Hacettepe Journal of Mathematics and Statistics
Volume 40 (6) (2011), 871 – 883

COMMON FIXED POINTS AND INVARIANT APPROXIMATION FOR BANACH OPERATOR PAIRS WITH ĆIRIĆ TYPE NONEXPANSIVE MAPPINGS

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Received 19 : 02 : 2011 : Accepted 20 : 06 : 2011

Abstract

Some common fixed point theorems for Banach operator pairs with Ćirić type nonexpansive mappings, and the existence of common fixed points of best approximation have been proved in the framework of convex metric spaces. The results proved in the paper generalize and extend some of the results of N. Hussain (*Common fixed points in best approximation for Banach operator pairs with Ćirić type I-contractions*, J. Math. Anal. Appl. **338**, 1351–1363, 2008) from the Banach space framework into the framework of convex metric spaces.

Keywords: Banach operator pair, Best approximation, Convex metric space, Star-shaped set, Convex set, Contractive jointly continuous family.

2000 AMS Classification: 41 A 50, 41 A 65, 47 H 10, 54 H 25.

1. Introduction and Preliminaries

For a metric space (X, d), a continuous mapping $W : X \times X \times [0, 1] \to X$ is said to be a *convex structure* on X if for all $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

holds for all $u \in X$. The metric space (X, d) together with a convex structure is called a *convex metric space* [27].

A subset K of a convex metric space (X, d) is said to be a *convex set* [27] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$. The set K is said to be *p-starshaped* [13] if there exists

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a $p \in K$ such that $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in [0, 1]$ i.e. the segment $[p, x] = \{W(x, p, \lambda) : \lambda \in [0, 1]\}$ joining p to x is contained in K for all $x \in K$.

Clearly, each convex set is starshaped but not conversely.

A convex metric space (X, d) is said to satisfy *Property (I)* [13] if for all $x, y, q \in X$ and $\lambda \in [0, 1]$,

$$d(W(x, q, \lambda), W(y, q, \lambda)) \leq \lambda d(x, y).$$

A normed linear space and each of its convex subsets are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces (see [4, 13, 27]). Property (I) is always satisfied in a normed linear space.

1.1. Example. [20] Consider a closed subset X of the unit ball $S = \{\|x\| = 1\}$ in a Hilbert space H, such that X has diameter $\delta(X) \leq \sqrt{2}$ and is geodesically connected, i.e., the point $W(x, y, \lambda) = \frac{\lambda x + (1-\lambda) y}{\|\lambda x + (1-\lambda) y\|}$ lies in X whenever $x, y \in X$ and $\lambda \in [0, 1]$. The metric space we obtain by measuring distances in X through central angles, i.e., with the metric $d[x, y] = \cos^{-1}(x, y)$ for every $x, y \in X$, turns out be a convex metric space (whose convex sets are exactly the goedesically connected subsets of X).

For a non-empty subset M of a metric space (X, d) and $x \in X$, an element $y \in M$ is said to be a *best approximant* to x or a *best M-approximant* to x if $d(x, y) = dist(x, M) \equiv \inf\{d(x, y) : y \in M\}$. The set of all such $y \in M$ is denoted by $P_M(x)$.

For a convex subset M of a convex metric space (X, d), a mapping $g : M \to X$ is said to be *affine* if for all $x, y \in M$, $g(W(x, y, \lambda)) = W(gx, gy, \lambda)$ for all $\lambda \in [0, 1]$. g is said to be *affine with respect to* $p \in M$ if $g(W(x, p, \lambda)) = W(gx, gp, \lambda)$ for all $x \in M$ and $\lambda \in [0, 1]$.

Suppose (X, d) is a metric space, M a nonempty subset of X, and S, T self mappings of M. T is said to be

(i) An *S-contraction* if there exists a $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(Sx, Sy)$,
(ii) *S-nonexpansive* if $d(Tx, Ty) \leq d(Sx, Sy)$ for all $x, y \in M$.

If $S =$ identity mapping, then T is called a *contraction*, *nonexpansive* respectively in (i) and (ii).

A point $x \in M$ is a *common fixed (coincidence) point* of S and T if $x = Sx = Tx$ ($Sx = Tx$). The set of fixed points (respectively, coincidence points) of S and T is denoted by $F(S, T)$ (respectively, $C(S, T)$). The pair (S, T) is said to be *commuting* on M if $STx = TSx$ for all $x \in M$.

The ordered pair (T, I) of two self maps of a metric space (X, d) is called a *Banach operator pair* [5], if the set $F(I)$ of fixed points of I is T-invariant, i.e. $T(F(I)) \subseteq F(I)$. Obviously, a commuting pair (T, I) is a Banach operator pair but not conversely (see [5]). If (T, I) is a Banach operator pair then (I, T) need not be a Banach operator pair (see [5]). If the self maps T and I of X satisfy $d(ITx, Tx) \leq kd(Ix, x)$, for all $x \in X$ and for some $k \geq 0$, $ITx = TIx$ whenever $x \in F(I)$ i.e. $Tx \in F(I)$, then (T, I) is a Banach operator pair. In particular, when $I = T$ the above inequality can be rewritten as $d(T^2x, Tx) \leq kd(Tx, x)$ for all $x \in X$. Such a T is called a *Banach operator of type k* (see [25, 26]).

This class of non-commuting mappings is different from the known classes of non-commuting mappings viz. R-weakly commuting, R-subweakly commuting, compatible, weakly compatible and C_q-commuting etc. (see [5, 16]). Hence the concept of Banach operator pair is of basic importance for the study of common fixed points.

1.2. Example. Let $X = \mathbb{R}$ with its usual metric and $K = [1, \infty)$. Let $T(x) = x^3$ and $I(x) = 2x - 1$, for all $x \in K$. Then $F(I) = \{1\}$. Here (T, I) is a Banach operator pair but T and I are not commuting.

