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THE GELIN-CESÀRO IDENTITY IN SOME CONDITIONAL SEQUENCES

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Abstract

In this paper, we deal with two families of conditional sequences. The first family consists of generalizations of the Fibonacci sequence. We show that the Gelin-Cesàro identity is satisfied. Also, we define a family of conditional sequences $\{u_n\}$ by the recurrence relation $u_n = au_{n-1} + bu_{n-2}$ if n is even, $u_n = cu_{n-1}+du_{n-2}$ if n is odd, with initial conditions $u_0 = 0$ and $u_1 = 1$, where a, b, c and d are non-zero numbers. Many sequences in the literature are special cases of this sequence. We find the generating function of the sequence and Binet's formula for odd and even subscripted sequences. Then we show that the Catalan and Gelin-Cesàro identities are satisfied by the indices of this generalized sequence.

Keywords: Generating function, Fibonacci sequence, Conditional sequence.

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1. Introduction

The sequence F_n of Fibonacci numbers is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with initial conditions $F_0 = 0$ and $F_1 = 1$. This famous sequence appears in many areas of mathematics. The Fibonacci sequence has been generalized in many ways. Fibonacci and generalized Fibonacci identities have been studied by many mathematicians for many years. For example, the Gelin-Cesàro identity [1] states that

$$F_n^4 - F_{n-1}F_{n-2}F_{n+1}F_{n+2} = 1.$$

Also, Melham *et. al.* and Howard obtained generalizations of the Gelin-Cesàro identity in [7] and [4] respectively. In this paper, we deal with two families of conditional sequences. The first family consists of the sequences denoted by $\{q_n\}$ and studied in [2]. We show that the Gelin-Cesàro identity is satisfied by the sequence $\{q_n\}$. Also, we define a family of conditional sequences $\{u_n\}$ by the recurrence relation $u_n = au_{n-1} + bu_{n-2}$ if n is even,

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$u_n = cu_{n-1} + du_{n-2}$ if n is odd, with initial conditions $u_0 = 0$ and $u_1 = 1$, where a, b, c and d are non-zero numbers. Many sequences in the literature are special cases of this generalized sequence. We find the generating function for the sequence $\{u_n\}$ and Binet's formula for even indices of $\{u_n\}$. Then we show that Catalan and Gelin-Cesàro identities are satisfied by even indices of this generalized sequence.

2. The first family of conditional sequences

Recently, the authors introduced in [2] a further generalization of the Fibonacci sequence, namely the generalized Fibonacci sequence defined by

$$q_0 = 0,\ q_1 = 1,\ q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even,} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd,} \end{cases}$$

for any two non-zero real numbers a and b.

2.1. Theorem. [Catalan Identity for $\{q_n\}$] *For any nonnegative integers n and r, we have*

$$\left(a^{\mu(n-r)}b^{1-\mu(n-r)}\right) q_{n-r}q_{n+r} - \left(a^{\mu(n)}b^{1-\mu(n)}\right) q_n^2 = a^{\mu(r)}b^{1-\mu(r)} (-1)^{n+1-r}\, q_r^2,$$

where $\mu(m) = \begin{cases} 0, & \text{if } m \text{ is even,} \\ 1, & \text{if } m \text{ is odd.} \end{cases}$

Proof. See [2]. □

2.2. Theorem. [Gelin-Cesàro Identity] *For any non-negative integers n, we have*

$$a^{2\mu(n)-1}b^{1-2\mu(n)}q_n^4 - q_{n-2}q_{n-1}q_{n+1}q_{n+2} = (-1)^{n+1}\left(\frac{a}{b}\right)^{\mu(n)} q_n^2 (ab-1) + a^2.$$

Proof. For $r = 1$ and $r = 2$, we get respectively

$$q_{n-1}q_{n+1} = a^{2\mu(n)-1}b^{1-2\mu(n)}q_n^2 + (-1)^n \left(\frac{a}{b}\right)^{\mu(n)}$$

and

$$q_{n-2}q_{n+2} = q_n^2 + (-1)^{n-1} a^2 b \frac{1}{a^{\mu(n)}b^{1-\mu(n)}}$$

by using Theorem 2.1 and the property $\mu(n) = \mu(n-2)$. So,

$$\begin{aligned} q_{n-2}q_{n-1}q_{n+1}q_{n+2} &= \left(a^{2\mu(n)-1}b^{1-2\mu(n)}q_n^2 + (-1)^n \left(\frac{a}{b}\right)^{\mu(n)}\right) \\ &\qquad \times \left(q_n^2 + (-1)^{n-1} a^2 b \frac{1}{a^{\mu(n)}b^{1-\mu(n)}}\right) \\ &= a^{2\mu(n)-1}b^{1-2\mu(n)}q_n^4 \\ &\qquad - (-1)^{n+1} q_n^2 \left(a^{1+\mu(n)}b^{1-\mu(n)} - \left(\frac{a}{b}\right)^{\mu(n)}\right) - a^2. \end{aligned}$$

