

THE HAZARD RATE PROPERTIES OF PARALLEL AND SERIES SYSTEMS FOR BIVARIATE EXPONENTIAL DISTRIBUTIONS

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Abstract

In this paper, two-component systems (parallel and series) with stochastically dependent components are considered. The aim is to investigate an ordering relation according to their hazard rates among the lifetime of the system with dependent and independent components, and the lifetime of the component. In addition, monotonicity of the hazard rate of two-component system for a bivariate exponential family of distributions is examined. Moreover, some general properties of the hazard rates of the systems for the Clayton's Distribution Family are given.

Keywords: Hazard rate, Hazard rate ordering, Systems with dependent components.

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1. Introduction

Consider a system consisting of several components. Suppose there is a system whose components are working under the same environment, or subjected to the same set of stresses and sharing the load. Generally, the lifetimes of the components are dependent. For example, consider a squad with two dealers in a sales department. If the success of the team depends on the marked sales for both dealers then the success of one may encourage the success of the other. Therefore, the amount of the individual sales of each dealer will be affected by the other. For a multicomponent system, it is desired to discuss the monotonicity of the hazard rates of such systems at least in two-component systems. That is, we are going to investigate whether the system lifetime behaves like its components or not when the component lifetimes have an increasing (or decreasing) hazard rate.

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Furthermore, we are going to discuss possible ordering relations among the lifetimes. That is, the system lifetime will be compared with both component lifetimes and the lifetime of the system with independent components according to the hazard rate ordering. A transformation is defined from the survival function of the component lifetime to the survival function of the system lifetime. In order to make an ordering for the lifetimes, this transformation is a useful tool for some bivariate families. Since the exponential distribution is widely used in reliability, we only focus on the exponential families of distributions, such as Clayton's Bivariate Distribution, Cowan's Bivariate Exponential Distribution, Gumbel's Bivariate Exponential Distribution, Gumbel's Type III Exponential Distribution and the FGM distribution. We are going to first introduce some general properties for Clayton's Family, then conditions for the hazard rate ordering will be given for these families with exponential marginals. The lifetimes of the series system have a bathtub shape property for the FGM family with exponential marginals (Gupta *et al.* [1]). Here, a generalization of the properties of the hazard rates of the system lifetime for a special case of the FGM family is considered. In contrast to the earlier works, we observe that the hazard rate of the lifetime of the series system is increasing for $\alpha = -1$.

A brief summary of the relevant literature is given below:

Navarro and Shaked [10] studied the monotonicity of the hazard rate of the order statistics. Navarro *et al.* [7] studied the basic properties for bivariate systems with dependent exchangeable components when the failure of the other components is known. Navarro *et al.* [8] obtained the basic reliability properties for k -out-of- n systems (order statistics) and, in particular, for the series and the parallel systems when the components are exchangeable and have Gumbel's Exponential for the joint distribution. Navarro and Lai [6] studied how the dependency effects the performance of the system. They also extend some comparison results in the case of independent components to the case of two dependent components with the help of diversity properties of the exponential parameters of the two components. Navarro *et al.* [9] examined the basic reliability properties of the systems with two exchangeable Pareto components. Navarro and Spizzichino [12] obtained some results for a stochastic comparison of the coherent systems with independent or dependent component lifetimes based on the copula representation. Navarro *et al.* [5] obtained general properties of the hazard rate of mixtures of systems. Zhang [14] investigated hazard rate ordering properties of the conditional lives of the coherent system with dependent exchangeable component lifetimes. Navarro and Shaked [11] studied the monotonicity of the hazard rate of the order statistics having the joint log-concave reliability function. Zhao and Balakrishnan [15] have some results for the hazard rate ordering of the series and the parallel systems with independent and heterogeneous exponential components based on majorization of the exponential parameters of the components. Joo and Mi [2] have studied the hazard rate properties for the parallel systems with two dependent components with exponential marginals.

In this study, we obtain results analogous to those of Navarro and Shaked [10] by defining a convex transformation. In this connection, two definitions for the hazard rate and the hazard rate ordering, respectively, are given below. For the comparisons of the lifetimes, we introduce two useful lemmas which are stated below.

2. Motivations

2.1. Definition. Let T be the lifetime of a component, and $S(t)$ the survival function of the component. If the probability density function of T , $f(t)$ exists, then the hazard rate $h(t)$ of a component is $\frac{f(t)}{S(t)}$ (Lai and Xie, [4, p.9–11]).

2.2. Definition. Let X and Y be two random variables with absolutely continuous distribution functions and hazard rate functions $h_X(t)$ and $h_Y(t)$, respectively, such that $h_X(t) \geq h_Y(t)$, for all $t \geq 0$. Then X is said to be *smaller* than Y with respect to the hazard rate ordering (denoted as $X \leq_{hr} Y$) (Shaked and Shanthikumar, [13, p. 12]).