W. G. Dotson Jr. [10] proved some results concerning the existence of fixed points of nonexpansive mappings on a certain class of nonconvex sets. In order to prove these results, which extended his previous work [9] on starshaped sets, he introduced the following class of nonconvex sets:

Suppose M is a subset of a normed linear space E, and let $\mathfrak{F} = \{f_\alpha\}_{\alpha \in M}$ be a family of functions from $[0, 1]$ into M having the property that for each $\alpha \in M$ we have $f_\alpha(1) = \alpha$. Such a family $\mathfrak{F}$ is said to be *contractive* provided there exists a function $\varphi : (0, 1) \to (0, 1)$ such that for all α, β in M and for all t in $(0, 1)$ we have

$$\|f_\alpha(t) - f_\beta(t)\| \leq \varphi(t)\, \|\alpha - \beta\| .$$

Such a family $\Im$ is said to be *jointly continuous* provided that if $t \to t_\circ$ in $[0, 1]$ and $\alpha \to \alpha_\circ$ in M then $f_\alpha(t) \to f_{\alpha_\circ}(t_\circ)$ in M.

Suppose that $H = \{f_\alpha\}_{\alpha \in M}$ is a family of functions from $[0, 1]$ into M having the property that for each sequence $\langle \lambda_n \rangle$ in $(0, 1]$, with $\lambda_n \to 1$, we have

$$f_\alpha(\lambda_n) = \lambda_n \alpha. \tag{1.1}$$

We observe that $H \subseteq \mathfrak{F}$ and it has the additional property that it is contractive and jointly continuous.

If for a subset M of E, there exists a contractive jointly continuous family of functions $\mathfrak{F} = \{f_\alpha\}_{\alpha \in M}$ (respectively, a family of functions satisfying (1.1)), then we say that S has the property of *contractiveness and joint continuity* (respectively, the property (1.1)) (see [3, 19]).

1.3. Example. Any subspace, a convex set with 0, a starshaped subset with center 0 and a cone of a normed linear space have a family of functions associated with them which satisfy condition (1.1).

In fact, it is easy to observe that if M is a q-starshaped subset of a normed linear space X and $f_x(t) = (1-t)q + tx$, for each x, $q \in M$ and $t \in [0, 1]$, then $\mathfrak{F}$ is a contractive jointly continuous family with $\varphi(t) = t$, $t \in (0, 1)$. Also if S is affine self mapping of M and $S(q) = q$, we have $S(f_x(t)) = S((1-t)q + tx) = (1-t)S(q) + tS(x) = (1-t)q + tS(x) = f_{S(x)}(t)$, for all x, $q \in M$ and $t \in [0, 1]$. Thus the class of subsets of X with the property of contractive and joint continuity contains the class of starshaped sets, which in turn contains the class of convex sets.

In recent years, many results related to Gregus's Theorem [12] have appeared. Fisher and Sessa [11], Jungck [17], Ćirić [6], [7] and many others (see [4, 8, 16] and references cited therein) have generalized the theorem of Gregus with more contractive conditions. The purpose of this paper is to prove some similar results for a class of Banach operator pairs without the assumptions of linearity or affinity of either T or S. We prove some common fixed point theorems for Banach operator pairs with Ćirić type nonexpansive mappings and the existence of common fixed points from the set of best approximation in the framework of convex metric spaces. The results proved in this paper generalize, unify and extend some of the results of [2, 3, 4, 5, 14, 15, 16, 18, 22, 23, 24, 26] and of a few others.

2. Main results

2.1. Common fixed point theorems with Ćirić type nonexpansive mappings. In this section we prove some common fixed point theorems for Banach operator pairs with Ćirić type nonexpansive mappings in convex metric spaces.

The following result of Ćirić [7] will be used in our first theorem:

2.1. Lemma. *Let C be a closed convex subset of a complete convex metric space (X, d) and $T : C \to C$ a mapping satisfying, for all $x, y \in C$*

$$d(Tx, Ty) \le a \max\{d(x, y), c[d(x, Ty) + d(y, Tx)]\} + b \max\{d(x, Tx), d(y, Ty)\},$$

where $0 < a < 1$, $a + b = 1$, $c \le \frac{4-a}{8-a}$. Then T has a unique fixed point. □

The following result extends and improves [16, Lemma 2.1] of Hussain.

2.2. Theorem. *Let M be a closed convex subset of a complete convex metric space (X, d), and (T, I) a Banach operator pair on M. Assume that T and I satisfy*

$$d(Tx, Ty) \le a \max\{d(Ix, Iy),\ c[d(Ix, Ty) + d(Iy, Tx)]\} + b \max\{d(Tx, Ix),\ d(Ty, Iy)\}$$

for all $x, y \in M$, where $0 < a < 1$, $a + b = 1$, $0 \le c < \eta$, where $\eta = \min\{\frac{2+a}{5+a}, \frac{2-a}{4}, \frac{4}{9+a}\}$. If I is continuous, $F(I)$ is nonempty and convex, then there is a unique common fixed point of T and I.

Proof. Since (T, I) is a Banach operator pair, $T(F(I)) \subseteq F(I)$. The continuity of I implies that $F(I)$ is closed. Thus $F(I)$ is a nonempty, closed and convex subset of M. Further for all $x, y \in F(I)$, we have

$$\begin{aligned} d(Tx, Ty) &\le a \max\{d(Ix, Iy),\ c[d(Ix, Ty) + d(Iy, Tx)]\} \\ &\qquad + b \max\{d(Tx, Ix),\ d(Ty, Iy)\} \\ &\le a \max\{d(x, y),\ c[d(x, Ty) + d(y, Tx)]\} + b \max\{d(x, Tx),\ d(y, Ty)\}. \end{aligned}$$

So by Lemma 2.1, T has a unique fixed point z in $F(I)$ and consequently $M \cap F(T) \cap F(I)$ is a singleton. □

The following theorem extends Hussain [16, Theorem 2.2]:

