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SURFACES IN THE EUCLIDEAN SPACE $\mathbb{E}^4$ WITH POINTWISE 1-TYPE GAUSS MAP

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Abstract

In this article we study surfaces in Euclidean space $\mathbb{E}^4$ with pointwise 1-type Gauss map. We give a characterization of surfaces in $\mathbb{E}^4$ with a pointwise 1-type Gauss map of the first kind. We conclude that an oriented non-minimal surface M in $\mathbb{E}^4$ has a pointwise 1-type Gauss map of the first kind if and only if M is a surface in a 3-sphere of $\mathbb{E}^4$ with constant mean curvature. We also obtain a characterization for non-planar minimal surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map of the second kind. Further we give a partial classification of surfaces in $\mathbb{E}^4$ in terms of the pointwise 1-type Gauss map of the second kind.

Keywords: Minimal surface, Normal bundle, Mean curvature, Pointwise 1-type, Gauss map.

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1. Introduction

A submanifold M of a Euclidean space E^m is said to be of *finite type* if its position vector x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $x = x_0 + x_1 + \cdots + x_k$, where x_0 is a constant map, $x_1, \ldots, x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$.

If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all different, then M is said to be of k-type (cf. [7, 8]). In [9], this definition was similarly extended to differentiable maps, in particular, to Gauss maps of submanifolds.

The notion of a finite type Gauss map is especially a useful tool in the study of submanifolds (cf. [2, 3, 4, 5, 9, 16]). In [9], Chen and Piccinni made a general study on

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compact submanifolds of Euclidean spaces with finite type Gauss map, and for hypersurfaces they proved that a compact hypersurface M of E^{n+1} has 1-type Gauss map if and only if M is a hypersphere in E^{n+1}.

If a submanifold M of a Euclidean space has 1-type Gauss map ν, then $\Delta\nu = \lambda(\nu + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C. However, the Laplacian of the Gauss map of several surfaces such as the helicoid, catenoid and right cones in $\mathbb{E}^3$, and also some hypersurfaces take the form

$$\Delta\nu = f(\nu + C) \tag{1.1}$$

for some smooth function f on M and some constant vector C. A submanifold of a Euclidean space is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1.1) for some smooth function f on M and some constant vector C. A submanifold with pointwise 1-type Gauss map is said to be of the *first kind* if the vector C in (1.1) is the zero vector. Otherwise, a submanifold with pointwise 1-type Gauss map is said to be of the *second kind*.

Surfaces in Euclidean spaces and in pseudo-Euclidean spaces with pointwise 1-type Gauss map were recently studied in [1, 10, 11, 13, 14, 15, 17]. Also, hypersurfaces of the Euclidean space E^{n+1} with pointwise 1-type Gauss map were studied in [12].

In this paper we give a characterization of a surface in $\mathbb{E}^4$ with pointwise 1-type Gauss map of the first kind in terms of M being minimal or non-minimal. We conclude that an oriented non-minimal surface in $\mathbb{E}^4$ has pointwise 1-type Gauss map of the first kind if and only if M is a surface in a 3-sphere of $\mathbb{E}^4$ with constant mean curvature.

On the other hand we give a characterization for non-planar minimal surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map of the second kind. Further, for an oriented surface M in $\mathbb{E}^4$ with non-parallel mean curvature direction, non-zero constant mean curvature, and $\dim(N_1(M)) = 1$ we prove that M has pointwise 1-type Gauss map of the second kind if and only if M is an open portion of a helical cylinder in $\mathbb{E}^4$, where $N_1(M)$ is the first normal space of M in $\mathbb{E}^4$.

2. Preliminaries

Let M be an oriented n-dimensional submanifold in an $(n+2)$-dimensional Euclidean space $\mathbb{E}^{n+2}$. We choose an oriented local orthonormal frame $\{e_1, \ldots, e_{n+2}\}$ on M such that $e_1, \ldots, e_n$ are tangent to M and e_{n+1}, e_{n+2} are normal to M. We use the following convention on the range of indices: $1 \le i, j, k, \ldots \le n$, $n+1 \le r, s, t, \ldots \le n+2$.

