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# TWO FORMULAS CONTIGUOUS TO A QUADRATIC TRANSFORMATION DUE TO KUMMER WITH AN APPLICATION

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**Abstract**

We aim at establishing two identities contiguous to Kummer's transformation:

$$(1-z)^{-a}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}a,\ \frac{1}{2}a+\frac{1}{2}\,;\\ b+\frac{1}{2}\,;\end{matrix}\ \left(\frac{z}{1-z}\right)^2\right] = {}_2F_1\left[\begin{matrix}a,\ b\,;\\ 2b\,;\end{matrix}\ 2z\right]$$

by using two different methods. They are further applied to prove two summation formulas for the series ${}_3F_2(1)$, closely related to the classical Watson's theorem due to Lavoie.

**Keywords:** Gamma function, Hypergeometric function, Generalized hypergeometric function, Kampé de Fériet function, Kummer's second theorem, Dixon and Whipple's summation theorems.

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## 1. Introduction and preliminaries

The *generalized hypergeometric series* ${}_pF_q$ is defined by (see [13, p. 73]):

$$(1.1)\qquad {}_pF_q\left[\begin{matrix}\alpha_1,\ \ldots,\ \alpha_p\,;\\ \beta_1,\ \ldots,\ \beta_q\,;\end{matrix}\ z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\,\frac{z^n}{n!} = {}_pF_q(\alpha_1,\ \ldots,\ \alpha_p;\ \beta_1,\ \ldots,\ \beta_q;\ z),$$

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where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [16, p. 2 and p. 6]):

$$(\lambda)_n := \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1,\, 2,\, 3,\, \ldots\}) \end{cases} = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \tag{1.2}$$

and $\mathbb{Z}_0^-$ denotes the set of nonpositive integers, $\mathbb{C}$ the set of complex numbers, and $\Gamma(\lambda)$ is the familiar Gamma function. Here $p$ and $q$ are positive integers or zero (interpreting an empty product as 1), and we assume (for simplicity) that the variable $z$, the numerator parameters $\alpha_1, \ldots, \alpha_p$, and the denominator parameters $\beta_1, \ldots, \beta_q$ take on complex values, provided that no zeros appear in the denominator of (1.1), that is, that

$$(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ j = 1,\, \ldots,\, q). \tag{1.3}$$

It is noted in passing that the Gamma function $\Gamma$ and its related constant, asymptotic formulas, and inequalities have been investigated by many authors, for example, see [2, 3].

It should also be remarked here that whenever the hypergeometric function ${}_2F_1$ and the generalized hypergeometric functions ${}_pF_q$ are expressed in terms of the Gamma function, the results are usually important, in particular, from the application point of view. Therefore, the well known summation theorems such as those of Gauss, Gauss's second, Bailey and Kummer for the series ${}_2F_1$ and Watson, Dixon and Whipple for the series ${}_3F_2$ and their extensions and generalizations (see [8, 9, 10, 11, 12]) play an important role in the theory of generalized hypergeometric series. For applications of the above-mentioned classical summation theorems, we refer to [1, 5, 6, 11, 12, 13, 14].

Moreover, it is well known that, if the product of two hypergeometric series can be expressed as a hypergeometric series with argument $x$, the coefficient of $x^n$ in the product must be expressible in terms of Gamma functions. Here, we mention some of the above summation theorems and their special cases so that the paper may be self-contained.

*Gauss's summation theorem* (see [13, p. 49])

$${}_2F_1\left[\begin{matrix} a,\, b\,; \\ c\,; \end{matrix}\; 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \quad (\Re(c-a-b) > 0); \tag{1.4}$$

*Special cases* (see [13, p. 49])

$${}_2F_1\left[\begin{matrix} -\frac{1}{2}n,\, -\frac{1}{2}n+\frac{1}{2}\,; \\ b+\frac{1}{2}\,; \end{matrix}\; 1\right] = \frac{2^n\,(b)_n}{(2b)_n} = \left(1+\frac{n}{2b}\right)\,\frac{2^n\,(b)_n}{(2b+1)_n} \tag{1.5}$$

and

$${}_2F_1\left[\begin{matrix} -\frac{1}{2}n+\frac{1}{2},\, -\frac{1}{2}n+1\,; \\ b+\frac{3}{2}\,; \end{matrix}\; 1\right] = \left(1+\frac{1}{2b}\right)\,\frac{2^n\,(b)_n}{(2b+1)_n}; \tag{1.6}$$