2.3. Theorem. *Let M be a closed convex subset of a complete convex metric space (X, d) with Property (I), and S, T continuous self maps of M. Suppose that $F(S)$ is nonempty and convex and that (T, S) is a Banach operator pair on M. If $\mathrm{cl}(T(M))$ is compact and satisfies*

$$d(Tx, Ty) \le \max\{d(Sx, Sy),\ c[\mathrm{dist}(Sx, [p, Ty]) + \mathrm{dist}(Sy, [p, Tx])]\} + \tfrac{1-\lambda}{\lambda} \max\{\mathrm{dist}(Sx, [p, Tx]),\ \mathrm{dist}(Sy, [p, Ty])\}$$

for some $p \in F(S) \cap M$, for all $x, y \in M$, $0 < \lambda < 1$, $0 \le c < \frac{1}{4}$, then there is a common fixed point of T and S.

Proof. For each n, define $T_n : M \to M$ by $T_n x = W(Tx, p, \lambda_n)$, $x \in M$, where $< \lambda_n >$ is a sequence in $(0, 1)$ such that $\lambda_n \to 1$. Now as (T, S) is a Banach operator pair and $F(S)$ is convex, $T_n(x) = W(Tx, p, \lambda_n) \in F(S)$ for each $x \in F(S)$ since $Tx \in F(S)$. Thus

(T_n, S) is a Banach operator pair for each n. Consider

$$\begin{aligned} d(T_n x, T_n y) &= d(W(Tx, p, \lambda_n), W(Ty, p, \lambda_n)) \\ &\leq \lambda_n d(Tx, Ty), \text{by Property (I)} \\ &\leq \lambda_n \{\max\{d(Sx, Sy),\ c[\operatorname{dist}(Sx, [p, Ty]) + \operatorname{dist}(Sy, [p, Tx])]\} \\ &+ \tfrac{1-\lambda_n}{\lambda_n} \max\{\operatorname{dist}(Sx, [p, Tx]),\ \operatorname{dist}(Sy, [p, Ty])\}\} \\ &\leq \lambda_n \{\max\{d(Sx, Sy),\ c[d(Sx, T_n y) + d(Sy, T_n x)]\} \\ &+ (1 - \lambda_n) \max\{[d(Sx, T_n x),\ d(Sy, T_n y)]\} \end{aligned}$$

for all $x, y \in M$,$0 \leq c < \frac{1}{4}$. Therefore, by Theorem 2.2, there exists some $x_n \in M$ such that $F(T_n) \bigcap F(S) = \{x_n\}$ for each $n \geq 1$. The compactness of $\operatorname{cl}(T(M))$ implies that there exists a subsequence $\{Tx_{n_i}\}$ of $\{Tx_n\}$ and $z \in \operatorname{cl}(T(M))$ such that $Tx_{n_i} \to z \in M$. Now $x_{n_i} = T_{n_i}(x_{n_i}) = W(Tx_{n_i}, p, k_{n_i}) \to W(z, p, 1) = z$. By the continuity of T and S, we have $z \in F(T) \cap F(S)$. Hence $M \cap F(T) \cap F(S) \neq \emptyset$. □

2.4. Example. Consider $M = R^2$ with the usual metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$, $(x_1, y_1), (x_2, y_2) \in R^2$. Define T and S on M as $T(x, y) = (\frac{x-2}{2}, \frac{x^2+y-4}{2})$ and $S(x, y) = (\frac{x-2}{2}, x^2 + y - 4)$. Obviously, T is S-nonexpansive but S is not linear. Moreover, $F(T) = (-2, 0)$, $F(S) = \{(-2, y) : y \in R\}$ and $C(S, T) = \{(x, y) : y = 4 - x^2, x \in R\}$. Thus (T, S) is a continuous Banach operator pair, $F(S)$ is convex and $(-2, 0)$ is a common fixed point of S and T.

2.2. Common fixed point theorems and invariant approximation. In this section we prove the existence of some common fixed points of best approximation for Banach operator pairs with Ćirić type nonexpansive mappings.

We begin the section with the following result.

2.5. Proposition. *If C is a closed and convex subset of a convex metric space (X, d) and $x \in X$ then $P_C(x)$ is closed and convex.*

Proof. Let $y, z \in P_C(x)$ and $\lambda \in [0, 1]$. Consider

$$\begin{aligned} d(x, W(y, z, \lambda)) &\leq \lambda d(x, y) + (1 - \lambda) d(x, z) \\ &= \lambda \operatorname{dist}(x, C) + (1 - \lambda) \operatorname{dist}(x, C) \\ &= \operatorname{dist}(x, C) \\ &\leq d(x, W(y, z, \lambda)) \text{ as } W(y, z, \lambda) \in C. \end{aligned}$$

Therefore, $d(x, W(y, z, \lambda)) = \operatorname{dist}(x, C)$ and so $W(y, z, \lambda) \in P_C(x)$. Thus $P_C(x)$ is convex and it is easy to prove that it is closed. □

The following theorem extends/generalizes the corresponding theorems of [2, 3, 4, 5, 18, 22, 23, 24], and [26].

2.6. Theorem. *Let C be a closed and convex subset of a complete convex metric space (X, d) with Property (I), and $S, T : X \to X$ mappings such that $u \in F(S) \cap F(T)$ for some $u \in X$ and $T(\partial C \cap C) \subseteq C$ (∂C denotes the boundary of C). Suppose that $D = P_C(u)$ is nonempty and $F(S)$ is nonempty and convex, S is continuous on $P_C(u)$, and $S(D) \subseteq D$. If T is continuous,* $\operatorname{cl}(T(D)$ *is compact and (T, S) is a Banach operator pair on D that satisfies*

$$d(Tx, Ty) \leq \begin{cases} d(Sx, Su) & \text{if } y = u, \\ M(x, y) & \text{if } y \in D, \end{cases} \tag{$*$}$$

where

$$M(x,y) = \max\{d(Sx,Sy),\ c[\text{dist}(Sx,[p,Ty]) + \text{dist}(Sy,[p,Tx])]\} + \tfrac{1-\lambda}{\lambda}\max\{\text{dist}(Sx,[p,Tx]),\ \text{dist}(Sy,[p,Ty])\},$$

for some $p \in F(S) \cap D$, for all $x, y \in C$, $0 < \lambda < 1$, $0 \le c < \frac{1}{4}$, then there is a common fixed point of T, S and D.