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\mathbb{E}^{n+2}$ and ∇ the induced connection on M. Denote by $\{\omega^1, \ldots, \omega^{n+2}\}$ the dual frame and by $\{\omega_B^A\}$, $A, B = 1, \ldots, n+2$, the connection forms associated to $\{e_1, \ldots, e_{n+2}\}$. Then we have

$$\begin{aligned}
&\widetilde{\nabla}_{e_k} e_i = \sum_{j=1}^{n} \omega_i^j(e_k) e_j + \sum_{r=n+1}^{n+2} h_{ik}^r e_r,\\
&\widetilde{\nabla}_{e_k} e_s = -A_r(e_k) + \sum_{r=n+1}^{n+2} \omega_s^r(e_k) e_r, \text{ and}\\
&D_{e_k} e_s = \sum_{r=n+1}^{n+2} \omega_s^r(e_k) e_r,
\end{aligned}$$

where D is the normal connection, h_{ij}^r the coefficients of the second fundamental form h, and A_r the Weingarten map in the direction e_r.

The mean curvature vector H and the squared length $\|h\|^2$ of the second fundamental form h are defined, respectively, by

$$H = \frac{1}{n}\sum_{r,i} h^r_{ii} e_r \tag{2.1}$$

and

$$\|h\|^2 = \sum_{r,i,j} h^r_{ij} h^r_{ji}. \tag{2.2}$$

The Codazzi equation of M in E^{n+2} is given by

$$\begin{aligned} &h^r_{ij,k} = h^r_{jk,i},\\ &h^r_{jk,i} = e_i(h^r_{jk}) + \sum_{s=n+1}^{n+2} h^s_{jk}\omega^r_s(e_i) - \sum_{\ell=1}^{n}\left(\omega^\ell_j(e_i)h^r_{\ell k} + \omega^\ell_k(e_i)h^r_{j\ell}\right). \end{aligned} \tag{2.3}$$

Also, from the Ricci equation of M in E^{n+2}, we have

$$R^D(e_j, e_k; e_r, e_s) = \langle [A_{e_r}, A_{e_s}](e_j), e_k\rangle = \sum_{i=1}^{n}\left(h^r_{ik}h^s_{ij} - h^r_{ij}h^s_{ik}\right), \tag{2.4}$$

where R^D is the normal curvature tensor.

The first normal space $N_1(M)$ of M at each point $p \in M$ in $\mathbb{E}^{n+2}$ is defined as the orthogonal complement of the space $\{\xi \in T^\perp_p M \mid A_\xi = 0\}$ in the normal space $T^\perp_p M$.

Let $G(m-n, m)$ denote the Grassmannian manifold consisting of all oriented $(m-n)$-planes through the origin of $\mathbb{E}^m$. Let M be an oriented n-dimensional submanifold of a Euclidean space $\mathbb{E}^m$. The Gauss map $\nu : M \to G(m-n, m)$ of M is a smooth map which carries a point $p \in M$ into the oriented $(m-n)$-plane through the origin of E^m obtained by the parallel translation of the normal space of M at p in E^m.

Since $G(m-n, m)$ is canonically embedded in $\bigwedge^{m-n}\mathbb{E}^m = \mathbb{E}^N$, $N = \binom{m}{m-n}$, the notion of the type of the Gauss map is naturally defined. If $\{e_{n+1}, e_{n+2}, \ldots, e_m\}$ is an oriented orthonormal normal frame on M, then the Gauss map $\nu : M \to G(m-n, m) \subset \mathbb{E}^N$ is given by $\nu(p) = (e_{n+1} \wedge e_{n+2} \wedge \cdots \wedge e_m)(p)$.

