Hacettepe Journal of Mathematics and Statistics
Volume 40 (6) (2011), 863 – 869

SOME MORE GENERALIZATIONS OF THE INTEGRAL INEQUALITIES OF HARDY AND HILBERT

Wen-wen Wei*† and Xiao-yu Lei*

Received 01 : 02 : 2011 : Accepted 31 : 05 : 2011

Abstract

In the present paper, we establish several new Hardy-Hilbert integral inequalities, and give some applications to other integral inequalities.

Keywords: Hardy-Hilbert's inequality, Integral inequality.

2000 AMS Classification: 26 D 15.

1. Introduction

If f, g are measurable real functions such that

$$0 < \int_0^\infty f^2(x)\,dx < \infty \text{ and } 0 < \int_0^\infty g^2(x)\,dx < \infty,$$

then we have the following well known Hilbert integral inequality [2],

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2},$$

where π is the best possible.

If $f, g \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and

$$0 < \int_0^\infty f^p(x)\,dx < \infty \text{ and } 0 < \int_0^\infty g^q(x)\,dx < \infty,$$

then the following Hardy-Hilbert integral inequality (see [2]), which is important in analysis and applications, holds

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x)dx \right)^{1/p} \left(\int_0^\infty g^q(x)dx \right)^{1/q}, \tag{1.1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

*College of Mathematics and Information Science, Henan Normal University, Henan Province, 453007, China.
E-mail: (W. W. Wei) weiwenwen4912341@gmail.com (X. Y. Lei) leixiaoyulxy@gmail.com
†Corresponding Author.

Other mathematicians have presented generalizations or new kinds of the above Hardy-Hilbert inequalities, as follows:

1.1. Theorem. [7] *Let* $f, g > 0$*. If* $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} \geq 1$*, and* $0 < \lambda = 2 - \frac{1}{p} + \frac{1}{q} \leq 1$*, then one has*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\, dx\, dy \leq k \left(\int_0^\infty f^p(x)\, dx \right)^{1/p} \left(\int_0^\infty g^q(x)\, dx \right)^{1/q}. \tag{1.2}$$

□

Here, k depends on p and q; only if $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda = 2 - \frac{1}{p} + \frac{1}{q} = 1$, k is the best possible.

1.2. Theorem. [6] *If* $f, g \geq 0$, $\lambda > 0$, $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *are such that*

$$0 < \int_0^\infty t^{p-1-\lambda} f^p(x)\, dx < \infty \text{ and } 0 < \int_0^\infty t^{q-1-\lambda} g^q(x)\, dx < \infty,$$

then one has

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}}\, dx\, dy \\ &\qquad < \frac{pq}{\lambda} \left(\int_0^\infty t^{p-1-\lambda} f^p(x)\, dx \right)^{1/p} \left(\int_0^\infty t^{q-1-\lambda} g^q(x)\, dx \right)^{1/q}, \end{aligned} \tag{1.3}$$

where the constant factor $\frac{pq}{\lambda}$ *is the best possible.* □

Recently, Du and Miao [1, 4] have studied the function $\frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \min\{x,y\}}$ with positive numbers α, β, γ, and given the following extended analogue of Hilbert's inequalities,

1.3. Theorem. [1] *Let* f, g *be real functions such that* $0 < \int_0^\infty f^2(x)dx < \infty$ *and* $0 < \int_0^\infty g^2(x)dx < \infty$*. Furthermore, let* $A \in (0, \infty)$*. Then we have*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \min\{x,y\}} f(x)g(y)\, dx\, dy \\ &\qquad < A \left(\int_0^\infty f^2(x)\, dx \right)^{1/2} \left(\int_0^\infty g^2(x)\, dx \right)^{1/2}, \end{aligned}$$

where A *is defined as*

$$A := \int_0^1 \frac{2^{\gamma+1} |\log t|^\gamma}{t^2(1+\beta) + \alpha}\, dt + \int_0^1 \frac{2^{\gamma+1} |\log t|^\gamma}{t^2(1+\alpha) + \beta}\, dt.$$

Here α, β, γ *are any positive real numbers.* □

In the present paper, based on the above works, we establish several new Hardy-Hilbert integral inequalities. What's more, as applications, some specific integral inequalities are deduced.