Proof. Let $x \in P_C(u)$, then for any $\lambda \in (0,1)$, we have

$$d(W(u,x,\lambda),u) \le \lambda d(u,u) + (1-\lambda)d(x,u) = (1-\lambda)d(x,u) < dist(u,C).$$

It follows (see [1, Lemma 3.2]) that the line segment $\{W(u,x,\lambda) : 0 \le \lambda \le 1\}$ and the set C are disjoint. Thus x is not in the interior of C and so $x \in \partial C \cap C$. Since $T(\partial C \cap C) \subset C$, Tx must be in C. Also since $Sx \in P_C(u)$, $u \in F(T) \cap F(S)$, and from Inequality $(*)$, we have

$$d(Tx,u) = d(Tx,Tu) \le d(Sx,Su) = d(Sx,u) = \text{dist}(u,C).$$

This implies that $Tx \in P_C(u)$. Consequently, $P_C(u)$ is T-invariant, closed and convex. Hence the result follows from Theorem 2.3. □

2.7. Corollary. [16, Theorem 2.5 (i)] *Let C be a subset of a Banach space X, and $S, T : X \to X$ mappings such that $u \in F(S) \cap F(T)$ for some $u \in X$ and $T(\partial C \cap C) \subseteq C$. Suppose that $D = P_C(u)$ and $F(S)$ are nonempty and convex, S is continuous on $P_C(u)$, and $S(P_C(u)) \subseteq P_C(u)$. If T is continuous, $\text{cl}(T(P_C(u))$ is compact and (T,S) is a Banach operator pair on $P_C(u)$ which satisfies*

$$\|Tx - Ty\| \le \begin{cases} \|Sx - Su\| & \text{if } y = u, \\ M(x,y) & \text{if } y \in D, \end{cases}$$

where

$$M(x,y) = \max\{\|Sx - Sy\|,\ c[\text{dist}(Sx,[p,Ty]) + \text{dist}(Sy,[p,Tx]]\} + \tfrac{1-\lambda}{\lambda}\max\{\text{dist}(Sx,[p,Tx]),\ \text{dist}(Sy,[p,Ty])\},$$

for some $p \in F(S) \cap C$, for all $x, y \in C$, $0 < \lambda < 1$, $0 \le c < \frac{1}{4}$, then there is a unique common fixed point of T, S and $P_C(u)$. □

Let $G_\circ$ denote the class of closed convex subsets containing a point $x_\circ$ of a convex metric space (X,d) with Property (I). For $M \in G_\circ$ and $p \in X$, let $M_p = \{x \in M : d(x,x_\circ) \le 2d(p,x_\circ)\}$. For $h \ge 0$, let $P_M(p) = \{x \in M : d(p,x) = d(p,M)\}$ - the set of best approximants to p in M, $C_M^S(p) = \{x \in M : Sx \in P_M(p)\}$ and $D_M^{h,S}(p) = P_M(p) \cap G_M^{h,S}(p)$, where $G_M^{h,S}(p) = \{x \in M : d(Sx,p) \le (2h+1)\text{dist}(p,M)\}$.

2.8. Theorem. *Let M be a convex subset of a complete convex metric space (X,d) with Property (I), and $S, T : X \to X$ mappings such that $u \in F(S) \cap F(T)$ for some $u \in X$ and $T(\partial M \cap M) \subseteq M$. Suppose that S is continuous on the closed convex set $D_M^{h,S}(u)$, $D_M^{h,S}(u) \cap F(S)$ is nonempty and $S(D_M^{h,S}(u)) \subseteq D_M^{h,S}(u)$. If $\text{cl}(T(D_M^{h,S}(u))$ is compact, T is continuous and the pair (T,S) satisfies*

(i) *$d(STx,Tx) \le hd(Sx,x)$ for all $x \in D_M^{h,S}(u)$ and $h \ge 0$;*
(ii) *For all $x \in D_M^{h,S}(u) \cup \{u\}$,*

$$d(Tx, Ty) \leq \begin{cases} d(Sx, Su) & \textit{if } y = u, \\ M(x, y) & \textit{if } y \in D_M^{h,S}, \end{cases}$$

where

$$M(x, y) = \max\{d(Sx, Sy),\ c[\mathrm{dist}(Sx, [p, Ty]) + \mathrm{dist}(Sy, [p, Tx])]\} + \tfrac{1-\lambda}{\lambda} \max\{\mathrm{dist}(Sx, [p, Tx]),\ \mathrm{dist}(Sy, [p, Ty])\},$$

for some $p \in F(S) \cap P_M(u)$, $0 < \lambda < 1$, $0 \leq c < \frac{1}{4}$,

then $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$.