Throughout the paper, the lifetimes of parallel (series) systems will be denoted by $T_{2:2}$ ($T_{1:2}$) for systems whose components are stochastically dependent on each other. On the other hand, $T_{2:2}^0$ ($T_{1:2}^0$) will denote the lifetimes of parallel (series) systems with independent components. Also, T is the component lifetime. Accordingly, $S_{2:2}(t)$ ($S_{1:2}(t)$), $S_{2:2}^0(t)$ ($S_{1:2}^0(t)$) and $S(t)$ denote survival functions for the system and component lifetimes, respectively.

Now, assume that the lifetimes of the components are continuous random variables and identically distributed. Then a transformation can be defined as follows:

$$\psi_{2:2(1:2)}(u) = S_{2:2(1:2)} S^{-1}(u) : [0, 1] \rightarrow [0, 1].$$

Here, we are going to obtain equivalent conditions for the hazard rate ordering (Shaked and Shanthikumar, [13, Chapter I; p. 13, 28]). Some of the properties of the transformation are given below.

- P1) $\psi_{1:2}(u)$ and $\psi_{2:2}(u)$ are both increasing in u with $\psi_{1:2}(0) = \psi_{2:2}(0) = 0$ and $\psi_{1:2}(1) = \psi_{2:2}(1) = 1$.
- P2) $\psi_{1:2}^0(u) = S_{1:2}^0 S^{-1}(u) = u^2$ and $\psi_{2:2}^0(u) = S_{2:2}^0 S^{-1}(u) = 1 - (1 - u)^2$.
- P3) $\psi_{2:2}(u) = 2u - \psi_{1:2}(u)$.
- P4) $\psi_{1:2}(u)$ lies on the downside of the diagonal line and $\psi_{2:2}(u)$ lies on the upper side of the diagonal line.
- P5) $\psi_{1:2}^0(u)$ is convex and hence $\psi_{2:2}^0(u)$ is a concave function.

2.3. Lemma. Consider a two-component system (parallel or series) whose component lifetimes are stochastically dependent and identically distributed. If $\psi_{1:2}(u)$ is convex, then $T_{1:2} \leq_{hr} T$ and $T \leq_{hr} T_{2:2}$.

Proof. The first statement is obvious from (1.C.3), (1.B.6) and Theorem 1.C.1 (Shaked and Shanthikumar, [13]). The second follows from (P3). By assuming exchangeability of the component lifetimes, the result can also be obtained from (2.2) in Navarro and Shaked [10]. \square

This ordering seems to be a reasonable because we know that the lifetime of a series system can not be longer than its components. However this is not always the case. One can see a counter example (Example 2.2) in Navarro and Shaked [10].

Navarro and Lai [6], Joo and Mi [2] give some conditions for the hazard rate ordering of the systems with two dependent exponential components under the effect of the dependency of the component lifetimes by assuming dispersivity and majorization on the exponential parameters.

2.4. Lemma. If $\frac{\psi_{2:2(1:2)}(u)}{\psi_{2:2(1:2)}^0(u)} \uparrow u$, then $T_{2:2(1:2)} \leq_{hr} T_{2:2(1:2)}^0$.

Proof. Clear from (1.B.2) and (1.B.6) in (Shaked and Shanthikumar, [13]). \square

Now, we are going to investigate the hazard rate ordering, " \leq_{hr} " between the lifetimes when the joint distribution of the component lifetimes belongs to some family of bivariate exponential distributions.

3. Hazard rate ordering for dependent series (parallel) systems

3.1. Clayton’s Bivariate Distribution. The joint survival function of the component lifetimes (T_1, T_2) is given by

$$P(T_1 > t, T_2 > t) = \left[S_1(t)^{-\theta} + S_2(t)^{-\theta} - 1 \right]^{\frac{-1}{\theta}} \quad \theta > 0,$$

(Kotz *et al.*, [3, p. 414]). As we noted earlier, if the components have the same marginals, then the survival function of the series system can be written as

$$S_{1:2}(t) = P(T_1 > t, T_2 > t) = \left[2S(t)^{-\theta} - 1 \right]^{\frac{-1}{\theta}}.$$

Therefore, the transformation defined on the survival function can be written as

$$\psi_{1:2}(u) = u \left[2 - u^\theta \right]^{\frac{-1}{\theta}}.$$

The first derivative of $\psi_{1:2}(u)$ is

$$(1) \quad \frac{d\psi_{1:2}(u)}{du} = \frac{2}{(2 - u^\theta)^{\frac{1}{\theta} + 1}},$$

which is a nondecreasing function of u . Therefore $\psi_{1:2}(u)$ is a convex function on $(0, 1)$. Hence, the ordering follows from an application of Lemma 2.3

$$(2) \quad T_{1:2} \leq_{hr} T \leq_{hr} T_{2:2}.$$

In order to make a similar comparison between $T_{1:2}^0$ and $T_{1:2}$, we will have to check the monotonicity property of the ratio $\frac{\psi_{1:2}(u)}{\psi_{1:2}^0(u)} = \frac{1}{u(2-u^\theta)^{\frac{1}{\theta}}}$. This ratio can also be written as $\frac{1}{[1-(1-u^\theta)^2]^{\frac{1}{\theta}}}$ which is a nondecreasing function. According to Lemma 2.4, we have