*Watson's theorem* (see [16, p. 251])

$${}_3F_2\left[\begin{matrix} a,\, b,\, c\,; \\ \frac{1}{2}(a+b+1),\, 2c\,; \end{matrix}\; 1\right] \quad (\Re(2c-a-b) > -1) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}a+c\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}b+c\right)}; \tag{1.7}$$

*Contiguous Watson's theorem* (see [8])

$$(1.8)\quad {}_3F_2\left[\begin{matrix} a,\, b,\, c\,; \\ \frac{1}{2}(a+b+1),\, 2c+1\,; \end{matrix}\; 1\right] \quad (\Re(2c-a-b)>-3)$$
$$= \frac{2^{a+b-2}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\,\Gamma\left(c+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma(a)\,\Gamma(b)}$$
$$\cdot\left\{\frac{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a+1\right)\,\Gamma\left(c-\frac{1}{2}b+1\right)}\right\}$$

and

$$(1.9)\quad {}_3F_2\left[\begin{matrix} a,\, b,\, c\,; \\ \frac{1}{2}(a+b+1),\, 2c-1\,; \end{matrix}\; 1\right] \quad (\Re(2c-a-b)>1)$$
$$= \frac{2^{a+b-2}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma(a)\,\Gamma(b)}$$
$$\cdot\left\{\frac{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}b-\frac{1}{2}\right)} + \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a\right)\,\Gamma\left(c-\frac{1}{2}b\right)}\right\}.$$

From the theory of differential equations, *Kummer* obtained the following interesting and useful *quadratic transformation* (see [13, p. 65]):

$$(1.10)\quad (1-z)^{-a}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a,\, \frac{1}{2}a+\frac{1}{2}\,; \\ b+\frac{1}{2}\,; \end{matrix}\; \left(\frac{z}{1-z}\right)^2\right] = {}_2F_1\left[\begin{matrix} a,\, b\,; \\ 2b\,; \end{matrix}\; 2z\right],$$

valid when $2b \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $|z|<\frac{1}{2}$, and $\left|\frac{z}{1-z}\right|<1$. As shown in Rainville [13, p. 65], the result (1.10) can be derived, without recourse to the differential equation, by mainly using Gauss's summation theorem (1.4). In (1.10), if we take $z=\frac{x}{1+x}$, we have the following form

$$(1.11)\quad (1+x)^{a}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a,\, \frac{1}{2}a+\frac{1}{2}\,; \\ b+\frac{1}{2}\,; \end{matrix}\; x^2\right] = {}_2F_1\left[\begin{matrix} a,\, b\,; \\ 2b\,; \end{matrix}\; \frac{2x}{1+x}\right],$$

which can be rewritten in the form

$$(1.12)\quad (1+x)^{-a}\,{}_2F_1\left[\begin{matrix} a,\, b\,; \\ 2b\,; \end{matrix}\; \frac{2x}{1+x}\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}a,\, \frac{1}{2}a+\frac{1}{2}\,; \\ b+\frac{1}{2}\,; \end{matrix}\; x^2\right].$$

It is also interesting to note that in (1.12) if we replace $x$ by $\frac{x}{a}$ and let $a\to\infty$, we get *Kummer's second theorem* (see [13, p. 126])

$$(1.13)\quad e^{-x}\,{}_1F_1\left[\begin{matrix} b\,; \\ 2b\,; \end{matrix}\; 2x\right] = {}_0F_1\left[\begin{matrix} -\,; \\ b+\frac{1}{2}\,; \end{matrix}\; \frac{x^2}{4}\right].$$

The objective of this paper is to establish two formulas contiguous to Kummer's quadratic transformation (1.10) by using two different methods. They are further applied to prove two summation formulas for the series ${}_3F_2(1)$, closely related to the classical Watson theorem derived earlier by Lavoie [7], who used a different elementary method. In order to establish our formulas we also need the following results.

