and Owa, S. *Sufficient conditions for univalence of certain integral operators*, Indian J. Math. **42** (3), 347–35, 2000.

[23] Pfaltzgraff, J. A. *Univalence of the integral of* $(f'(z))^{\lambda}$, Bull. London Math. Soc. **7** (3), 254–256, 1975.

[24] Rønning, F. *On starlike functions associated with parabolic regions*, Ann. Univ. Mariae Curie-SkÃlodowska Sect. A. **45** (14), 117–122, 1991.

[25] Wiatrowski, P. *The coefficients of a certain family of holomorphic functions*, Zeszyty Nauk. Uniw.Lodz. Nauki Mat. Pryrod. Ser. **3**, 75–85, 1971.