$T_{1:2}^0 \leq_{hr} T_{1:2}$. These two orderings will imply

$$(3) \quad T_{1:2}^0 \leq_{hr} T_{1:2} \leq_{hr} T \leq_{hr} T_{2:2}.$$

Now, it remains to check the place of $T_{2:2}^0$ in (3). According to the properties (P2) and (P5), since the function $\psi_{2:2}^0(u) = 2u - u^2$ is concave, we can conclude $T \leq_{hr} T_{2:2}^0$ from Lemma 2.3. According to Lemma 2.4, we need to check the monotonicity property of the ratio $\frac{\psi_{2:2}(u)}{\psi_{2:2}^0(u)}$. A slightly modified version of this ratio is as follows

$$(4) \quad \frac{\psi_{2:2}(u)}{\psi_{2:2}^0(u)} = \frac{2 - [2 - u^\theta]^{\frac{-1}{\theta}}}{(2 - u)}.$$

The sign of the first derivative will determine whether the ratio is increasing or decreasing. The first derivative of $\frac{\psi_{2:2}(u)}{\psi_{2:2}^0(u)}$ is obtained as

$$(5) \quad \frac{d}{du} \left[\frac{\left(2 - [2 - u^\theta]^{\frac{-1}{\theta}} \right)}{(2 - u)} \right] = \frac{2 \left[1 - [2 - u^\theta]^{\frac{-1}{\theta} - 1} (u^{\theta-1} + 1 - u^\theta) \right]}{(2 - u)^2}.$$

Consider the numerator in (5). Since $[2 - u^\theta]^{\frac{-1}{\theta} - 1} \leq 1$ for all $u \in [0, 1]$, and $u^{\theta-1} + 1 - u^\theta \leq 2 - u^\theta$ for $\theta \geq 1$, then

$$1 - [2 - u^\theta]^{\frac{-1}{\theta} - 1} (u^{\theta-1} + 1 - u^\theta) \geq 1 - [2 - u^\theta]^{\frac{-1}{\theta}} \geq 0.$$

Therefore, $\frac{\psi_{2:2}(u)}{\psi_{2:2}^0(u)}$ is nondecreasing for $\theta \geq 1$. By using Lemma 2.4, we have

$$(6) \quad T_{2:2} \leq_{hr} T_{2:2}^0.$$

The combination of (6) and (3) implies that the following ordering holds for the Clayton Distribution Family for any marginal with $\theta \geq 1$,

$$T \leq_{hr} T_{2:2} \leq_{hr} T_{2:2}^0.$$

3.2. Cowan’s Bivariate Exponentials. The joint distribution function of the component lifetimes T_1 and T_2 , distributed as Cowan’s Bivariate Exponential, is given by

$$P(T_1 \leq t_1, T_2 \leq t_2) = 1 - e^{-t_1} - e^{-t_2} + e^{\left\{ \frac{-1}{2} (t_1 + t_2 + \sqrt{t_1^2 + t_2^2 - 2t_1 t_2 \cos \alpha}) \right\}},$$

with $t_1, t_2 > 0$ and $0 \leq \alpha \leq \pi$ (Kotz *et al.* [3, p. 385]). The survival function of the series system whose component lifetimes have an exponential distribution with $\lambda = 1$ can be given by

$$S_{1:2}(t) = e^{\left\{ \frac{-1}{2} (2t + \sqrt{2t^2 - 2t^2 \cos \alpha}) \right\}} = e^{-t \left(1 + \frac{\sqrt{2}}{2} \sqrt{1 - \cos \alpha} \right)} = e^{-tc_\alpha},$$

where $c_\alpha = \left(1 + \frac{\sqrt{2}}{2} \sqrt{1 - \cos \alpha} \right)$. Then the hazard rate of the series system can be written as

$$h_{1:2}(t) = c_\alpha.$$

It is clear that the lifetime of a series system has the Constant Hazard Rate (CHR) property. On the other hand, since $h(t) = 1$ and $h_{1:2}^0(t) = 2$, and $1 \leq c_\alpha \leq 2$, we can write

$$T_{1:2}^0 \leq_{hr} T_{1:2} \leq_{hr} T.$$

Up to this point, we have considered systems serially connected to each other, and have obtained an ordering according to their hazard rates. Now, we will consider systems connected to each other in parallel. The survival function of the lifetime of a parallel system is given by

$$S_{2:2}(t) = 2e^{-t} - e^{-tc_\alpha},$$

and the corresponding transformation $\psi_{2:2}(u)$ is as follows

$$\psi_{2:2}(u) = 2u - u^{c_\alpha}.$$

As can be easily seen, since $c_\alpha > 1$, $\psi_{2:2}(u)$ is concave, which implies

$$(7) \quad T \leq_{hr} T_{2:2}.$$

In order to decide the place of $T_{2:2}^0$, we consider the ratio $\frac{\psi_{2:2}(u)}{\psi_{2:2}^0(u)}$. We have to investigate three different cases according to c_α .