2. Main results

Before starting our work, we recall some results and definitions about the Gamma function $\Gamma(p)$ and Beta function $B(p, q)$ as follows,

$$\begin{aligned} \Gamma(p) &= \int_0^\infty t^{p-1} e^{-t}\, dt, \ p > 0 \\ B(p,q) &= \int_0^1 t^{p-1}(1-t)^{q-1}\, dt = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\, dt, \ p, q > 0. \end{aligned} \tag{2.1}$$

2.1. Lemma. [5] *Let $p, q > 0$. Then*

$$\Gamma(p) = \mathrm{e}\int_0^1 \frac{x^{-p-1}\mathrm{e}^{-\frac{1}{x}}}{(1-x)^{1-p}}\,dx = \mathrm{e}\int_1^\infty \frac{\mathrm{e}^{-x}}{(x-1)^{1-p}}\,dx \tag{2.2}$$
$$B(p,q) = \int_1^\infty \frac{x^{-p-q}}{(x-1)^{1-p}}\,dx. \qquad \square$$

Furthermore, for convenience, we state the definition of homogeneous function: The function $F(x,y)$ is said to be homogeneous of degree λ, $(\lambda > 0)$, if $F(tx,ty) = t^\lambda F(x,y)$ for all $(x,y) \in D$ and $(tx,ty) \in D$, where D denotes the domain of the function $F(x,y)$.

Now we can give the following main results in this paper.

2.2. Theorem. *Assume that $f, g, h, k \geq 0$, $h = h(x,y) : R_+ \times R_+ \to R_+$, $k = k(t) : R_+ \to R_+$, h is homogeneous of degree λ and k is nondecreasing; $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

(a) *For $k(t) \neq 1$, (or, in general, $k(t) \neq c$, where c is some constant),*

$$\int_0^\infty y^{(p-1)(\lambda-1)}\left(\int_0^\infty \frac{f(x)}{h(x,y)\max\{k(\frac{x}{y}), k(\frac{y}{x})\}}\,dx\right)^p dy \leq C^p \int_0^\infty x^{1-\lambda} f^p(x)\,dx, \tag{2.3}$$

where $C = I_1 + I_2$,

$$I_1 = \int_0^1 \frac{dx}{h(x,1)k(x^{-1})},\ I_2 = \int_1^\infty \frac{dx}{h(x,1)k(x)}.$$

(b) *For $k(t) = 1$ (in general a and b are both arbitrary constants),*

$$\int_0^\infty y^{[b(q-1)+\lambda-1-a](p-1)}\left(\int_0^\infty \frac{f(x)\,dx}{h(x,y)}\right)^p dy \leq C^p \int_0^\infty x^{1+b-\lambda-a(p-1)} f^p(x)\,dx, \tag{2.4}$$

where $C = K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$,

$$K_1 = \int_0^\infty \frac{t^b dt}{h(1,t)},\ K_2 = \int_0^\infty \frac{t^a dt}{h(t,1)}.$$

Here we assume that all the integrals on the RHS do exist.

Proof. (a) According to Holder's inequality, it is easy to see that

$$\int_0^\infty \frac{f(x)\,dx}{h(x,y)\max\{k(\frac{x}{y}), k(\frac{y}{x})\}} \leq \left(\int_0^\infty \frac{f^p(x)dx}{h(x,y)\max\{k(\frac{x}{y}), k(\frac{y}{x})\}}\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{dx}{h(x,y)\max\{k(\frac{x}{y}), k(\frac{y}{x})\}}\right)^{\frac{1}{q}},$$

which yields

$$\left(\int_0^\infty \frac{f(x)\,dx}{h(x,y)\max\{k(\frac{x}{y}), k(\frac{y}{x})\}}\right)^p \leq \int_0^\infty \frac{f^p(x)dx}{h(x,y)\max\{k(\frac{x}{y}), k(\frac{y}{x})\}}\left(\int_0^\infty \frac{dx}{h(x,y)\max\{k(\frac{x}{y}), k(\frac{y}{x})\}}\right)^{\frac{p}{q}}. \tag{2.5}$$

We first consider the following integral

$$\begin{aligned}
&\int_0^\infty \frac{dx}{h(x,y)\max\{k(\frac{x}{y}),k(\frac{y}{x})\}}\\
&\qquad = \int_0^y \frac{dx}{h(x,y)\max\{k(\frac{x}{y}),k(\frac{y}{x})\}} + \int_y^\infty \frac{dx}{h(x,y)\max\{k(\frac{x}{y}),k(\frac{y}{x})\}}\\
&\qquad = M_1 + M_2.
\end{aligned}$$