We define

$$A = \left(a^{1+\mu(n)}b^{1-\mu(n)} - \left(\frac{a}{b}\right)^{\mu(n)}\right).$$

If n is even then $A = ab - 1$, else $A = a^2 - \frac{a}{b} = \frac{a}{b}(ab-1)$. So we can rewrite the product $q_{n-2}q_{n-1}q_{n+1}q_{n+2}$ as

$$q_{n-2}q_{n-1}q_{n+1}q_{n+2} = a^{2\mu(n)-1}b^{1-2\mu(n)}q_n^4 - (-1)^{n+1} q_n^2 \left(\frac{a}{b}\right)^{\mu(n)} (ab-1) - a^2.$$

As a result,

$$\begin{aligned} & a^{2\mu(n)-1} b^{1-2\mu(n)} q_n^4 - q_{n-2} q_{n-1} q_{n+1} q_{n+2} \\ & \qquad = (-1)^{n+1} \left(\frac{a}{b}\right)^{\mu(n)} q_n^2 \,(ab-1) + a^2. \end{aligned}$$

□

Note that when $a = b = 1$ the above result reduces to the Gelin-Cesàro identity for the Fibonacci numbers:

$$F_n^4 - F_{n-2} F_{n-1} F_{n+1} F_{n+2} = 1.$$

3. The second family of conditional sequences

There exists a generalization of the sequence $\{q_n\}$ in the literature (see [8]). Here, we define a new generalization of this sequence. Let us denote this sequence by $\{u_n\}$ which is defined recursively by

$$u_0 = 0,\ u_1 = 1,\ u_n = \begin{cases} au_{n-1} + bu_{n-2}, & \text{if } n \text{ is even}, \\ cu_{n-1} + du_{n-2}, & \text{if } n \text{ is odd}, \end{cases}$$

where a, b, c and d are indeterminates. For example, the first six terms of the sequence are

$$\{0,\ 1,\ a,\ ac + d,\ a^2c + ad + ab,\ a^2c^2 + 2acd + abc + d^2\}.$$

For $a = b = c = d = 1$, we get the ordinary Fibonacci sequence, when $a = c = 2$, $b = d = 1$, we have the Pell sequence, when $a = c = k$, $b = d = 1$, we obtain a k-Fibonacci sequence, etc. Also, when $b = d = 1$, we get the sequence which is defined in [2].

In this study, first we obtain the generating function and then Binet's formula for the even indices of the sequence $\{u_n\}$. Finally, we show some properties, for example, the Catalan identity, divisibility and the gcd property, etc. are satisfied by the even indices of the sequence.

Generating functions are very useful as a means of counting, but they can also be used in proofs. To find the generating function of the sequence $\{u_n\}$, we need some properties of this sequence. We give these properties in the following lemma.

3.1. Lemma. *For the sequence $\{u_n\}$, the following properties are satisfied*

(i) $u_{2n+1} = (ac + b + d)u_{2n-1} - (bd)u_{2n-3}$,
(ii) $u_{2n} = (ac + b + d)u_{2n-2} - (bd)u_{2n-4}$.

Proof. (i) Since $2n + 1$ is odd, we get

$$u_{2n+1} = cu_{2n} + du_{2n-1}$$

by the definition of the sequence. Since $u_{2n} = au_{2n-1} + bu_{2n-2}$ ($2n$ is even), we get

$$\begin{aligned} u_{2n+1} &= c(au_{2n-1} + bu_{2n-2}) + du_{2n-1} \\ &= (ac + d)u_{2n-1} + bcu_{2n-2}. \end{aligned}$$

Substituting $cu_{2n-2} = u_{2n-1} - du_{2n-3}$ (by the definition of the sequence) in above equality, we obtain the desired result as follows:

$$\begin{aligned} u_{2n+1} &= (ac + d)u_{2n-1} + b(u_{2n-1} - du_{2n-3}) \\ &= (ac + b + d)u_{2n-1} - bdu_{2n-3}. \end{aligned}$$

(ii) The proof is similar to (i). □

Indeed, Lemma 3.1 reduces the odd and even subscripted sequences to the kind of generalized Fibonacci number studied in [3], [5] and [6].

Now, we can give the generating function of the sequence.