(i) If $c_0 = 1$, then $\frac{\psi_{2:2}(u)}{\psi_{2:2}^0(u)} = \frac{1}{2-u}$ is a nondecreasing function of u and therefore, $T_{2:2} \leq_{hr} T_{2:2}^0$.

(ii) If $c_\pi = 2$, then the component lifetimes are independent, which is a trivial case.

(iii) If $1 < c_\alpha < 2$, then

$$\frac{\psi_{2:2}(u)}{\psi_{2:2}^0(u)} = \frac{2u - u^{c_\alpha}}{2u - u^2} \text{ with } \frac{\psi_{2:2}(0)}{\psi_{2:2}^0(0)} = 1 \text{ and } \frac{\psi_{2:2}(1)}{\psi_{2:2}^0(1)} = 1$$

imply that this ratio cannot be monotonic from the Rolle’s Theorem. Thus, $T_{2:2}$ and $T_{2:2}^0$ cannot be compared in the sense of hazard rate ordering. Conversely, since $T \leq_{hr} T_{2:2}^0$ then from (7) and (i) we have

$$T \leq_{hr} T_{2:2} \leq_{hr} T_{2:2}^0 \text{ for } c_\alpha = 1.$$

3.3. Gumbel’s Bivariate Exponentials. The joint distribution function of the component lifetimes T_1 and T_2 , distributed as Gumbel’s Bivariate Exponential, is given by

$$P(T_1 \leq t_1, T_2 \leq t_2) = 1 - e^{-t_1} - e^{-t_2} + e^{-(t_1+t_2+\theta t_1 t_2)},$$

$0 \leq \theta \leq 1$ (Kotz *et al.*, [3, p.350]). Here, the lifetimes of the components have an exponential distribution with $\lambda = 1$. Then the survival function of the series system $S_{1:2}(t)$ and its hazard rate can be written as

$$S_{1:2}(t) = e^{-(2t+\theta t^2)}, \quad h_{1:2}(t) = 2(1 + \theta t),$$

respectively. Since $h_{1:2}(t) \geq h_{1:2}^0(t) = 2$ and $h_{1:2}^0(t) \geq h(t) = 1$, we have

$$T_{1:2} \leq_{hr} T_{1:2}^0 \leq_{hr} T.$$

A similar result for the generalized Gumbel’s Exponential distribution can be found in Navarro *et al.* [8] (see Proposition 19.5.1 (3)). In Proposition 19.5.1 (3), if we take $b = 0$, the same ordering we found above is valid.

According to the property (P3), the survival function of the parallel system is $S_{2:2}(t) = 2e^{-t} - e^{-(2t+\theta t^2)}$, then the hazard rate of the system will be

$$h_{2:2}(t) = 1 - \frac{(1 + 2\theta t)e^{-(t+\theta t^2)}}{2 - e^{-(t+\theta t^2)}}.$$

Since $h_{2:2}(t) \leq h(t)$, then $T \leq_{hr} T_{2:2}$. The same problem arises as in the Clayton Family when trying to arrange the lifetimes $T_{2:2}^0$ and $T_{2:2}$. To overcome this problem, the following ratio is reconsidered,

$$(8) \quad \frac{S_{2:2}(t)}{S_{2:2}^0(t)} = \frac{2 - e^{-(t+\theta t^2)}}{2 - e^{-t}}.$$

The problem is cleared up by checking the sign of the first derivative of (8). The numerator of the first derivative of (8) can be arranged as

$$(9) \quad \varphi_\theta(t) = (1 + 2\theta t) e^{-(t+\theta t^2)} (2 - e^{-t}) - e^{-t} (2 - e^{-(t+\theta t^2)}).$$

Note that $(2 - e^{-t}) \leq 2 - e^{-(t+\theta t^2)}$ for all $t > 0$ and $\theta \in [0, 1]$. Conversely, if $(1 + 2\theta t) e^{-\theta t^2} \leq 1$, then $\varphi_\theta(t)$ is negative. Accordingly, from the first four terms of Maclaurin’s series expansion of $e^{\theta t^2}$, we have $e^{\theta t^2} \geq 1 + \theta t^2$. On the other hand, for $t \geq 2$, we also have $1 + \theta t^2 \geq 1 + 2\theta t$. Hence, the inequality $(1 + 2\theta t) \leq e^{\theta t^2}$ is valid for $t \geq 2$. Since $\varphi_\theta(t) \leq 0$, the ratio in (8) is decreasing. According to Definition 2.2, $T_{2:2}$ and $T_{2:2}^0$ cannot be compared for all t . That is, we can only talk about orderings amongst T , $T_{2:2}$ and $T_{2:2}^0$ as $T \leq_{hr} T_{2:2}$ and $T \leq_{hr} T_{2:2}^0$.