For the case M_1, since $x \le y$ implies $\frac{x}{y} \le \frac{y}{x}$, hence $k(\frac{x}{y}) \le k(\frac{y}{x})$, then we have

$$M_1 = \int_0^y \frac{dx}{h(x,y)k(\frac{y}{x})}.$$

Let $u = \frac{x}{y}$, then

$$M_1 = \int_0^y \frac{dx}{h(xy^{-1}y,y)k(\frac{y}{x})} = y^{1-\lambda}\int_0^1 \frac{du}{h(u,1)k(u^{-1})} = I_1 y^{1-\lambda},$$

and

$$\begin{aligned}
M_2 &= \int_y^\infty \frac{dx}{h(x,y)\max\{k(\frac{x}{y}),k(\frac{y}{x})\}}\\
&= \int_y^\infty \frac{dx}{h(x,y)k(\frac{x}{y})} = y^{1-\lambda}\int_1^\infty \frac{du}{h(u,1)k(u)} = I_2 y^{1-\lambda},
\end{aligned}$$

which implies

$$\int_0^\infty \frac{dx}{h(x,y)\max\{k(\frac{x}{y}),k(\frac{y}{x})\}} = (I_1+I_2)y^{1-\lambda} = Cy^{1-\lambda}. \tag{2.6}$$

Therefore, from (2.5) and (2.6), we have

$$\left(\int_0^\infty \frac{f(x)dx}{h(x,y)\max\{k(\frac{x}{y}),k(\frac{y}{x})\}}\right)^p \le C^{\frac{p}{q}} y^{\frac{p}{q}(1-\lambda)} \int_0^\infty \frac{f^p(x)dx}{h(x,y)\max\{k(\frac{x}{y}),k(\frac{y}{x})\}}.$$

Now since

$$\begin{aligned}
&\int_0^\infty y^{(p-1)(\lambda-1)} C^{\frac{p}{q}} y^{(1-\lambda)\frac{p}{q}} \int_0^\infty \frac{f^p(x)dx}{h(x,y)\max\{k(\frac{x}{y}),k(\frac{y}{x})\}}\,dy\\
&\qquad = C^{\frac{p}{q}} \int_0^\infty \int_0^\infty \frac{f^p(x)\,dx}{h(x,y)\max\{k(\frac{x}{y}),k(\frac{y}{x})\}}\,dy\\
&\qquad = C^{p-1}\int_0^\infty f^p(x)dx \int_0^\infty \frac{dy}{h(x,y)\max\{k(\frac{x}{y}),k(\frac{y}{x})\}}\\
&\qquad = C^{p-1}\int_0^\infty Cx^{1-\lambda} f^p(x)\,dx\\
&\qquad = C^p \int_0^\infty x^{1-\lambda} f^p(x)\,dx,
\end{aligned}$$

then the inequality (2.3) holds.

(b) Similarly, according to Holder's inequality, we have

$$\begin{aligned}
&\int_0^\infty \frac{f(x)\,dx}{h(x,y)} \\
&= \int_0^\infty \frac{f(x)y^{\frac{b}{p}}}{h^{\frac{1}{p}}(x,y)x^{\frac{a}{q}}} \cdot \frac{x^{\frac{a}{q}}}{h^{\frac{1}{q}}(x,y)y^{\frac{b}{p}}}\,dx \\
&\le \left(\int_0^\infty \frac{f^p(x)y^b dx}{h(x,y)x^{\frac{ap}{q}}}\right)^{\frac{1}{p}} \left(\int_0^\infty \frac{x^a\,dx}{h(x,y)y^{\frac{qb}{p}}}\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.7}$$