Proof. Let $x \in D_M^{h,S}(u)$. Then $x \in P_M(u)$, and as in the proof of Theorem 2.6, we see that Tx is in M. Since $Sx \in P_M(u)$, $u \in F(S) \cap F(T)$ and S and T satisfy (i) and (ii), it follows that $d(Tx, u) = d(Tx, Tu) \leq d(Sx, Su) = d(Sx, u) = \mathrm{dist}(u, M)$. Now

$$\begin{aligned} d(STx, u) &\leq d(STx, Tx) + d(Tx, Tu) \\ &\leq hd(Sx, x) + d(Tx, u) \\ &\leq h[d(Sx, u) + d(u, x)] + d(Tx, u) \\ &\leq h[\mathrm{dist}(u, M) + \mathrm{dist}(u, M)] + \mathrm{dist}(u, M) \\ &\leq (2h + 1)\mathrm{dist}(u, M). \end{aligned}$$

This implies that $Tx \in G_M^{h,S}(u)$. Consequently, $Tx \in D_M^{h,S}(u)$ and so $T(D_M^{h,S}(u)) \subseteq D_M^{h,S}(u)$. Inequality (i) also implies that (T, S) is a Banach operator pair. As M is convex, $D_M^{h,S}(u)$ is convex. Thus by Theorem 2.3, we obtain that $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$. □

2.9. Remark. Let $C_M^S(u) = \{x \in M : Sx \in P_M(u)\}$. Then $S(P_M(u)) \subset P_M(u)$ implies that $P_M(u) \subset C_M^S(u) \subset D_M^{h,S}(u)$ and hence $D_M^{h,S}(u) = P_M(u)$. Consequently, Theorem 2.8 remains valid when $D_M^{h,S}(u) = P_M(u)$ and the pair (T, S) is a Banach operator pair on $P_M(u)$ instead of satisfying (i), which in turn extends/generalizes the corresponding results in [2, 5, 14, 22, 24] and [26].

2.10. Corollary. [16, Theorem 2.6 (i)] *Let M be a subset of a Banach space X, and $S, T : X \to X$ mappings such that $u \in F(S) \cap F(T)$ for some $u \in X$ and $T(\partial M \cap M) \subseteq M$. Suppose that S is continuous on the closed convex set $D_M^{h,S}(u)$, $D_M^{h,S}(u) \cap F(S)$ is nonempty convex and $S(D_M^{h,S}(u)) \subseteq D_M^{h,S}(u)$. If $\mathrm{cl}(T(D_M^{h,S}(u))$ is compact, T is continuous and the pair (T, S) satisfies*

(i) $\|STx - Tx\| \leq h\,\|Sx - x\|$ *for all* $x \in D_M^{h,S}(u)$ *and* $h \geq 0$*;*
(ii) *For all* $x \in D_M^{h,S}(u) \cup \{u\}$*, and* $0 < \lambda < 1$*,*

$$\|Tx - Ty\| \leq \begin{cases} \|Sx - Sy\| & \textit{if } y = u, \\ M(x, y) & \textit{if } y \in D_M^{h,S}, \end{cases}$$

where

$$M(x, y) = \max\{\|Sx - Sy\|,\ c[\mathrm{dist}(Sx, [p, Ty]) + \mathrm{dist}(Sy, [p, Tx]]\} + \tfrac{1-\lambda}{\lambda} \max\{\mathrm{dist}(Sx, [p, Tx]),\ \mathrm{dist}(Sy, [p, Ty])\}$$

for some $p \in F(S) \cap P_M(u)$*, for all* $x, y \in M$*,* $0 < \lambda < 1$*,* $0 \leq c < \frac{1}{4}$*,*

then $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$. □

The next two theorems extend and generalize the corresponding results of [2, 4, 14, 16, 22] and [24].

2.11. Theorem. *Let S and T be self maps of a complete convex metric space (X,d) with Property (I), $u \in F(S) \cap F(T)$ and $M \in G_\circ$ such that $T(M_u) \subseteq S(M) \subseteq M$. Suppose that $\mathrm{cl}(S(M_u))$ is compact, $d(Sx,Su) \le d(x,u)$ for all $x \in M_u$, T, S are continuous on M_u, T satisfies $d(Tx,u) \le d(Sx,u)$ for all $x \in M_u$. Then*

(i) *$P_M(u)$ is nonempty, closed and convex,*
(ii) *$T(P_M(u)) \subseteq S(P_M(u)) \subseteq P_M(u)$, provided that $d(Sx,Sp) = d(x,u)$ for all $x \in C_M^S(u)$, and*
(iii) *$P_M(u) \cap F(S) \cap F(T) \neq \emptyset$ provided that $d(Sx,Su) = d(x,u)$ for all $x \in C_M^S(u)$, $F(S)$ is nonempty and convex, (T,S) is a Banach operator pair on $P_M(u)$ and T satisfies for some $p \in F(S) \cap P_M(u)$,*

$$d(Tx,Ty) \le \max\{d(Sx,Sy),\ c[\mathrm{dist}(Sx,[p,Ty]) + \mathrm{dist}(Sy,[p,Tx])]\} + \tfrac{1-\lambda}{\lambda}\max\{\mathrm{dist}(Sx,[p,Tx]),\ \mathrm{dist}(Sy,[p,Ty])\}$$

for all $x, y \in P_M(p)$, $0 < \lambda < 1$, and some $0 \le c < \frac{1}{4}$.

Proof. If $u \in M$ then all the arguments are obvious. So assume that $u \notin M$. If $x \in M \backslash M_u$, then $d(x,x_\circ) > 2d(u,x_\circ)$ and so $d(u,x) \ge d(x,x_\circ) - d(u,x_\circ) > d(u,x_\circ) \ge \mathrm{dist}(u,M)$. Thus $\alpha = \mathrm{dist}(u,M_u) = \mathrm{dist}(u,M) \le d(u,x_\circ)$. Since $\mathrm{cl}(S(M_u))$ is compact, and the distance function is continuous, there exists $z \in \mathrm{cl}(S(M_u))$ such that $\beta = \mathrm{dist}(u, \mathrm{cl}(S(M_u))) = d(u,z)$. Hence

$$\begin{aligned}\alpha &= \mathrm{dist}(u,M_u)\\ &\le \mathrm{dist}(u,\mathrm{cl}(S(M_u))) \text{ as } T(M_u) \subseteq S(M) \subseteq M \implies \mathrm{cl}S(M_u) \subseteq M\\ &= \beta \le \mathrm{dist}(u,S(M_u)) \le d(u,Sx)\\ &\le d(u,x)\end{aligned}$$

for all $x \in M_u$. Therefore $\alpha = \beta = \mathrm{dist}(u,M)$ i.e. $\mathrm{dist}(u,M) = \mathrm{dist}(u,\mathrm{cl}(S(M_u))) = d(u,z)$ i.e. $z \in P_M(u)$ and so $P_M(u)$ is nonempty. The closedness and convexity of $P_M(u)$ follows from that of M. This proves (i).