*Integral representation for* ${}_2F_1$ (see [13, p. 47])

$$(1.14)\quad {}_2F_1\left[\begin{matrix} a,\, b\,; \\ c\,; \end{matrix}\; z\right] = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1-zt)^{-a}\,dt$$
$$(\Re(c)>\Re(b)>0,\ \ |z|<1).$$

*Integral formula*

$$B(\rho,\,\sigma)\,{}_3F_2\left[\begin{matrix}\alpha,\,\beta,\,\rho\,;\\ \gamma,\,\sigma+\rho\,;\end{matrix}\;z\right]=\int_0^1 t^{\rho-1}\,(1-t)^{\sigma-1}\,{}_2F_1(\alpha,\,\beta\,;\,\gamma\,;\,tz)\,dt \tag{1.15}$$

$$\big(\Re(\rho)>0,\ \Re(\sigma)>0,\ |\arg(1-z)|<\pi\big),$$

which is a corrected version of the formula given in [4, p. 399, Entry (7)], and where $B(\rho,\,\sigma)$ is the Beta function (see [16, pp. 9–11]).

*Quadratic transformation* (see [13, p. 67])

$${}_2F_1\left[\begin{matrix}2a,\,2b\,;\\ a+b+\frac{1}{2}\,;\end{matrix}\;z\right]={}_2F_1\left[\begin{matrix}a,\,b\,;\\ a+b+\frac{1}{2}\,;\end{matrix}\;4z(1-z)\right] \tag{1.16}$$

$$\left(a+b+\frac{1}{2}\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ |z|<1,\ |4z(1-z)|<1\right).$$

*Definite integral*

$$\int_0^{2a} f(x)\,dx=\begin{cases}2\int_0^a f(x)\,dx & \text{if } f(2a-x)=f(x);\\ 0 & \text{if } f(2a-x)=-f(x).\end{cases} \tag{1.17}$$

*Known result* (see [16, p. 10, Eq. (64)])

$$\int_0^{\frac{\pi}{2}}\sin^m\theta\,\cos^n\theta\,d\theta=\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)} \tag{1.18}$$

$$(\Re(m)>-1,\ \Re(n)>-1).$$

$$\frac{1}{2}\left[(1+x)^{-a}+(1-x)^{-a}\right]={}_2F_1\left[\begin{matrix}\frac{1}{2}a,\,\frac{1}{2}a+\frac{1}{2}\,;\\ \frac{1}{2}\,;\end{matrix}\;x^2\right] \tag{1.19}$$

and

$$\frac{1}{2}\left[(1+x)^{-a}-(1-x)^{-a}\right]=-a\,x\,{}_2F_1\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2},\,\frac{1}{2}a+1\,;\\ \frac{3}{2}\,;\end{matrix}\;x^2\right]. \tag{1.20}$$

## 2. Main transformation formulas

The following two formulas closely related to Kummer's transformation (1.10) will be established.

**2.1. Theorem.** *Each of the following transformation formulas hold true.*

$$\begin{aligned}{}_2F_1\left[\begin{matrix}a,\,b\,;\\ 2b+1\,;\end{matrix}\;2z\right]&=(1-z)^{-a}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}a,\,\frac{1}{2}a+\frac{1}{2}\,;\\ b+\frac{1}{2}\,;\end{matrix}\;\left(\frac{z}{1-z}\right)^2\right]\\ &\quad-\frac{az}{2b+1}\,(1-z)^{-(a+1)}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2},\,\frac{1}{2}a+1\,;\\ b+\frac{3}{2}\,;\end{matrix}\;\left(\frac{z}{1-z}\right)^2\right]\end{aligned} \tag{2.1}$$

$$\left(2b+1\in\mathbb{C}\setminus\mathbb{Z}_0^-,\ |z|<\frac{1}{2},\ \left|\frac{z}{1-z}\right|<1\right)$$