It is easy to check

$$\begin{aligned}
\int_0^\infty \frac{x^a\,dx}{h(x,y)y^{\frac{qb}{p}}} &= y^{\frac{-qb}{p}} \int_0^\infty \frac{x^a\,dx}{h(x,y)} \\
&= y^{(1+a-\lambda-\frac{qb}{p})} \int_0^\infty \frac{u^a\,du}{h(u,1)} \\
&= K_2 y^{(1+a-\lambda-\frac{qb}{p})},
\end{aligned}$$

then we can obtain

$$\left(\int_0^\infty \frac{f(x)\,dx}{h(x,y)}\right)^p \le K_2^{\frac{p}{q}} y^{(1+a-\lambda-\frac{qb}{p})\frac{p}{q}} \int_0^\infty \frac{f^p(x)y^b}{h(x,y)x^{\frac{ap}{q}}}\,dx. \tag{2.8}$$

Therefore, by the inequality (2.8), we have

$$\begin{aligned}
&\int_0^\infty y^{[b(q-1)+\lambda-1-a](p-1)} \left(\int_0^\infty \frac{f(x)\,dx}{h(x,y)}\right)^p dy \\
&\le \int_0^\infty y^{[b(q-1)+\lambda-1-a](p-1)} \cdot y^{[-b(q-1)-\lambda+1+a](p-1)} \\
&\qquad \cdot K_2^{\frac{p}{q}} \int_0^\infty \frac{f^p(x)y^b}{h(x,y)x^{\frac{ap}{q}}}\,dx\,dy \\
&= K_2^{\frac{p}{q}} \int_0^\infty \int_0^\infty \frac{f^p(x)y^b}{h(x,y)x^{\frac{ap}{q}}}\,dx\,dy \\
&= K_2^{\frac{p}{q}} \cdot K_1 \int_0^\infty x^{1+b-\lambda-a(p-1)} f^p(x)\,dx \\
&\le C^p \int_0^\infty x^{1+b-\lambda-a(p-1)} f^p(x)\,dx,
\end{aligned}$$

where $C = K_1^{1/p} K_2^{1/q}$. By now ,we have completed the proof of the theorem. □

3. Applications

Firstly, we recall the fact: if $0 < p < 1$, then one has

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)} \text{ and } B(p, 1-p) = \frac{\pi}{\sin(p\pi)}.$$

3.1. Corollary. *Assume that* $f \ge 0$, $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}\,dx\right)^p dy \le \left[\frac{\pi}{\sin(\pi/p)}\right]^p \int_0^\infty f^p(x)\,dx \tag{3.1}$$

provided the integrals on the RHS exist.

Proof. The result is obtained from result (b) in Theorem 2.2 by putting

$$h(x,y)=x+y,\ a=\frac{1}{q}-1,\ b=\frac{1}{p}-1.$$

Thus we have

$$\begin{aligned}\int_0^\infty\left(\int_0^\infty\frac{f(x)\,dx}{x+y}\right)^p dy &\le \left[\left(\int_0^\infty\frac{t^{\frac{1}{p}-1}}{1+t}dt\right)^{\frac{1}{p}}\cdot\left(\int_0^\infty\frac{t^{\frac{1}{q}-1}}{1+t}\,dt\right)^{\frac{1}{q}}\right]^p\int_0^\infty f^p(x)\,dx\\ &\le\left[\Gamma\Big(\frac{1}{p}\Big)\Gamma\Big(1-\frac{1}{p}\Big)\right]^p\int_0^\infty f^p(x)dx\\ &\le\left[\frac{\pi}{\sin(\pi/p)}\right]^p\int_0^\infty f^p(x)dx.\end{aligned}$$

□

3.2. Corollary. *Assume that* $f,g\ge 0$, $\lambda>0$, $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Then*

(3.2)
$$\int_0^\infty y^{(p-1)(\lambda-1)}\left(\int_0^\infty\frac{f(x)dx}{|x-y|^{1-\lambda}\max\{(\frac{x}{y})^{2\lambda},(\frac{y}{x})^{2\lambda}\}}\right)^p dy \le[2B(\lambda,\lambda)]^p\int_0^\infty x^{1-\lambda}f^p(x)\,dx.$$

Proof. The result is obtained from result (a) in Theorem 2.2 by putting

$$h(x,y)=|x-y|^{1-\lambda},\ k(x)=x^{2\lambda}.$$

So we get

$$I_1=\int_0^1\frac{t^{2\lambda}}{(1-t)^{1-\lambda}}\,dt=\int_0^1\frac{t^{\lambda+1}t^{\lambda-1}}{(1-t)^{1-\lambda}}\,dt\le\int_0^1\frac{t^{\lambda-1}}{(1-t)^{1-\lambda}}\,dt=B(\lambda,\lambda),$$
$$I_2=\int_1^\infty\frac{t^{-2\lambda}}{(t-1)^{1-\lambda}}\,dt=B(\lambda,\lambda).$$