The product of a circular helix with non-zero torsion which lies in a 3-dimensional linear subspace E^3 of the Euclidean space $\mathbb{E}^4$ and a line of $\mathbb{E}^4$ is called a 2-dimensional helical cylinder in the Euclidean space $\mathbb{E}^4$.

3. Pointwise 1-type Gauss map of the first kind

In this section we investigate surfaces in the Euclidean space $\mathbb{E}^4$ with pointwise 1-type Gauss map of the first kind. However we prove the following lemma for n-dimensional submanifolds of the Euclidean space $\mathbb{E}^{n+2}$.

3.1. Lemma. *Let M be an n-dimensional submanifold of Euclidean space $\mathbb{E}^{n+2}$. Then, the Laplacian of the Gauss map $\nu = e_{n+1} \wedge e_{n+2}$ is given by*

$$\begin{aligned} \Delta\nu = {} & \|h\|^2\nu + 2\sum_{j<k} R^D(e_j, e_k; e_{n+1}, e_{n+2}) e_j \wedge e_k\\ & + n\sum_{j=1}^{n} \omega^{n+1}_{n+2}(e_j) e_j \wedge H + \nabla(\mathrm{tr}A_{n+1}) \wedge e_{n+2} - \nabla(\mathrm{tr}A_{n+2}) \wedge e_{n+1}, \end{aligned} \tag{3.1}$$

where $\|h\|^2$ is the squared length of the second fundamental form, R^D the normal curvature tensor and $\nabla\mathrm{tr}A_r$ the gradient of $\mathrm{tr}A_r$.

Proof. By regarding $\nu = e_{n+1} \wedge e_{n+2}$ as an $\mathbb{E}^N$-valued function with $N = \binom{n+2}{2}$ on M, we have

$$e_i\nu = -A_{n+1}(e_i) \wedge e_{n+2} - e_{n+1} \wedge A_{n+2}(e_i). \tag{3.2}$$

As the Laplacian of ν is defined by

$$\Delta\nu = -\sum_{i=1}^{n} \big(e_ie_i\nu - \nabla_{e_i}e_i\nu\big),$$

then, by using (3.2) we obtain

$$\begin{aligned}\Delta\nu = \sum_{i=1}^{n} \Big\{&e_{n+1} \wedge \Big(\nabla_{e_i}(A_{n+2}(e_i)) - A_{n+2}(\nabla_{e_i}e_i) - \omega_{n+2}^{n+1}(e_i)\, A_{n+1}(e_i)\Big)\\ &+ \Big(\nabla_{e_i}(A_{n+1}(e_i)) - A_{n+1}(\nabla_{e_i}e_i) - \omega_{n+1}^{n+2}(e_i)A_{n+2}(e_i)\Big) \wedge e_{n+2}\Big\}\\ &+ \sum_{i=1}^{n} \Big\{h(A_{n+1}(e_i), e_i) \wedge e_{n+2} + e_{n+1} \wedge h(A_{n+2}(e_i), e_i)\Big\}\\ &- 2\sum_{i=1}^{n} A_{n+1}(e_i) \wedge A_{n+2}(e_i).\end{aligned} \tag{3.3}$$

By a direct calculation, it is seen that

$$\sum_{i=1}^{n} h(A_{n+1}(e_i), e_i) \wedge e_{n+2} + e_{n+1} \wedge h(A_{n+2}(e_i), e_i) = \|h\|^2\nu,$$

where $\|h\|^2 = \mathrm{tr}A_{n+1}^2 + \mathrm{tr}A_{n+2}^2$,

$$\sum_{i=1}^{n} A_{n+1}(e_i) \wedge A_{n+2}(e_i) = -\sum_{j<k} R^D(e_j, e_k; e_{n+1}, e_{n+2})e_j \wedge e_k$$

and

$$\nabla_{e_i}(A_r(e_i)) - A_r(\nabla_{e_i}e_i) - \sum_{s=n+1}^{n+2} \omega_r^s(e_i)A_s(e_i) = \sum_{j=1}^{n} h_{ij,\,i}^r\, e_j, \quad r = n+1, n+2.$$