*and*

$$\begin{aligned}{}_2F_1\left[\begin{matrix}a,\,b\,;\\ 2b-1\,;\end{matrix}\;2z\right]&=(1-z)^{-a}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}a,\,\frac{1}{2}a+\frac{1}{2}\,;\\ b-\frac{1}{2}\,;\end{matrix}\;\left(\frac{z}{1-z}\right)^2\right]\\ &\quad+\frac{az}{2b-1}\,(1-z)^{-(a+1)}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2},\,\frac{1}{2}a+1\,;\\ b+\frac{1}{2}\,;\end{matrix}\;\left(\frac{z}{1-z}\right)^2\right]\end{aligned} \tag{2.2}$$

$$\left(2b-1 \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ |z| < \frac{1}{2},\ \left|\frac{z}{1-z}\right| < 1\right).$$

*Proof.* We prove our main results in two ways.

**Method 1.** Starting with the integral representation (1.14) for ${}_2F_1$ and replacing $c$ by $2b+1$ and $z$ by $2z$, we have

$$\text{(2.3)} \qquad {}_2F_1\left[\begin{matrix} a,\ b\,; \\ 2b+1\,; \end{matrix}\ 2z\right] = \frac{1}{B(b,\ b+1)} \int_0^1 t^{b-1}\,(1-t)^b\,(1-2zt)^{-a}\,dt.$$

In (2.3), if we put $t = \sin^2\theta$, after a little simplification, we obtain

$$\text{(2.4)} \qquad \begin{aligned} &{}_2F_1\left[\begin{matrix} a,\ b\,; \\ 2b+1\,; \end{matrix}\ 2z\right] \\ &\quad = \frac{(1-z)^{-a}}{B(b,\ b+1)\,2^{2b-1}} \int_0^{\frac{\pi}{2}} \sin^{2b-1} 2\theta\,(1+\cos 2\theta)\,(1+\xi\,\cos 2\theta)^{-a}\,d\theta, \end{aligned}$$

where, for simplicity, $\xi = z/(1-z)$. Setting $2\theta = \phi$, we get

$$\text{(2.5)} \qquad \begin{aligned} &{}_2F_1\left[\begin{matrix} a,\ b\,; \\ 2b+1\,; \end{matrix}\ 2z\right] \\ &\quad = \frac{(1-z)^{-a}}{B(b,\ b+1)\,2^{2b}} \int_0^{\pi} \sin^{2b-1}\phi\,(1+\cos\phi)\,(1+\xi\,\cos\phi)^{-a}\,d\phi. \end{aligned}$$

In exactly the same manner, if we put $t = \cos^2\theta$, after a little simplification, we obtain

$$\text{(2.6)} \qquad \begin{aligned} &{}_2F_1\left[\begin{matrix} a,\ b\,; \\ 2b+1\,; \end{matrix}\ 2z\right] \\ &\quad = \frac{(1-z)^{-a}}{B(b,\ b+1)\,2^{2b}} \int_0^{\pi} \sin^{2b-1}\phi\,(1-\cos\phi)\,(1-\xi\,\cos\phi)^{-a}\,d\phi. \end{aligned}$$

Adding (2.5) and (2.6), we have

$$\text{(2.7)} \qquad \begin{aligned} &{}_2F_1\left[\begin{matrix} a,\ b\,; \\ 2b+1\,; \end{matrix}\ 2z\right] = \frac{(1-z)^{-a}}{B(b,\ b+1)\,2^{2b+1}} \\ &\quad \cdot \int_0^{\pi} \sin^{2b-1}\phi\,\left\{(1+\cos\phi)\,(1+\xi\,\cos\phi)^{-a} + (1-\cos\phi)\,(1-\xi\,\cos\phi)^{-a}\right\}\,d\phi. \end{aligned}$$