3.4. Gumbel’s Bivariate Exponentials: Model III. Consider the joint distribution of the lifetimes of the components T_1 and T_2 for $m \geq 1$ (Kotz *et al.* [3, p. 355]) given by

$$P(T_1 \leq t_1, T_2 \leq t_2) = 1 - e^{-t_1} - e^{-t_2} + e^{\left\{ -\left(t_1^m + t_2^m \right)^{\frac{1}{m}} \right\}}.$$

Since the lifetimes of the components are distributed exponentially with $\lambda = 1$, the survival function of the series system is defined as $S_{1:2}(t) = e^{-2\frac{1}{m}t}$, and $h_{1:2}(t) = 2\frac{1}{m}$ is obtained. It can be easily seen that $h_{1:2}(t) \geq h(t) = 1$ and this yields $T_{1:2} \leq_{hr} T$. Since $h_{1:2}^0(t) = 2$, then the inequality $h_{1:2}^0(t) \geq h_{1:2}(t)$ is obviously valid, and hence $T_{1:2}^0 \leq_{hr} T_{1:2}$ holds. Therefore, the ordering

$$T_{1:2}^0 \leq_{hr} T_{1:2} \leq_{hr} T$$

is valid. From the joint distribution, the survival function of the parallel system is given as $S_{2:2}(t) = 2e^{-t} - e^{-ct}$, where $c = 2^{\frac{1}{m}}$. Therefore, the transformation $\psi_{2:2}(u) = 2u - u^c$ can be defined. Since $c > 1$, then $\psi_{2:2}(u)$ is concave. From Lemma 2.3, there exists a relation between T and $T_{2:2}$, namely $T \leq_{hr} T_{2:2}$. Next, in order to compare $T_{2:2}$ and $T_{2:2}^0$, consider the ratio $\frac{\psi_{2:2}(u)}{\psi_{2:2}^0(u)} = \frac{2-u^{c-1}}{2-u}$. This ratio has the same end points, i.e. $\frac{\psi_{2:2}(0)}{\psi_{2:2}^0(0)} = \frac{\psi_{2:2}(1)}{\psi_{2:2}^0(1)} = 1$, which implies that the ratio is not monotonic. Therefore, the lifetimes $T_{2:2}^0$ and $T_{2:2}$ cannot be compared in the sense of the hazard rate ordering.

3.5. Farlie Gumbel Morgenstern Bivariate Distribution. The *FGM* distribution function of the component lifetimes T_1 and T_2 for which their marginals are identically distributed as an exponential with mean $\frac{1}{\lambda}$ is given below:

$$P(T_1 \leq t_1, T_2 \leq t_2) = (1 - e^{-\lambda t_1})(1 - e^{-\lambda t_2}) \left[1 + \alpha(e^{-\lambda(t_1+t_2)}) \right],$$

$\alpha \in [-1, 1]$ (Lai and Xie, [4]). Consider a series system having the survival function $S_{1:2}(t)$, where

$$S_{1:2}(t) = e^{-2\lambda t} \left[1 + \alpha(1 - e^{-\lambda t})^2 \right].$$

Based on this survival function the transformation $\psi_{1:2}(u)$ is defined as

$$\psi_{1:2}(u) = u^2 \left[1 + \alpha(1 - u)^2 \right].$$

In order to check the convexity of $\psi_{1:2}(u)$, we look at whether the second derivative is positive or not. The second derivative of $\psi_{1:2}(u)$ is obtained as

$$\begin{aligned} \frac{d^2\psi_{1:2}(u)}{du^2} &= 2 + 2\alpha - 12\alpha u + 12\alpha u^2 \\ (10) \qquad \qquad &= 12\alpha \left(\frac{1}{2} - u \right)^2 + 2 - \alpha. \end{aligned}$$