The desired result can now be obtained. □

3.3. Corollary. *Assume that* $f,g\ge 0$, $0<\lambda<2$, $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Then*

(3.3)
$$\int_0^\infty y^{(p-1)(\lambda-1)}\left(\int_0^\infty\frac{f(x)\,dx}{(x^\lambda+y^\lambda)\max\{(\frac{x}{y})^{1-\frac{\lambda}{2}},(\frac{y}{x})^{1-\frac{\lambda}{2}}\}}\right)^p dy \le\left[\frac{\pi}{2\lambda}+\frac{2}{\lambda}\int_0^1\frac{y^{\frac{4}{\lambda}-2}}{y^2+1}\,dy\right]^p\int_0^\infty x^{1-\lambda}f^p(x)\,dx.$$

In particular, when $m:=\frac{4}{\lambda}-2$ *is a positive integer, we have*

(3.4)
$$\int_0^\infty y^{(p-1)(\lambda-1)}\left(\int_0^\infty\frac{f(x)dx}{(x^\lambda+y^\lambda)\max\{(\frac{x}{y})^{1-\frac{\lambda}{2}},(\frac{y}{x})^{1-\frac{\lambda}{2}}\}}\right)^p dy \le\left[\frac{\pi}{2\lambda}+\frac{2}{\lambda}\sum_{k=0}^\infty(-1)^k\frac{1}{m+2k+1}\right]^p\int_0^\infty x^{1-\lambda}f^p(x)\,dx.$$

Proof. The result is obtained from result (a) in Theorem 2.2 by putting

$$h(x,y) = x^{\lambda} + y^{\lambda},\ k(x) = x^{1-\frac{\lambda}{2}}.$$

Therefore, we can obtain (by letting $y = u^{\frac{\lambda}{2}}$),

$$I_1 = \int_0^1 \frac{u^{1-\frac{\lambda}{2}}}{u^{\lambda}+1}\,du = \frac{2}{\lambda}\int_0^1 \frac{y^{\frac{4}{\lambda}-2}}{y^2+1}\,dy,$$

$$I_2 = \int_1^{\infty} \frac{u^{\frac{\lambda}{2}-1}}{u^{\lambda}+1}\,du = \frac{2}{\lambda}\int_1^{\infty} \frac{1}{y^2+1}\,dy = \frac{\pi}{2\lambda},$$

which yields $C = \frac{\pi}{2\lambda} + \frac{2}{\lambda}\int_0^1 \frac{y^{\frac{4}{\lambda}-2}}{y^2+1}\,dy$. In particular, when $m := \frac{4}{\lambda} - 2$ is a positive integer, on the basis of the table of integrals, we have

$$I_1 = \frac{2}{\lambda}\int_0^1 \frac{y^m}{y^2+1}dy = \frac{2}{\lambda}\sum_{k=0}^{\infty}(-1)^k \frac{1}{m+2k+1}.$$

So the proof of the desired result can be completed. □

Acknowledgement

The authors are very grateful to the referee for his/her valuable report which improved the presentation of this work.

References

[1] Du, H. X and Miao, Y. *Several extended analogues of Hilbert's inequalities*, Demonstratio Math. **XLII** (2), 297–302, 2009.

[2] Hardy, G. H., Littlewood, J. E. and Polya, G. *Inequalities* (Cambridge University Press, Cambridge, 1952).

[3] Kuang, J. C. *Applied Inequalities. Third Edition* (Shandong Science and Technology Press, Jinan, 2004).

[4] Miao, Y. and Du, H. X. *A note of Hilbert type integral inequality*, Inequality Theory & Applications **6**, 261–267, 2010.

[5] Sulaiman, W. T. *New kinds of Hardy-Hilbert's integral inequalities*, Appl. Math. Lett. **23**, 361–365, 2010.

[6] Sun, B. *Best generalization of Hilbert type inequality*, J. Inequal. Pure Appl. Math. **7** (3), Art 113, 2006.

[7] Yang, B. C. *The norm of operator and Hilbert-type inequalities* (Science Press, Beijing, 2009).