Thus, we get

$$\begin{aligned}\Delta\nu = &\sum_{i,\,j} h_{ij,\,i}^{n+1} e_j \wedge e_{n+2} + \sum_{i,\,j} h_{ij,\,i}^{n+2} e_{n+1} \wedge e_j + \|h\|^2\nu\\ &+ 2\sum_{j<k} R^D(e_j, e_k; e_{n+1}, e_{n+2})e_j \wedge e_k.\end{aligned} \tag{3.4}$$

Using the Codazzi equation (2.3) we have

$$\begin{aligned}\sum_{i=1}^{n} h_{ij,\,i}^r = \sum_{i=1}^{n} h_{ii,\,j} &= \sum_{i=1}^{n} \Big\{e_j(h_{ii}^r) + \sum_{s=n+1}^{n+2} h_{ii}^s\omega_s^r(e_j) - 2\sum_{\ell=1}^{n} \omega_i^\ell(e_j)h_{\ell i}^r\Big\}\\ &= e_j\Big(\sum_{i=1}^{n} h_{ii}^r\Big) + \sum_{s=n+1}^{n+2} \omega_s^r(e_j)\sum_{i=1}^{n} h_{ii}^s - 2\sum_{i<\ell} \Big(\omega_i^\ell(e_j) + \omega_\ell^i(e_j)\Big)h_{\ell i}^r\\ &= e_j\big(\mathrm{tr}A_r\big) + \sum_{s=n+1}^{n+2} \omega_s^r(e_j)\mathrm{tr}A_s\end{aligned} \tag{3.5}$$

for $r = n+1, n+2$. Since $\nabla(\mathrm{tr}A_r) = \sum_{j=1}^{n} e_j(\mathrm{tr}A_r)e_j$, then substituting (3.5) into (3.4) for $r = n+1$ and $r = n+2$ we obtain (3.1). □

Now we give a characterization of a surface in $\mathbb{E}^4$ with pointwise 1-type Gauss map of the first kind according to M being minimal or non-minimal.

3.2. Theorem. *An oriented non-minimal surface M in the Euclidean space $\mathbb{E}^4$ has a pointwise 1-type Gauss map of the first kind if and only if M has parallel mean curvature vector in $\mathbb{E}^4$.*

Proof. Since M is non-minimal, i.e., the mean curvature $\alpha \neq 0$, then we can choose a local orthonormal normal frame $\{e_3, e_4\}$ such that $e_3 = H/\alpha$, which implies that $\mathrm{tr}A_3 = 2\alpha$ and $\mathrm{tr}A_4 = 0$.

Suppose that M has pointwise 1-type Gauss map of the first kind in $\mathbb{E}^4$. From (1.1) and (3.1) we have

$$\|h\|^2\nu + 2R^D e_1 \wedge e_2 + 2\sum_{j=1}^{2}\omega_4^3(e_j)e_j \wedge H + 2\nabla\alpha \wedge e_4 = f\nu$$

for some differentiable function f on M, where $R^D = R^D(e_1, e_2; e_3, e_4)$ is the normal curvature of M. Hence, we get $R^D = 0$, $\omega_4^3 = 0$ and α is a non-zero constant. Therefore, the normal bundle is flat and the vector e_3 is parallel, i.e., the mean curvature vector $H = \alpha e_3$ is parallel.