3.2. Theorem. [Generating function] *The generating function for the sequence* $\{u_n\}$ *is*

$$F(x) = \frac{x\left(1+ax-bx^2\right)}{1-(ac+b+d)x^2+bdx^4}.$$

Proof. Let

$$F(x) = u_0 + u_1 x + u_2 x^2 + \cdots + u_k x^k + \cdots = \sum_{m=0}^{\infty} u_m x^m,$$

which is the formal power series of the generating function for u_n. We know that

$$\sum_{m=0}^{\infty} u_m x^m = \sum_{m=0}^{\infty} u_{2m} x^{2m} + \sum_{m=0}^{\infty} u_{2m+1} x^{2m+1}.$$

Now let us define

$$f_1(x) = \sum_{m=0}^{\infty} u_{2m} x^{2m} \text{ and } f_2(x) = \sum_{m=0}^{\infty} u_{2m+1} x^{2m+1}.$$

So, if we can find $f_1(x)$ and $f_2(x)$, we get the desired generating function $F(x)$.

Note that,

$$\begin{aligned}
f_1(x) &= \sum_{m=0}^{\infty} u_{2m} x^{2m} = u_0 x^0 + u_2 x^2 + \sum_{m=2}^{\infty} u_{2m} x^{2m}, \\
(ac+b+d)x^2 f_1(x) &= \sum_{m=1}^{\infty} (ac+b+d)\, u_{2m-2} x^{2m} \\
&= (ac+b+d)\, u_0 x^2 + \sum_{m=2}^{\infty} (ac+b+d)\, u_{2m-2} x^{2m}, \textit{and} \\
bdx^4 f_1(x) &= \sum_{m=2}^{\infty} bd u_{2m-4} x^{2m}.
\end{aligned}$$

So,

$$\begin{aligned}
&\left(1-(ac+b+d)\,x^2+bdx^4\right) f_1(x) \\
&\qquad = ax^2 + \sum_{m=2}^{\infty} \left(u_{2m} - (ac+b+d)\,u_{2m-2} + bd u_{2m-4}\right) x^{2m}.
\end{aligned}$$

we have $u_{2n} - (ac+b+d)u_{2n-2} + (bd)u_{2n-4} = 0$ by lemma 3.1, so we get

$$f_1(x) = \frac{ax^2}{1-(ac+b+d)\,x^2+bdx^4}.$$

Similarly, we can get

$$f_2(x) = \frac{x-bx^3}{1-(ac+b+d)\,x^2+bdx^4}.$$

As a result, we can obtain the generating function as follows

$$\begin{aligned}
F(x) &= f_1(x) + f_2(x) \\
&= \frac{x\left(1+ax-bx^2\right)}{1-(ac+b+d)x^2+bdx^4}.
\end{aligned}$$

□

In fact, we can give Binet's formula for even indices of the sequence as follows.

3.3. Theorem. [Binet's formula for the even indices of the sequence] *For the even indices of the sequence u_n, we have*

$$u_{2n} = a\frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where α and β are roots of the polynomial $x^2 - (ac + b + d)\,x + bd$, that is,

$$\alpha = \frac{ac + b + d + \sqrt{a^2c^2 + b^2 + d^2 + 2abc + 2acd - 2bd}}{2}$$

and

$$\beta = \frac{ac + b + d - \sqrt{a^2c^2 + b^2 + d^2 + 2abc + 2acd - 2bd}}{2}.$$

Proof. We shall prove Binet's formula by induction, making use of the above form. First we need to show that the formula holds for $n = 0$ and $n = 1$.

$$\begin{aligned} u_0 &= a\frac{\alpha^0 - \beta^0}{\alpha - \beta} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} u_2 &= a\frac{\alpha^1 - \beta^1}{\alpha - \beta} \\ &= a \end{aligned}$$

Then for any $k \geqslant 1$, assume that the formula holds for all $n \leqslant k$, in particular for $n = k$ and $n = k - 1$. So,

$$u_{2k} = a\frac{\alpha^k - \beta^k}{\alpha - \beta}$$

and

$$u_{2(k-1)} = a\frac{\alpha^{k-1} - \beta^{k-1}}{\alpha - \beta}.$$

By Lemma 3.1, we can write

$$\begin{aligned} u_{2(k+1)} &= (ac + b + d)\,u_{2k} - bdu_{2(k-1)} \\ &= (ac + b + d)\,a\frac{\alpha^k - \beta^k}{\alpha - \beta} - bda\frac{\alpha^{k-1} - \beta^{k-1}}{\alpha - \beta}. \end{aligned}$$

Now, we use the fact that α and β are roots of $x^2 - (ac + b + d)\,x + bd$, which gives $(ac + b + d) = \alpha + \beta$ and $bd = \alpha\beta$. Hence

$$\begin{aligned} u_{2(k+1)} &= \frac{a\,(\alpha + \beta)\left(\alpha^k - \beta^k\right)}{(\alpha - \beta)} - \frac{a\alpha\beta\left(\alpha^{k-1} - \beta^{k-1}\right)}{(\alpha - \beta)} \\ &= a\left(\frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}\right), \end{aligned}$$

which is Binet's expression for $n = k + 1$. This completes the inductive step and so Binet's formula holds for $n \in \mathbb{N}$. □

Similarly, if α and β are the roots of $x^2 - (ac + b + d)\,x + bd$, we can prove

$$u_{2n+1} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} - b\frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Note that the formula for u_{2n+1} gives $au_{2n+1} = u_{2n+2} - bu_{2n}$.