Using (1.17), we get

$$\text{(2.8)} \qquad {}_2F_1\left[\begin{matrix} a,\ b\,; \\ 2b+1\,; \end{matrix}\ 2z\right] = I_1 + I_2,$$

where, for convenience,

$$I_1 := \frac{(1-z)^{-a}}{B(b,\ b+1)\,2^{2b}} \int_0^{\frac{\pi}{2}} \sin^{2b-1}\phi\,\left\{(1+\xi\,\cos\phi)^{-a} + (1-\xi\,\cos\phi)^{-a}\right\}\,d\phi$$

and

$$I_2 := \frac{(1-z)^{-a}}{B(b,\ b+1)\,2^{2b}} \int_0^{\frac{\pi}{2}} \sin^{2b-1}\phi\,\cos\phi\,\left\{(1+\xi\,\cos\phi)^{-a} - (1-\xi\,\cos\phi)^{-a}\right\}\,d\phi.$$

Now we will compute the integrals $I_1$ and $I_2$. Using (1.19), we find

$$\text{(2.9)} \qquad I_1 = \frac{(1-z)^{-a}}{B(b,\ b+1)\,2^{2b-1}} \int_0^{\frac{\pi}{2}} \sin^{2b-1}\phi\ {}_2F_1\left[\begin{matrix} \frac{1}{2}a,\ \frac{1}{2}a+\frac{1}{2}\,; \\ \frac{1}{2}\,; \end{matrix}\ \xi^2\,\cos^2\phi\right]\,d\phi.$$

Expressing ${}_2F_1$ as a series, changing the order of integration and summation, which is guaranteed due to the uniform convergence of the series involved in the process, then after a little simplification, we have

$$I_1 = \frac{(1-z)^{-a}}{B(b,\, b+1)\, 2^{2b-1}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_n \left(\frac{1}{2}a+\frac{1}{2}\right)_n}{\left(\frac{1}{2}\right)_n\, n!}\, \xi^{2n} \int_0^{\frac{\pi}{2}} \sin^{2b-1}\phi\, \cos^{2n}\phi\, d\phi. \tag{2.10}$$

Using (1.18), after a little simplification, we get

$$\begin{aligned} I_1 &= (1-z)^{-a} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_n \left(\frac{1}{2}a+\frac{1}{2}\right)_n}{n!\, \left(b+\frac{1}{2}\right)_n}\, \xi^{2n} \\ &= (1-z)^{-a}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}a,\, \frac{1}{2}a+\frac{1}{2}\,; \\ b+\frac{1}{2}\,; \end{matrix}\ \xi^2\right]. \end{aligned} \tag{2.11}$$

A similar argument will lead us to

$$I_2 = -\frac{az}{2b+1}(1-z)^{-a-1}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2},\, \frac{1}{2}a+1\,; \\ b+\frac{3}{2}\,; \end{matrix}\ \xi^2\right]. \tag{2.12}$$

Using (2.11) and (2.12) in (2.8), we arrive at the result (2.1). In exactly the same manner, the result (2.2) can also be established. This completes the proof.

**Method 2.** Let us denote the right-hand side of (2.1) by $S_1 - S_2$, where $S_1$ is the first and $S_2$ the second sum. Then we find

$$S_1 = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_k \left(\frac{1}{2}a+\frac{1}{2}\right)_k}{\left(b+\frac{1}{2}\right)_k\, k!}\, z^{2k}\, (1-z)^{-(a+2k)}.$$

Using the binomial theorem

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n\, z^n}{n!} \quad (|z|<1,\ \ \alpha \in \mathbb{C}),$$

we have

$$S_1 = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_k \left(\frac{1}{2}a+\frac{1}{2}\right)_k}{\left(b+\frac{1}{2}\right)_k\, k!}\, \frac{(a+2k)_n}{n!}\, z^{n+2k}.$$

Using $(a)_{2k} = 2^{2k}\, \left(\frac{1}{2}a\right)_k\, \left(\frac{1}{2}a+\frac{1}{2}\right)_k$, we get

$$S_1 = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{2k}\, (a+2k)_n\, z^{n+2k}}{2^{2k}\, \left(b+\frac{1}{2}\right)_k\, k!\, n!}.$$

Using $(a)_{2k}\, (a+2k)_n = (a)_{n+2k}$, we obtain

$$S_1 = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{n+2k}\, z^{n+2k}}{2^{2k}\, \left(b+\frac{1}{2}\right)_k\, k!\, n!}.$$