For $\alpha \geq 0$, it can be seen from (10) that $\frac{d^2\psi_{1:2}(u)}{du^2}$ is positive. If $\alpha < 0$, then

$$12\alpha \left(\frac{1}{2} - u \right)^2 + 2 - \alpha \geq 2(1 + \alpha) \geq 0$$

can be written because $\left(\frac{1}{2} - u\right)^2 \leq \frac{1}{4}$. Therefore the function $\psi_{1:2}(u)$ is convex for $\alpha \in [-1, 1]$. According to Lemma 2.3, using the convexity of $\psi_{1:2}(u)$, we can write

$$T_{1:2} \leq_{hr} T.$$

Now, we are going to investigate ordering relations of $T_{1:2}^0$ with $T_{1:2}$ and T . Since $\psi_{1:2}^0(u) = u^2$ is convex, it is obvious from (P5) and Lemma 2.3 that $T_{1:2}^0 \leq_{hr} T$. In order to compare $T_{1:2}^0$ with $T_{1:2}$, we consider the ratio

$$\frac{\psi_{1:2}(u)}{\psi_{1:2}^0(u)} = \left[1 + \alpha(1 - u)^2 \right].$$

This ratio is an increasing function of u when $\alpha < 0$, and a decreasing function for $\alpha > 0$. Therefore, we have to consider two different cases, namely

$$(11) \quad \begin{aligned} T_{1:2} &\leq_{hr} T_{1:2}^0 \leq_{hr} T \text{ for } \alpha < 0 \\ T_{1:2}^0 &\leq_{hr} T_{1:2} \leq_{hr} T \text{ for } \alpha > 0. \end{aligned}$$

Consider a parallel system with survival function

$$S_{2:2}(t) = 2e^{-\lambda t} - e^{-2\lambda t} \left[1 + \alpha \left(1 - e^{-\lambda t} \right)^2 \right],$$

and corresponding transformation

$$\begin{aligned} \psi_{2:2}(u) &= 2u - u^2 [1 + \alpha(1 - u)^2] \\ &= 2u - \psi_{1:2}(u). \end{aligned}$$

Since the transformation $\psi_{1:2}(u)$ is convex, then $\psi_{2:2}(u)$ is concave, which implies from Lemma 2.3 that

$$T \leq_{hr} T_{2:2}$$

is valid for $\alpha \in [-1, 1]$. According to (P5), $T_{2:2}^0$ is placed on the right hand side of T . But we still have to compare $T_{2:2}$ and $T_{2:2}^0$. In order to compare these lifetimes, we consider the ratio as

$$\frac{\psi_{2:2}(u)}{\psi_{2:2}^0(u)} = 1 - \alpha \frac{u(1-u)^2}{2-u}.$$

Here, it can easily be seen that $\frac{\psi_{2:2}(0)}{\psi_{2:2}^0(0)} = 1$ and $\frac{\psi_{2:2}(1)}{\psi_{2:2}^0(1)} = 1$. Therefore, $T_{2:2}$ and $T_{2:2}^0$ cannot be compared, as in the Gumbel III model.

4. Monotonicity of a system with dependent components

From now on, for the families under discussion, we are going to look at whether the hazard rate of the system will behave like its components or not.

4.1. Clayton’s Bivariate Distribution. Here a general characterization will be given for this family. Afterwards, we are going to consider systems with exponentially distributed component lifetimes. The hazard rate function of the series system $h_{1:2}(t)$ can be rewritten as

$$h_{1:2}(t) = h(t) \left[\frac{2}{2 - S(t)^\theta} \right].$$

Since the ratio in brackets, i.e. $\frac{2}{2 - S(t)^\theta}$, is a decreasing function of t , $h_{1:2}(t)$ is also decreasing by assuming that the hazard rate of the component $h(t)$ is decreasing. Therefore, the lifetime of the series system has the DHR property when the component lifetimes have the DHR property. If the lifetimes of the components have an exponential distribution with parameter λ then $h(t) = \lambda$ but the system lifetime still has the DHR property as in the general case. For parallel systems, the hazard rate function of the parallel system can be written as

$$(12) \quad h_{2:2}(t) = h(t) \left[1 - \frac{\frac{S^\theta(t)}{2 - S^\theta(t)}}{2 [2 - S^\theta(t)]^{\frac{1}{\theta}} - 1} \right]$$

after some rearrangement. The term $\frac{S^\theta(t)}{2 - S^\theta(t)}$ in the numerator decreases with t . Since $2 - S^\theta(t) \geq 1$ is valid, then $2 [2 - S^\theta(t)]^{\frac{1}{\theta}} - 1$ is a positive and increasing function of t . Therefore the ratio $\frac{\frac{S^\theta(t)}{2 - S^\theta(t)}}{2 [2 - S^\theta(t)]^{\frac{1}{\theta}} - 1}$ decreases with t . In conclusion, while $h(t)$ is nondecreasing, $h_{2:2}(t)$ is increasing. Now, we can say that if the components are IHR, then the parallel system also preserves the IHR property. As a result, according to (12), we can also say that if the lifetimes of the components have exponential distribution with parameter λ , then the parallel system has the IHR property.