To prove (ii) let $z \in P_M(u)$. Then $d(Sz,u) = d(Sz,Su) \le d(z,u) = \mathrm{dist}(u,M)$. This implies that $Sz \in P_M(u)$ and so $S(P_M(u)) \subseteq P_M(u)$. Let $y \in T(P_M(u))$. Since $T(M_u) \subseteq S(M)$ and $P_M(u) \subseteq M_u$, there exists $z \in P_M(u)$ and $x_1 \in M$ such that $y = Tz = Sx_1$. Further, we have $d(Sx_1,u) = d(Tz,u) \le d(Sz,u) \le d(z,u) = dist(u,M)$. Thus $Sx_1 \in P_M(u)$ and $x_1 \in C_M^S(u)$. Also, as $Sx_1 \in M$ and $\mathrm{dist}(u,M) \le d(Sx_1,u)$, it follows that $\mathrm{dist}(u,M) = d(Sx_1,u)$. Since $d(x_1,u) = d(Sx_1,u) = \mathrm{dist}(u,M)$, $x_1 \in P_M(u)$ and $y = Sx_1 \in S(P_M(u))$. Hence $T(P_M(u)) \subseteq S(P_M(u))$ and so (ii) holds.

By (ii), the compactness of $cl(S(M_u))$ implies that $\mathrm{cl}(T(P_M(u)))$ is compact and hence complete. Hence the conclusion (iii) follows from Theorem 2.3 applied to $P_M(u)$. □

2.12. Corollary. [16, Theorem 2.7] *Let S and T be self maps of a Banach space X with $u \in F(S) \cap F(T)$ and $C \in G_\circ$ such that $T(C_u) \subseteq S(C) \subseteq C$. Suppose that $\mathrm{cl}(S(Cu))$ is compact, $\|Sx - u\| = \|x - u\|$ for all $x \in C_u$, T, S are continuous on C_u, and T satisfies $\|Tx - u\| \le \|Sx - u\|$ for all $x \in C_u$. Then*

(i) *$P_C(u)$ is nonempty, closed and convex,*
(ii) *$T(P_C(u)) \subseteq S(P_C(u)) \subseteq P_C(u)$, provided that $\|Sx - u\| = \|x - u\|$ for all $x \in C_C^S(u)$, and*
(iii) *$P_C(u) \cap F(S) \cap F(T) \neq \emptyset$ provided that $\|Sx - u\| = \|x - u\|$ for all $x \in C_C^S(u)$, $F(S)$ is nonempty and convex, (T,S) is Banach operator pair on $P_C(u)$ and T*

satisfies for some $p \in F(S) \cap P_C(u)$,

$$\|Tx - Ty\| \le \max\{\|Sx - Sy\|,\ c[\mathrm{dist}(Sx, [p, Ty]) + \mathrm{dist}(Sy, [p, Tx]]\} + \tfrac{1-\lambda}{\lambda} \max\{\mathrm{dist}(Sx, [p, Tx]),\ \mathrm{dist}(Sy, [p, Ty])\},$$

for all $x, y \in P_C(u)$, $0 < \lambda < 1$, *and some* $0 \le c < \frac{1}{4}$. □

2.13. Theorem. *Let* S *and* T *be self maps of a complete convex metric space* (X, d) *with Property (I),* $u \in F(S) \cap F(T)$ *and* $M \in G_\circ$ *such that* $T(M_u) \subseteq S(M) \subseteq M$. *Suppose that* $\mathrm{cl}(T(M_u))$ *is compact,* $d(Sx, Su) \le d(x, u)$ *for all* $x \in M_u$, T, S *are continuous on* M_u, *and* T *satisfies* $d(Tx, u) \le d(Sx, u)$ *for all* $x \in M_u$. *Then*

(i) $P_M(u)$ *is nonempty, closed and convex,*
(ii) $T(P_M(u)) \subseteq S(P_M(u)) \subseteq P_M(u)$, *provided that* $d(Sx, Su) = d(x, u)$ *for all* $x \in C_M^S(u)$, *and*
(iii) $P_M(u) \cap F(S) \cap F(T) \neq \emptyset$ *provided that* $d(Sx, Su) = d(x, u)$ *for all* $x \in C_M^S(u)$, $F(S)$ *is nonempty and convex,* (T, S) *is a Banach operator pair on* $P_M(u)$ *and* T *satisfies*

$$d(Tx, Ty) \le \max\{d(Sx, Sy),\ c[\mathrm{dist}(Sx, [p, Ty]) + \mathrm{dist}(Sy, [p, Tx])]\} + \tfrac{1-\lambda}{\lambda} \max\{\mathrm{dist}(Sx, [p, Tx]),\ \mathrm{dist}(Sy, [p, Ty])\}$$

for all $p \in F(S) \cap P_M(u)$, $x, y \in P_M(u)$, $0 < \lambda < 1$, *and some* $0 \le c < \frac{1}{4}$.