Applying a well-known formal manipulation of a double series (see [13, p. 57]):

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} A(k, n-2k),$$

we have

$$S_1 = \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(a)_n\, z^n}{2^{2k}\, \left(b+\frac{1}{2}\right)_k\, k!\, (n-2k)!}.$$

Using $(n-2k)! = n!/(-n)_{2k}$ and $(-n)_{2k} = 2^{2k}\,(-n/2)_k\,(-n/2+1/2)_k$, we find

$$S_1 = \sum_{n=0}^{\infty} \frac{(a)_n}{n!}\, z^n \sum_{k=0}^{[\frac{n}{2}]} \frac{\left(-\frac{n}{2}\right)_k \left(-\frac{n}{2}+\frac{1}{2}\right)_k}{\left(b+\frac{1}{2}\right)_k\, k!} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!}\, z^n\, {}_2F_1\left[\begin{matrix} -\frac{n}{2},\, -\frac{n}{2}+\frac{1}{2}\,; \\ b+\frac{1}{2}\,; \end{matrix}\; 1\right].$$

Using (1.5), we get

$$S_1 = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \frac{2^n\,(b)_n\, z^n}{(2b+1)_n} \left(1+\frac{n}{2b}\right).$$

In a similar way, we obtain

$$S_2 = -\sum_{n=0}^{\infty} \frac{(a)_n}{n!} \frac{2^n\,(b)_n\, z^n}{(2b+1)_n} \frac{n}{2b}.$$

Finally it is easy to see that

$$S_1 - S_2 = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \frac{2^n\,(b)_n}{(2b+1)_n}\, z^n = {}_2F_1\left[\begin{matrix} a,\, b\,; \\ 2b+1\,; \end{matrix}\; 2z\right],$$

which is equal to the left-hand side of (2.1). This completes the proof of (2.1). In exactly the same manner, the identity (2.2) can also be established by the two methods enumerated above. □

If we replace $z$ by $x/(1+x)$ in (2.1) and (2.2), we get the following alternative and slightly modified forms.

**2.2. Corollary.** *Each of the following formulas hold true.*

(2.13)
$$(1+x)^{-a}\, {}_2F_1\left[\begin{matrix} a,\, b\,; \\ 2b+1\,; \end{matrix}\; \frac{2x}{1+x}\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}a,\, \frac{1}{2}a+\frac{1}{2}\,; \\ b+\frac{1}{2}\,; \end{matrix}\; x^2\right] - \frac{ax}{2b+1}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2},\, \frac{1}{2}a+1\,; \\ b+\frac{3}{2}\,; \end{matrix}\; x^2\right]$$

$$\left(2b+1 \in \mathbb{C}\setminus\mathbb{Z}_0^-,\ \left|\frac{x}{1+x}\right| < \frac{1}{2},\ |x|<1\right),$$

*and*

(2.14)
$$(1+x)^{-a}\, {}_2F_1\left[\begin{matrix} a,\, b\,; \\ 2b-1\,; \end{matrix}\; \frac{2x}{1+x}\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}a,\, \frac{1}{2}a+\frac{1}{2}\,; \\ b-\frac{1}{2}\,; \end{matrix}\; x^2\right] + \frac{ax}{2b-1}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2},\, \frac{1}{2}a+1 \\ b+\frac{1}{2} \end{matrix}\; ;\; x^2\right]$$

$$\left(2b-1 \in \mathbb{C}\setminus\mathbb{Z}_0^-,\ \left|\frac{x}{1+x}\right| < \frac{1}{2},\ |x|<1\right).$$ □

Further, in (2.13) and (2.14), if we replace $x$ by $\frac{x}{a}$ and let $a \to \infty$, we get the following identities.