4.2. Cowan’s Bivariate Exponentials. For the series system, we know that $h_{1:2}(t) = c_\alpha$ and monotonicity is preserved in the sense of CHR. Now, consider a parallel system with the hazard rate function, after some rearrangement, in the form

$$h_{2:2}(t) = c_\alpha - \frac{2(c_\alpha - 1)}{2 - e^{-t(c_\alpha - 1)}}.$$

Since $c_\alpha - 1 > 0$, then $2 - e^{-t(c_\alpha - 1)}$ is nondecreasing in t , and therefore $h_{2:2}(t)$ is nondecreasing in t . In this case, the lifetimes of the components have the CHR property but the system has IHR property. For the case $c_\alpha = 1$, the hazard rate function $h_{2:2}(t)$ is constant and therefore the monotonicity is preserved towards the CHR.

4.3. Gumbel’s Bivariate Exponentials. For the series system, since $h_{1:2}(t) = 2(1 + \theta t)$ the system has the IHR property, which implies that the monotonicity of the hazard rate cannot hold. Now, let us examine this for the parallel system: The hazard function can be written using $A_\theta(t) = t + \theta t^2$ as

$$h_{2:2}(t) = 1 - \frac{A'_\theta(t)e^{-A_\theta(t)}}{2 - e^{-A_\theta(t)}}.$$

Now, we are going to look at the first derivative of $h_{2:2}(t)$ to investigate whether it decreases or not. The first derivative of $h_{2:2}(t)$ is given by

$$\frac{dh_{2:2}(t)}{dt} = \frac{2e^{-A_\theta(t)}}{(2 - e^{-A_\theta(t)})^2} [\theta e^{-A_\theta(t)} - 2\theta + A'_\theta(t)^2].$$

Since $e^{-A_\theta(t)} \geq 1 - A_\theta(t)$, then

$$\theta e^{-A_\theta(t)} - 2\theta + A'_\theta(t)^2 \geq -\theta A_\theta(t) + A'_\theta(t)^2 - \theta$$

is valid. The right hand side of this inequality is equivalent to a quadratic form in t which is given by

$$(13) \quad 3\theta^2 t^2 + 3\theta t + 1 - \theta.$$

The roots of this quadratic form are

$$t_1 = \frac{-\frac{1}{2} + \frac{1}{6}\sqrt{12\theta - 3}}{\theta}, \quad t_2 = \frac{-\frac{1}{2} - \frac{1}{6}\sqrt{12\theta - 3}}{\theta}.$$

For $\theta < \frac{1}{4}$, both t_1 and t_2 are not real and therefore $3\theta^2 t^2 + 3\theta t + 1 - \theta \geq \frac{1}{4} - \theta > 0$. For $\theta \geq \frac{1}{4}$, we have $t_2 \leq 0$ and since $\sqrt{12\theta - 3} \leq 3$, t_1 is negative. Thus the quadratic form in (13) has no positive root. The quadratic form is both convex and nondecreasing. Moreover its value is $1 - \theta$ at the point 0. Therefore, the quadratic form in (13) cannot be negative for $t \geq 0$. Thus, $\frac{dh_{2:2}(t)}{dt} \geq 0$ for $\theta \in [0, 1]$. Thence $h_{2:2}(t)$ is nondecreasing in t . Namely, system has the IHR property. A similar result has been obtained in Navarro *et al.* [8, Proposition 19.5.1 (4)].

4.4. Gumbel’s Bivariate Exponentials: Model III. For the series system we found that $h_{1:2}(t) = 2^{\frac{1}{m}}$ and therefore the system preserves monotonicity itself as CHR. For the parallel system, consider the hazard rate function as

$$h_{2:2}(t) = 1 - \frac{(c - 1)e^{-(c-1)t}}{2 - e^{-(c-1)t}}.$$

It is obvious that $(c - 1)e^{-(c-1)t}$ and $2 - e^{-(c-1)t}$ are both positive and $(c - 1)e^{-(c-1)t}$ is decreasing, while $2 - e^{-(c-1)t}$ is increasing. Therefore the ratio is nonincreasing. As a result, $h_{2:2}(t)$ is nondecreasing and hence the system has the IHR property.

4.5. Farlie Gumbel Morgenstern Bivariate Distribution. Here we are first going to investigate a special case of the lifetimes of the series system. A general result for the series system has been investigated by Gupta *et al.* [1]. They showed that the shape of the hazard function of the system is like a bathtub. However, we find that the shape of the hazard function does not seem to be a bathtub for $\alpha = -1$. In order to check the monotonicity property of the series system, we write the hazard function as

$$(14) \quad h_{1:2}(t) = 2\lambda e^{-\lambda t} \left[\frac{1}{e^{-\lambda t}} - \alpha \frac{(1 - e^{-\lambda t})}{1 + \alpha(1 - e^{-\lambda t})^2} \right].$$

The first derivative of $h_{1:2}(t)$ may be helpful in deciding the monotonicity of $h_{1:2}(t)$ and is given by

$$(15) \quad h'_{1:2}(t) = \frac{\alpha 2\lambda^2 e^{-\lambda t} [\alpha(1 - e^{-\lambda t})^2 + 2(1 - e^{-\lambda t}) - 1]}{[1 + \alpha(1 - e^{-\lambda t})^2]^2}.$$

The first derivative is positive for $\alpha = -1$ because

$$h'_{1:2}(t)|_{\alpha=-1} = \frac{2\lambda^2 e^{-\lambda t}}{[2 - e^{-\lambda t}]^2} \geq 0.$$

Therefore, $h_{1:2}(t)$ is increasing in t . In other words, the lifetime of the series system has the IHR property for $\alpha = -1$. The same result has been obtained by Joo and Mi [2, Theorem 3.3].