Conversely, assume that M has parallel mean curvature vector H in $\mathbb{E}^4$. Then, α is a non-zero constant and $e_3 = H/\alpha$ is parallel in the normal bundle, i.e., $\omega_4^3 = 0$. Since the codimension is two, then the normal vector e_4 is parallel too. Thus, the normal bundle is flat, that is, $R^D = 0$. Consequently, equation (3.1) for $n = 2$ implies that $\Delta\nu = \|h\|^2\nu$, i.e., M has a pointwise 1-type Gauss map of the first kind. □

Considering [6, Theorem 2.1, p.106] we have

3.3. Corollary. *An oriented non-minimal surface M in the Euclidean space $\mathbb{E}^4$ has pointwise 1-type Gauss map of the first kind if and only if M is a surface in a 3-sphere $S^3(a)$ of $\mathbb{E}^4$ with constant mean curvature.* □

For instance, all minimal surfaces of $S^3(a) \subset \mathbb{E}^4$ have pointwise 1-type Gauss map of the first kind. Also, a torus $T^2 = S^1(a) \times S^1(b)$ in $S^3(\sqrt{a^2 + b^2}) \subset \mathbb{E}^4$ has 1-type Gauss map of the first kind.

3.4. Theorem. *An oriented minimal surface M in the Euclidean space $\mathbb{E}^4$ has pointwise 1-type Gauss map of the first kind if and only if M has a flat normal bundle.*

Proof. Immediately follows from Definition (1.1) and Lemma 3.1. □

We give the following example for Theorem 3.4.

3.5. Example. Let M be a surface in $\mathbb{E}^4$ with the parametrization

$$x(u, v) = (u\cos v, u\sin v, v, v)$$

which lies in $\mathbb{E}^4$. The surface M, which is called a helicoid in $\mathbb{E}^4$, is minimal, and its Gauss map ν is of pointwise 1-type of the first kind, i.e., $\Delta\nu = \frac{4}{(u^2+2)^2}\nu$.

4. Pointwise 1-type Gauss map of the second kind

In this section we partially classify surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map of the second kind. For a characterization of minimal surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map of the second kind we prove

4.1. Theorem. *A non-planar minimal oriented surface M in the Euclidean space $\mathbb{E}^4$ has pointwise 1-type Gauss map of the second kind if and only if, with respect to some suitable local orthonormal frame $\{e_1, e_2, e_3, e_4\}$ on M, the shape operators of M are given by $A_3 = \operatorname{diag}(\rho, -\rho)$ and $A_4 = \operatorname{adiag}(\pm\rho, \pm\rho)$, where ρ is a smooth non-zero function on M and $\operatorname{adiag}(a, b)$ means a 2×2 anti-diagonal matrix.*

Proof. Suppose that M is a non-planar minimal oriented surface in $\mathbb{E}^4$ with pointwise 1-type Gauss map of the second kind. Then, the mean curvature vector H is zero, and from (3.1) we have $\Delta\nu = \|h\|^2\nu + 2R^D e_1 \wedge e_2$ which implies $R^D = R^D(e_1, e_2; e_3, e_4) \neq 0$ on M because if $R^D = 0$, then M would have a pointwise 1-type Gauss map of the first kind. Considering (1.1) we have

$$\|h\|^2\nu + 2R^D e_1 \wedge e_2 = f(\nu + C)$$

for some smooth non-zero function f on M and some constant vector C. Writing $C = \sum_{1\leq A<B\leq 4} C_{AB}\, e_A \wedge e_B$, where $C_{AB} = \langle C, e_A \wedge e_B\rangle$, we get

$$\|h\|^2 = f(1 + C_{34}), \quad C_{34} \neq -1, \tag{4.1}$$

$$2R^D = fC_{12} \neq 0, \tag{4.2}$$

$$C_{13} = C_{14} = C_{23} = C_{24} = 0. \tag{4.3}$$

Assuming that e_1, e_2 are principal directions of A_3 and considering the minimality of M, then A_3 and A_4 can be expressed as follows:

$$A_3 = \begin{pmatrix} h^3_{11} & 0 \\ 0 & -h^3_{11} \end{pmatrix} \quad \text{and} \quad A_4 = \begin{pmatrix} h^4_{11} & h^4_{12} \\ h^4_{12} & -h^4_{11} \end{pmatrix}.$$