Proof. Similar to that of Theorem 2.11. □

2.14. Corollary. [16, Theorem 2.8] *Let* S *and* T *be be self maps of a Banach space* X *with* $u \in F(S) \cap F(T)$ *and* $C \in G_\circ$ *such that* $T(C_u) \subseteq S(C) \subseteq C$. *Suppose that* $\mathrm{cl}(T(C_u))$ *is compact,* $\|Sx - u\| = \|x - u\|$ *for all* $x \in C_u$, T, S *are continuous on* C_u, *and* T *satisfies* $\|Tx - u\| \le \|Sx - u\|$ *for all* $x \in C_u$. *Then*

(i) $P_C(u)$ *is nonempty, closed and convex,*
(ii) $T(P_C(u)) \subseteq S(P_C(u)) \subseteq P_C(u)$, *provided that* $\|Sx - u\| = \|x - u\|$ *for all* $x \in C_C^S(u)$, *and*
(iii) $P_C(u) \cap F(S) \cap F(T) \neq \emptyset$ *provided that* $\|Sx - u\| = \|x - u\|$ *for all* $x \in C_C^S(u)$, $F(S)$ *is nonempty and convex,* (T, S) *is a Banach operator pair on* $P_C(u)$ *and* T *satisfies*

$$\|Tx - Ty\| \le \max\{\|Sx - Sy\|,\ c[\mathrm{dist}(Sx, [p, Ty]) + \mathrm{dist}(Sy, [p, Tx]]\} + \tfrac{1-\lambda}{\lambda} \max\{\mathrm{dist}(Sx, [p, Tx]),\ \mathrm{dist}(Sy, [p, Ty])\},$$

for all $p \in F(S) \cap P_C(u)$, $x, y \in P_C(u)$, $0 < \lambda < 1$, *and some* $0 \le c < \frac{1}{4}$. □

2.3. Common fixed point theorem with generalized nonexpansive mappings. In this section we prove a common fixed point theorem for Banach operator pairs with generalized nonexpansive mappings in metric spaces.

The following lemma of Hussain [16] will be used in our next theorem.

2.15. Lemma. [16] *Let* C *be a nonempty subset of a metric space* (X, d), (T, f) *and* (T, g) *Banach operator pairs on* C. *Suppose that* $\mathrm{cl}(T(C))$ *is complete and* T, f, g *satisfy for all* $x, y \in C$ *and* $0 \le k < 1$,

$$d(Tx, Ty) \le k \max\{d(fx, gy),\ d(Tx, fx),\ d(Ty, gy),\ d(Tx, gy),\ d(Ty, fx)\}.$$

If f *and* g *are continuous and* $F(f) \cap F(g)$ *is nonempty, then there is a unique common fixed point of* T, f *and* g. □

The following result extends and improves the corresponding results of [2, 3, 5, 14, 16, 18, 21, 22] and [23].

2.16. Theorem. *Let C be a nonempty subset of a metric space (X,d), T,g and h self maps of C. Suppose that g and h are continuous and that C has a contractive jointly continuous family $\mathfrak{F}=\{f_x : x\in C\}$ with $gf_x(k)=f_{gx}(k)$ and $hf_x(k)=f_{hx}(k)$ for all $x\in M$, $k\in[0,1]$. If $\operatorname{cl}(T(C))$ is compact, T is continuous and (T,g), (T,h) are Banach operator pairs that satisfy*

$$\begin{aligned}&d(Tx,Ty)\\&\quad\le \max\{d(hx,gy),\ \operatorname{dist}(hx,Y_q^{Tx}),\ \operatorname{dist}(gy,Y_q^{Ty}),\ \operatorname{dist}(hx,Y_q^{Ty}),\ \operatorname{dist}(gy,Y_q^{Tx})\}\end{aligned}$$

for all $x,y\in C$, where $Y_q^{Tx}=\{f_{Tx}(k):0\le k\le 1\}$ and $q=f_{Tx}(0)$, then T,g and h have a common fixed point.

Proof. For each $n\in N$, define $T_n : C\to C$ by $T_n(x)=f_{Tx}(k_n)$ for each $x\in C$, where $\langle k_n\rangle$ is a sequence in $(0,1)$ such that $k_n\to 1$. Then each T_n is a self mapping of C. As (T,g) is a Banach operator pair, for $x\in F(g)$ we have $Tx\in F(g)$. Consider

$$g(T_nx)=gf_{Tx}(k_n)=f_{gTx}(k_n)=f_{Tx}(k_n)=T_nx.$$

Then $T_nx\in F(g)$. Thus for each n, (T_n,g) is a Banach operator pair on C. Similarly, (T_n,h) is a Banach operator pair on C. Consider

$$\begin{aligned}&d(T_nx,T_ny)\\&\quad=d(f_{Tx}(k_n),f_{Ty}(k_n))\\&\quad\le\varphi(k_n)d(Tx,Ty)\\&\quad\le\varphi(k_n)\max\{d(hx,gy),\ \operatorname{dist}(hx,Y_q^{Tx}),\ \operatorname{dist}(gy,Y_q^{Ty}),\ \operatorname{dist}(hx,Y_q^{Ty}),\\&\qquad\qquad\operatorname{dist}(gy,Y_q^{Tx})\}\\&\quad\le\varphi(k_n)\max\{d(hx,gy),\ d(hx,T_nx),\ d(gy,T_ny),\ d(hx,T_ny),\ d(gy,T_nx)\}\end{aligned}$$

for all $x,y\in C$. As $\operatorname{cl}(T(C))$ is compact, $\operatorname{cl}(T_n(C))$ is compact for each n and hence complete. Now by Lemma 2.15, there exists $x_n\in C$ such that x_n is a common fixed point of g,h and T_n for each n. The compactness of $\operatorname{cl}(T(C))$ implies there exists a subsequence $\{Tx_{n_i}\}$ of $\{Tx_n\}$ such that $Tx_{n_i}\to y\in C$. Now

$$x_{n_i}=T_{n_i}x_{n_i}=f_{Tx_{n_i}}(k_{n_i})\to f_y(1)=y,$$

and the result follows by using the continuity of T,h and g. □

If C is a q-starshaped subset of a convex metric space (X,d) with Property (I), and we define the family $\mathfrak{F}$ as $f_x(\alpha)=W(x,q,\alpha)$, then

$$\begin{aligned}d(f_x(\alpha),f_y(\alpha))&=d(W(x,q,\alpha),W(y,q,\alpha))\\&\le\alpha d(x,y),\end{aligned}$$

so taking $\varphi(\alpha)=\alpha$, $0<\alpha<1$, the family is a contractive jointly continuous family. Consequently, we have:

2.17. Corollary. *Let C be a nonempty q-starshaped subset of a convex metric space (X,d) with Property (I), and let T,g and h be self maps of C. Suppose that g and h are continuous and $F(g)$ and $F(h)$ are q-starshaped with $q\in F(g)\cap F(h)$. If $\operatorname{cl}(T(C))$ is compact, T is continuous and (T,g), (T,h) are Banach operator pairs that satisfy*

$$\begin{aligned}d(Tx,Ty)\le\max\{&d(hx,gy),\ \operatorname{dist}(hx,[q,Tx]),\ \operatorname{dist}(gy,[q,Ty]),\ \operatorname{dist}(hx,[q,Ty]),\\&\operatorname{dist}(gy,[q,Tx])\}\end{aligned}$$

for all $x,y\in C$, $k\in[0,1)$, then T,g and h have a common fixed point.

Proof. For each $n \in N$, define $T_n : C \to C$ by $T_n(x) = W(Tx, q, k_n)$ for each $x \in C$, where $\langle k_n \rangle$ is a sequence in $(0,1)$ such that $k_n \to 1$. Then each T_n is a self mapping of C. Since (T, g) is a Banach operator pair and $F(g)$ is q-starshaped, then for each $x \in F(g)$, $T_n(x) = W(Tx, q, k_n) \in F(g)$, since $Tx \in F(g)$. Thus (T_n, g) is a Banach operator pair for each n. Similarly, (T_n, h) is a Banach operator pair on C. Consider

$$\begin{aligned} d(T_n x, T_n y) &= d(W(Tx, q, k_n), W(Ty, q, k_n)) \\ &\le k_n d(Tx, Ty) \\ &\le k_n \max\{d(hx, gy),\, \mathrm{dist}(hx, [q, Tx]),\, \mathrm{dist}(gy, [q, Ty]), \\ &\qquad\qquad \mathrm{dist}(hx, [q, Ty]),\, \mathrm{dist}(gy, [q, Tx])\} \\ &\le k_n \max\{d(hx, gy),\, d(hx, T_n x),\, d(gy, T_n y),\, d(hx, T_n y),\, d(gy, T_n x)\} \end{aligned}$$

for all $x, y \in C$. As $\mathrm{cl}(T(C))$ is compact, $\mathrm{cl}(T_n(C))$ is compact for each n and hence complete. Now by Lemma 2.15, there exists $x_n \in C$ such that x_n is a common fixed point of g, h and T_n for each n. The compactness of $\mathrm{cl}(T(C))$ implies there exists a subsequence $\{Tx_{n_i}\}$ of $\{Tx_n\}$ such that $Tx_{n_i} \to y \in C$. Now, as $k_{n_i} \to 1$, we have

$$x_{n_i} = T_{n_i} x_{n_i} = W(Tx, q, k_{n_i}) \to y,$$

and the result follows by using the continuity of T, h and g. □

2.18. Corollary. [16, Theorem 2.11] *Let C be a nonempty q-starshaped subset of a normed linear space X, and let T, g and h be self maps of C. Suppose that g and h are continuous and that $F(g)$ and $F(h)$ are q-starshaped with $q \in F(g) \cap F(h)$. If $\mathrm{cl}(T(C))$ is compact, T is continuous and (T, g), (T, h) are Banach operator pairs that satisfy*

$$\begin{aligned} d(Tx, Ty) \le \max\{&d(hx, gy),\, \mathrm{dist}(hx, [Tx, q]),\, \mathrm{dist}(gy, [Ty, q]), \\ &\mathrm{dist}(hx, [Ty, q]),\, \mathrm{dist}(gy, [Tx, q])\} \end{aligned}$$

for all $x, y \in C$, $k \in [0, 1)$, then T, g and h have a common fixed point. □

2.19. Corollary. *Let C be a nonempty q-starshaped subset of a convex metric space (X, d) with Property (I), and let T, g and h be self maps of C. Suppose that g and h are continuous and affine with $q \in F(g) \cap F(h)$. If $\mathrm{cl}(T(C))$ is compact, T is continuous and (T, g), (T, h) are Banach operator pairs that satisfy*

$$\begin{aligned} d(Tx, Ty) \le \max\{&d(hx, gy),\, \mathrm{dist}(hx, [q, Tx]),\, \mathrm{dist}(gy, [q, Ty]),\, \mathrm{dist}(hx, [q, Ty]), \\ &\mathrm{dist}(gy, [q, Tx])\} \end{aligned}$$

for all $x, y \in C$, $k \in [0, 1)$, then T, g and h have a common fixed point.

Proof. Taking $k_n = \frac{n}{n+1}$, define $T_n(x) = W(Tx, q, k_n)$. Since (T, g) is a Banach operator pair, g is affine with $q \in F(g) \cap F(h)$ so for each $x \in F(g)$ we have $Tx \in F(g)$. Consider

$$g(T_n(x)) = g(W(Tx, q, k_n)) = W(g(Tx), g(q), k_n) = W(Tx, q, k_n) = T_n(x),$$

so that $T_n x \in F(g)$. Thus (T_n, g) is a Banach operator pair for each n. Similarly, (T_n, h) is a Banach operator pair on C for each n. Now proceeding as in Corollary 2.17, we get the result. □

2.20. Remark. As an application of Theorem 2.16, the analogues of [15, Theorem 2.3 (i), Corollary 2.6, Corollary 2.7, Theorem 2.8 (i), Corollary 2.9 (i), Corollary 2.10 (i)] can be established for Banach operator pairs (T, g) and (T, h) in convex metric spaces.

Acknowledgements.

The authors are thankful to the learned referee for the valuable comments and suggestions.

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