In order to investigate the monotonicity of the hazard function for the parallel system, we look at the sign of the first derivative of $h_{2:2}(t)$, which is given by

$$h'_{2:2}(t) = \frac{2\lambda^2 e^{-\lambda t}}{(-2 + e^{-\lambda t} + \alpha e^{-\lambda t} - 2\alpha e^{-2\lambda t} + \alpha e^{-3\lambda t})^2} \times [1 + \alpha + 2\alpha e^{-\lambda t} (5e^{-\lambda t} - e^{-2\lambda t} - 4) + \alpha^2 e^{-2\lambda t} (1 - 2e^{-\lambda t} + e^{-2\lambda t})].$$

Let $u = e^{-\lambda t}$ and

$$(16) \quad \begin{aligned} m(u) &= 1 + \alpha - 2\alpha u (u^2 - 5u + 4) + \alpha^2 u^2 (1 - 2u + u^2) \\ &= 1 + \alpha [1 + u(1 - u)(2u + \alpha u(1 - u) - 8)]. \end{aligned}$$

It is obvious that $2u + \alpha u(1 - u) - 8 \geq -8$ for $\alpha > 0$, and $-8u(1 - u) \geq -2$. Both inequalities imply that $m(u) \geq 1 - \alpha$. That is, $m(u)$ cannot be negative for $0 < \alpha \leq 1$ and thus $h_{2:2}(t)$ is nondecreasing. On the other hand, if $\alpha \leq 0$, then $2u + \alpha u(1 - u) - 8 \leq -6$. Moreover $1 - 6u(1 - u) \leq 1$, so these two inequalities together imply that $m(u) \geq 1 + \alpha$. That is, $m(u)$ cannot be negative for $-1 \leq \alpha \leq 0$ and thus $h_{2:2}(t)$ is nondecreasing. These two results imply that the lifetime of the parallel system has the IHR property for $\alpha \in [-1, 1]$. Joo and Mi [2] obtained the same results in a different way (see Theorem 3.5).

5. Conclusion

In this paper, an ordering of the lifetime of a system (series or parallel) has been observed according to their hazard rates. The results are summarized in Table 1 below. This table includes the possible orderings.

Table 1. Ordering Relation of the Lifetimes

Clayton's Bivariate Distribution (CBD)		
$\theta > 0$ $T_{1:2}^0 \leq_{hr} T_{1:2} \leq_{hr} T \leq_{hr} T_{2:2}$	$\theta > 1$ $T_{2:2} \leq_{hr} T_{2:2}^0$	
Cowan's Bivariate Exponentials (CWBED)		
$1 \leq c_\alpha \leq 2$ $T_{1:2}^0 \leq_{hr} T_{1:2} \leq_{hr} T$	$c_\alpha > 1$ $T \leq_{hr} T_{2:2}$	$c_\alpha = 1$ $T_{2:2} \leq_{hr} T_{2:2}^0$
Gumbel's Bivariate Exponentials (GBED)		
$\theta \in [0, 1]$ $T_{1:2} \leq_{hr} T_{1:2}^0 \leq_{hr} T \leq_{hr} T_{2:2}$		
Gumbel's Bivariate Exponentials III (GBED III)		
$m \geq 1$ $T_{1:2} \leq_{hr} T_{1:2}^0 \leq_{hr} T \leq_{hr} T_{2:2}$		
Farlie Gumbel Morgenstern Bivariate Distribution (FGMED)		
$\alpha < 0$ $T_{1:2} \leq_{hr} T_{1:2}^0 \leq_{hr} T$	$\alpha > 0$ $T_{1:2}^0 \leq_{hr} T_{1:2} \leq_{hr} T$	$\alpha \in [-1, 1]$ $T \leq_{hr} T_{2:2}$

The monotonicity properties of the hazard functions of the lifetimes are given in Table 2.

Table 2. Monotonicity Properties of the Systems

	Component	Series System	Parallel System
CBD	DHR	DHR*	- - -
	IHR	- - -	IHR*
CBED	CHR	- - -	IHR
CWBED	CHR	CHR*	IHR
GBED	CHR	IHR	IHR
GBED III	CHR	CHR*	IHR
FGMED	CHR	IHR ($\alpha = -1$)	IHR

Here, CBED denotes Clayton's Bivariate Exponential Distribution and the other abbreviations are as in Table 1. (*) indicates which systems preserve the monotonicity.

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