Thus we get $R^D = -2h^3_{11}h^4_{12} \neq 0$, that is, $h^3_{11} \neq 0$ and $h^4_{12} \neq 0$ on M. When we evaluate $e_k(C_{13}) = e_k\langle C, e_1 \wedge e_3\rangle = 0$ and $e_k(C_{14}) = e_k\langle C, e_1 \wedge e_4\rangle = 0$ for $k = 1, 2$ by using (4.3) we obtain

$$h^4_{11}C_{34} = 0, \tag{4.4}$$

$$h^4_{12}C_{34} - h^3_{11}C_{12} = 0, \tag{4.5}$$

$$h^3_{11}C_{34} - h^4_{12}C_{12} = 0, \tag{4.6}$$

$$h^4_{11}C_{12} = 0. \tag{4.7}$$

Equation (4.2) implies that $C_{12} \neq 0$. From (4.4), if $C_{34} = 0$, then (4.6) gives $h^4_{12} = 0$ as $C_{12} \neq 0$, which is not possible because $R^D = -2h^3_{11}h^4_{12} \neq 0$. Hence we get $h^4_{11} = 0$ by (4.4) or (4.7). Moreover, since $C_{12} \neq 0$ and $C_{34} \neq 0$, then (4.5) and (4.6) are satisfied if and only if $h^4_{12} = \pm h^3_{11}$. If we put $\rho = h^3_{11}$, then $h^4_{12} = \pm\rho$, and hence we obtain the diagonal and anti-diagonal shape operators.

Conversely, assume that $A_3 = \operatorname{diag}(\rho, -\rho)$ and $A_4 = \operatorname{adiag}(\pm\rho, \pm\rho)$. Since $\operatorname{tr}A_3 = 0$ and $\operatorname{tr}A_4 = 0$, M is minimal. Also $\|h\|^2 = \operatorname{tr}(A_3^2) + \operatorname{tr}(A_4^2) = 4\rho^2$ and $R^D = -2h^3_{11}h^4_{12} = -2\varepsilon\rho^2 \neq 0$, where $\varepsilon = \pm 1$. Hence $\Delta\nu = 4\rho^2(\nu - \varepsilon e_1 \wedge e_2)$ by (3.1). Let $f = 8\rho^2$ and $C = -\frac{\varepsilon}{2}e_1 \wedge e_2 - \frac{1}{2}e_3 \wedge e_4$. Considering the entries of A_3 and A_4 it can be shown that $e_k(C) = 0$ for $k = 1, 2$, i.e., C is a constant vector. Therefore it is easily seen that for the chosen f and C equation (1.1) holds, i.e., M has pointwise 1-type Gauss map of the second kind. □

We give the next example for Theorem 4.1.

4.2. Example. We consider the graph surface M in $\mathbb{E}^4$ defined by

$$x(u, v) = (u, v, u^2 - v^2, 2uv), \quad (u, v) \in \mathbb{R}^2,$$

where (u, v) is an isothermal coordinate system on M.

The unit vectors

$$e_1 = \frac{1}{\lambda}\frac{\partial}{\partial u}, \quad e_2 = \frac{1}{\lambda}\frac{\partial}{\partial v},\ e_3 = \frac{1}{\lambda}(-2u, 2v, 1, 0),\ e_4 = \frac{1}{\lambda}(-2v, -2u, 0, 1),$$

where $\lambda = \sqrt{1 + 4u^2 + 4v^2}$, form an orthonormal frame on M such that e_3, e_4 are normal to M.

From a direct calculation we obtain the shape operators A_3 and A_4 in the directions e_3 and e_4, respectively, as follows

$$A_3 = \frac{2}{\lambda^3}\mathrm{diag}(1, -1) \text{ and } A_4 = \frac{2}{\lambda^3}\mathrm{adiag}(1, 1).$$

Therefore, M is minimal, and it has a pointwise 1-type Gauss map of the second kind by Theorem 4.1. Furthermore the Gauss map $\nu = e_3 \wedge e_4$ satisfies (1.1) for $f = 32/(1 + 4u^2 + 4v^2)^3$ and the constant vector $C = -1/2e_1 \wedge e_2 - 1/2e_3 \wedge e_4 \in \mathbb{E}^6$.