3.4. Theorem. [Catalan's identity for $\{u_n\}$] *For any two nonnegative even integers m and s, with $m \geqslant s$, we have*

$$u_m^2 - u_{m-s}u_{m+s} = (bd)^{\frac{m-s}{2}} u_s^2.$$

Proof. Let $m = 2n$ and $s = 2r$ for nonnegative integers r and s. Using Binet's formula for the sequence u_{2n}, we get

$$\begin{aligned} u_m^2 - u_{m-s}u_{m+s} &= u_{2n}^2 - u_{2(n-r)}u_{2(n+r)} \\ &= \left(a\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^2 - a\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}a\frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta} \\ &= a^2\frac{\alpha^{2n}-2\alpha^n\beta^n+\beta^{2n}}{(\alpha-\beta)^2} \\ &\qquad - a^2\frac{\alpha^{2n}-\alpha^{n-r}\beta^{n+r}-\alpha^{n+r}\beta^{n-r}+\beta^{2n}}{(\alpha-\beta)^2} \\ &= a^2\frac{\alpha^{n-r}\beta^{n-r}\left(\alpha^{2r}+\beta^{2r}-2\alpha^r\beta^r\right)}{(\alpha-\beta)^2}. \end{aligned}$$

Since $\alpha\beta = bd$, we obtain the desired result as follows:

$$\begin{aligned} u_m^2 - u_{m-s}u_{m+s} &= a^2\frac{(bd)^{n-r}(\alpha^r-\beta^r)^2}{(\alpha-\beta)^2} \\ &= (bd)^{n-r} u_r^2 \\ &= (bd)^{\frac{m-s}{2}} u_s^2. \end{aligned}$$

□

The results obtained in [5] can be applied to $\{u_{2n}\}$ and $\{u_{2n+1}\}$ due to Lemma 3.1. For example, [5, Corollary 3.3] will give Catalan's identity (with $w_n = u_{2n}$). Similarly, if we use [5, Corollary 3.3] with $w_n = u_{2n+1}$, for any two nonnegative odd integers m and s, with $m \geqslant s$, we obtain

$$u_m^2 - (bd)^{\frac{m-s}{2}} u_s^2 = v_{\frac{m-s}{2}}\left((ac+d)\,u_{m+s-1} - (bd)\,u_{m+s-3}\right),$$

where $\{v_n\}$ is defined by

$$v_0 = 0,\ v_1 = 1 \text{ and } v_n = (ac+b+d)\,v_{n-1} - (bd)\,v_{n-2} \text{ for } n \geqslant 2.$$

3.5. Theorem. [Gelin-Cesàro Identity] *For any non-negative even integer $m \geqslant 4$, we have*

$$\begin{aligned} u_m^4 - u_{m-4}u_{m-2}u_{m+2}u_{m+4} = au_m^2 &\left(a\,(bd)^{\frac{m-2}{2}} + (ac+b+d)\,(bd)^{\frac{m-2}{4}}\right) \\ &- a^3\,(bd)^{\frac{3m-6}{4}}\,(ac+b+d). \end{aligned}$$

Proof. For $s = 2$ and $s = 4$ we get

$$u_{m-2}u_{m+2} = q_m^2 - a^2\,(bd)^{\frac{m-1}{2}}$$

and

$$u_{m-4}u_{m+4} = u_m^2 - a\,(bd)^{\frac{m-2}{4}}\,(ac+b+d)$$

respectively, by Theorem 3.4. So we can find

$$\begin{aligned} u_{m-4}u_{m-2}u_{m+2}u_{m+4} = u_m^4 - au_m^2 &\left(a\,(bd)^{\frac{m-2}{2}} + (ac+b+d)\,(bd)^{\frac{m-2}{4}}\right) \\ &+ a^3\,(bd)^{\frac{3m-6}{4}}\,(ac+b+d) \end{aligned}$$

with the aid of Maple. As a result, we get the desired result

$$u_m^4 - u_{m-4}u_{m-2}u_{m+2}u_{m+4} = au_m^2\left(a\left(bd\right)^{\frac{m-2}{2}} + \left(ac+b+d\right)\left(bd\right)^{\frac{m-2}{4}}\right) - a^3\left(bd\right)^{\frac{3m-6}{4}}\left(ac+b+d\right).$$

□

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