**2.3. Corollary.** *Each of the following formulas hold true.*

(2.15)
$$e^{-x}\, {}_1F_1\left[\begin{matrix} b\,; \\ 2b+1\,; \end{matrix}\; 2x\right] = {}_0F_1\left[\begin{matrix} -\,; \\ b+\frac{1}{2}\,; \end{matrix}\; \frac{x^2}{4}\right] - \frac{x}{2b+1}\, {}_0F_1\left[\begin{matrix} -\,; \\ b+\frac{3}{2}\,; \end{matrix}\; \frac{x^2}{4}\right]$$

*and*

(2.16)
$$e^{-x}\, {}_1F_1\left[\begin{matrix} b\,; \\ 2b-1\,; \end{matrix}\; 2x\right] = {}_0F_1\left[\begin{matrix} -\,; \\ b-\frac{1}{2}\,; \end{matrix}\; \frac{x^2}{4}\right] + \frac{x}{2b-1}\, {}_0F_1\left[\begin{matrix} -\,; \\ b+\frac{1}{2}\,; \end{matrix}\; \frac{x^2}{4}\right].$$ □

**2.4. Remark.** Results (2.15) and (2.16) were proved in [15] by a different method. They are also recorded in [5].

## 3. An application

Here, as an application of our main results (2.1) and (2.2), we will give an alternate proof of the contiguous Watson's formulas (1.8) and (1.9). For this, in (1.15), if we set $\alpha = a$, $\beta = b$, $\rho = c$, $\sigma = d - c$, and $\gamma = 2b + 1$, we have

$$B(c,\, d-c)\, {}_3F_2\left[\begin{matrix} a,\, b,\, c\,; \\ d,\, 2b+1\,; \end{matrix}\; z\right] = \int_0^1 t^{c-1}\,(1-t)^{d-c-1}\, {}_2F_1(a,\, b\,;\, 2b+1\,;\, z\,t)\, dt \tag{3.1}$$

$$\big(\Re(d) > \Re(c) > 0, \;\; |\arg(1-z)| < \pi\big).$$

Applying the result (2.1) in the integrand ${}_2F_1$, we obtain

$$B(c,\, d-c)\, {}_3F_2\left[\begin{matrix} a,\, b,\, c\,; \\ d,\, 2b+1\,; \end{matrix}\; z\right] = F_1 - F_2, \tag{3.2}$$

where, for convenience,

$$F_1 := \int_0^1 t^{c-1}\,(1-t)^{d-c-1}\left(1 - \frac{1}{2}zt\right)^{-a} {}_2F_1\left[\begin{matrix} \frac{1}{2}a,\, \frac{1}{2}a + \frac{1}{2}\,; \\ b + \frac{1}{2}\,; \end{matrix}\; \left(\frac{zt}{2-zt}\right)^2\right] dt$$

and

$$F_2 := \frac{az}{2(2b+1)} \int_0^1 t^{c}\,(1-t)^{d-c-1}\left(1 - \frac{1}{2}zt\right)^{-a-1} \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2}a + \frac{1}{2},\, \frac{1}{2}a + 1\,; \\ b + \frac{3}{2}\,; \end{matrix}\; \left(\frac{zt}{2-zt}\right)^2\right] dt.$$

In order to evaluate $F_1$, we express ${}_2F_1$ in the integrand as a series and change the order of integration and summation (guaranteed as discussed before). Then, after a little simplification, we get

$$F_1 = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_n \left(\frac{1}{2}a + \frac{1}{2}\right)_n}{n!\, \left(b + \frac{1}{2}\right)_n} \left(\frac{z}{2}\right)^{2n} \int_0^1 t^{c+2n-1}\,(1-t)^{d-c-1}\left(1 - \frac{1}{2}zt\right)^{-a-2n} dt.$$

Comparing the integral with (1.14), and after a little simplification, we have

$$F_1 = B(c,\, d-c) \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_n \left(\frac{1}{2}a + \frac{1}{2}\right)_n (c)_{2n}}{n!\, \left(b + \frac{1}{2}\right)_n (d)_{2n}} \left(\frac{z}{2}\right)^{2n} {}_2F_1\left[\begin{matrix} a+2n,\, c+2n\,; \\ d+2n\,; \end{matrix}\; \frac{1}{2}z\right]. \tag{3.3}$$