We need the following example for the proof of the next theorem. We show that a 2-dimensional helical cylinder M in $\mathbb{E}^4$ has a 1-type Gauss map of the second kind. It has also non-parallel mean curvature direction, constant mean curvature, and $\dim(N_1(M)) = 1$.

4.3. Example. Let M be a 2-dimensional helical cylinder in $\mathbb{E}^4$. Then, by a suitable choice of the Euclidean coordinates, M takes the following form

$$x(u, v) = (a\cos u, a\sin u, bu, v)\,,$$

for some constants $a \neq 0$ and b. If we put

$$e_1 = \frac{1}{c}\frac{\partial}{\partial u},\ e_2 = \frac{\partial}{\partial v},\ e_3 = (\cos u, \sin u, 0, 0)\,,\ e_4 = \frac{1}{c}(b\sin u, -b\cos u, a, 0)\,,$$

where $c = \sqrt{a^2 + b^2}$, then the dual forms are $\omega^1 = c\,du$, $\omega^2 = dv$, and by a direct calculation we obtain the connection forms ω_A^B of M as

$$\omega_2^1 = 0, \quad \omega_2^3 = \omega_1^4 = \omega_2^4 = 0, \quad \omega_1^3 = -\frac{a}{c^2}\,\omega^1, \quad \omega_4^3 = \frac{b}{c^2}\,\omega^1. \tag{4.8}$$

All these relations show that M^2 has a flat normal bundle, the mean curvature $\alpha = -a/(2c^2)$ is constant, and the mean curvature direction $e_3 = H/\alpha$ is non-parallel.

By a calculation we have

$$\Delta\nu = \frac{1}{c^2}\left(\nu - \frac{ab}{c^2}e_1 \wedge e_3 - \frac{b^2}{c^2}e_3 \wedge e_4\right)$$

which satisfies the definition (1.1) with $f(u, v) = 1/c^2$ and $C = -\frac{ab}{c^2}e_1 \wedge e_3 - \frac{b^2}{c^2}e_3 \wedge e_4$. We can see by a direct calculation that $e_k(C) = 0$ for $k = 1, 2$. Therefore the helical cylinder M^2 has 1-type Gauss map of the second kind as f is constant.

4.4. Theorem. *Let M be an oriented surface in the Euclidean space $\mathbb{E}^4$ with non-parallel mean curvature direction, non-zero constant mean curvature, and $\dim(N_1(M)) = 1$, where $N_1(M)$ denotes the first normal space of M. Then, M has pointwise 1-type Gauss map of the second kind if and only if M is an open portion of a helical cylinder in $\mathbb{E}^4$.*

Proof. From the hypotheses on M, we can choose a local orthonormal normal frame $\{e_3, e_4\}$ such that $e_3 = H/\alpha$, $\alpha \neq 0$, and $D_{e_i}e_3 = \omega_3^4(e_i)e_4 \neq 0$, i.e., $\omega_3^4(e_i) \neq 0$ at least for one $i \in \{1, 2\}$.

Thus, without losing generality we may assume that $\omega_4^3(e_1) \neq 0$ in the following calculation. From $\dim(N_1(M)) = 1$ we have $A_4 = 0$, i.e., $h_{ij}^4 = 0$, $i, j = 1, 2$ which implies $R^D = 0$ on M by (2.4). We choose a local orthonormal tangent frame $\{e_1, e_2\}$ on M such that $A_3 = \mathrm{diag}(h_{11}^3, h_{22}^3)$.

Now suppose M has a pointwise 1-type Gauss map of the second kind. Since $\mathrm{tr}A_3 = 2\alpha$ is constant and $\mathrm{tr}A_4 = 0$, then we have from (1.1) and (3.1)

$$(4.9)\qquad ||h||^2\nu + 2\alpha\sum_{i=1}^{2}\omega_4^3(e_i)e_i\wedge e_3 = f(\nu + C)$$

for some smooth function f on M and some constant vector C which can be written as

$$C = \sum_{1\leq A<B\leq 4} C_{AB}\, e_A\wedge e_B,$$

where $C_{AB} = \langle C, e_A\wedge e_B\rangle$. Equation (4.9) implies that

$$(4.10)\qquad ||h||^2 = f(1 + C_{34}),$$

$$(4.11)\qquad 2\,\alpha\,\omega_4^3(e_1) = f\,C_{13},$$

$$(4.12)\qquad 2\,\alpha\,\omega_4^3(e_2) = f\,C_{23},$$

$$(4.13)\qquad C_{14} = \langle C, e_1\wedge e_4\rangle = 0,\ C_{24} = \langle C, e_2\wedge e_4\rangle = 0,\ C_{12} = \langle C,\ e_1\wedge e_2\rangle = 0.$$

By evaluating $e_2(\langle C, e_1\wedge e_2\rangle) = e_2(0)$, $e_1(\langle C, e_2\wedge e_4\rangle) = e_1(0)$, and $e_1(\langle C, e_1\wedge e_4\rangle) = e_1(0)$, and using (4.13), we obtain the following equations:

$$(4.14)\qquad h_{22}^3\,C_{13} = 0,$$

$$(4.15)\qquad \omega_4^3(e_1)\,C_{23} = 0,$$

$$(4.16)\qquad h_{11}^3\,C_{34} + \omega_4^3(e_1)\,C_{13} = 0,$$

As $\omega_4^3(e_1) \neq 0$ we have $C_{13} \neq 0$ from (4.11). Thus, (4.16) implies that $C_{34} \neq 0$ and $h_{11}^3 \neq 0$. Also, (4.14) and (4.15) give, respectively, $h_{22}^3 = 0$ ($h_{11}^3 = 2\alpha \neq 0$) and $C_{23} = \langle C, e_2\wedge e_3\rangle = 0$. Moreover, we have $\omega_4^3(e_2) = 0$ by (4.12).

Now, when we evaluate $e_k(\langle C, e_2\wedge e_3\rangle) = e_k(0)$ for $k = 1, 2$ by using (4.13) and $h_{ij}^4 = 0$, we then have

$$(4.17)\qquad \omega_2^1(e_1)\,C_{13} = 0,$$

$$(4.18)\qquad \omega_2^1(e_2)\,C_{13} = 0.$$

These equations imply that $\omega_2^1(e_1) = \omega_2^1(e_2) = 0$, that is, M is flat.

By considering $C_{23} = 0$, $h_{ij}^4 = 0, i, j = 1, 2$, and (4.13) it is seen that $e_k(C_{13}) = 0$ and $e_k(C_{34}) = 0$ for $k = 1, 2$, that is, C_{13} and C_{34} are constant. Since $||h||^2 = (h_{11}^3)^2 = 4\alpha^2$ is constant, then the function f is constant because of (4.10). Moreover, Equation (4.11) implies that $\omega_4^3(e_1) = \frac{fC_{13}}{2\alpha}$ is a constant.

Consequently, we obtain

$$\omega_2^1 = \omega_2^3 = \omega_1^4 = \omega_2^4 = 0,\quad \omega_1^3 = 2\alpha\omega^1,\quad \omega_4^3 = \mu_0\omega^1,$$

where $\mu_0 = \frac{fC_{13}}{2\alpha}$. All these relations show that the connection forms ω_B^A of M coincide with the connection forms of the helical cylinder, which are given by (4.8). Therefore, by the fundamental theorem of submanifolds, M is locally isometric to a helical cylinder of $\mathbb{E}^4$.

The converse follows from Example 4.3.

Note that if $\omega_4^3(e_2) \neq 0$, we can obtain the same result by a similar argument. □

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