Similarly as in obtaining (3.3), we find

$$F_2 = -\frac{a}{2b+1}\, B(c,\, d-c) \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}a + \frac{1}{2}\right)_n \left(\frac{1}{2}a + 1\right)_n (c)_{2n+1}}{n!\, \left(b + \frac{3}{2}\right)_n (d)_{2n+1}} \left(\frac{z}{2}\right)^{2n+1} \cdot {}_2F_1\left[\begin{matrix} a+2n+1,\, c+2n+1\,; \\ d+2n+1\,; \end{matrix}\; \frac{1}{2}z\right]. \tag{3.4}$$

Now, setting (3.3) and (3.4) into (3.2), interchanging $b$ and $c$, and taking $d = \frac{1}{2}(a+b+1)$, we get

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,\, b,\, c\,; \\ \frac{1}{2}(a+b+1),\, 2c+1\,; \end{matrix}\, z\right] \\
&\quad = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_n \left(\frac{1}{2}a+\frac{1}{2}\right)_n (b)_{2n}}{n!\, \left(c+\frac{1}{2}\right)_n \left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)_{2n}} \left(\frac{z}{2}\right)^{2n} {}_2F_1\left[\begin{matrix} a+2n,\, b+2n\,; \\ \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}+2n\,; \end{matrix}\, \frac{1}{2}z\right] \\
&\qquad - \frac{a}{2c+1} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(\frac{1}{2}a+1\right)_n (b)_{2n+1}}{n!\, \left(c+\frac{3}{2}\right)_n \left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)_{2n+1}} \left(\frac{z}{2}\right)^{2n+1} \\
&\qquad\qquad \cdot {}_2F_1\left[\begin{matrix} a+2n+1,\, b+2n+1\,; \\ \frac{1}{2}a+\frac{1}{2}b+\frac{3}{2}+2n\,; \end{matrix}\, \frac{1}{2}z\right].
\end{aligned}
$$

Now if we use the identity (1.16), we have

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,\, b,\, c\,; \\ \frac{1}{2}(a+b+1),\, 2c+1\,; \end{matrix}\, z\right] \\
&\quad = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_n \left(\frac{1}{2}a+\frac{1}{2}\right)_n (b)_{2n}}{n!\, \left(c+\frac{1}{2}\right)_n \left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)_{2n}} \left(\frac{z}{2}\right)^{2n} \\
&\qquad \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2}a+n,\, \frac{1}{2}b+n\,; \\ \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}+2n\,; \end{matrix}\, 2z\left(1-\frac{z}{2}\right)\right] \\
&\qquad - \frac{a}{2c+1} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(\frac{1}{2}a+1\right)_n (b)_{2n+1}}{n!\, \left(c+\frac{3}{2}\right)_n \left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)_{2n+1}} \left(\frac{z}{2}\right)^{2n+1} \\
&\qquad\qquad \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}+n,\, \frac{1}{2}b+\frac{1}{2}+n\,; \\ \frac{1}{2}a+\frac{1}{2}b+\frac{3}{2}+2n\,; \end{matrix}\, 2z\left(1-\frac{z}{2}\right)\right].
\end{aligned}
$$

Finally setting $z = 1$, evaluating the two ${}_2F_1$ 's appearing on the right-hand side by using Gauss's theorem (1.4), and after a little simplification, we find

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,\, b,\, c\,; \\ \frac{1}{2}(a+b+1),\, 2c+1\,; \end{matrix}\, 1\right] \\
&\quad = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right) \Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} {}_2F_1\left[\begin{matrix} \frac{1}{2}a,\, \frac{1}{2}b\,; \\ c+\frac{1}{2}\,; \end{matrix}\, 1\right] \\
&\qquad - \frac{2}{2c+1} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right) \Gamma\left(\frac{1}{2}b\right)} {}_2F_1\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2},\, \frac{1}{2}b+\frac{1}{2}\,; \\ c+\frac{3}{2}\,; \end{matrix}\, 1\right].
\end{aligned}
$$

Applying again Gauss's theorem (1.4) to the two ${}_2F_1$ series, and after a little simplification, we get (1.8). In exactly the same manner, the result (1.9) can also be established.

**3.1. Remark.** The formulas (1.8) and (1.9) were proved by Lavoie [7] who used contiguous function relations in a very elementary way.

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