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FORMULA FOR SECOND REGULARIZED TRACE OF A PROBLEM WITH SPECTRAL PARAMETER DEPENDENT BOUNDARY CONDITION

Mehmet Bayramoglu* and Nigar Aslanova*†

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Abstract

In the paper we establish a formula for the second regularized trace of the problem generated by a Sturm – Liouville operator equation and with a spectral parameter dependent boundary condition.

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1. Introduction

Let H be a separable Hilbert space. In the Hilbert space $L_2([0,\pi],H)$ we consider the following boundary value problem

(1.1) $\quad -y''(t) + Ay(t) + q(t)\,y(t) = \lambda y(t)\,,$

(1.2) $\quad y(0) = 0,$

(1.3) $\quad y'(\pi) - \lambda y(\pi) = 0.$

Here A is a selfadjoint positive definite operator ($A > E$, E is the identity operator in H) with a compact inverse, $q(t)$ is a selfadjoint operator-valued function in H for each t. Also let $q(t)$ be weakly measurable with the properties:

*Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku 9, F.Agayev str., AZ1141, Baku, Azerbaijan. E-mail:
(M. Bayramoglu) mamed.bayramoglu@yahoo.com (N. Aslanova) nigar.aslanova@yahoo.com
†Corresponding Author.

1. It has a fourth order weak derivative on $[0,\pi]$, $q^{(l)}(t) \in \sigma_1(H)$ and $\|q^{(l)}(t)\|_{\sigma_1(H)} \leq$ const for each $t \in [0,\pi]$, $(l = \overline{0,4})$, $Aq^{(l)}(t) \in \sigma_1(H)$, $\|Aq^{(l)}(t)\| \leq$ const for $l = 0, 2$. Here $\sigma_1(H)$ is a trace class (see [12, p.521], also [9, p.88]) of compact operators in separable Hilbert space, whose singular values form convergent series. It should be noted that in [12] this class is denoted by $\mathcal{B}_1(H)$ while in [9] by $\sigma_1(H)$. We will use the last notation;
2. $q'(0) = q'(\pi) = q(\pi) = 0$;
3. $\int\limits_0^\pi (q(t) f, f)\, dt = 0$ for each $f \in H$.

In the direct sum $\mathcal{L}_2 = L_2([0,\pi], H) \oplus H$ let us associate with problem (ref1)–(1.3) for $q(t) \equiv 0$, the operator L_0 defined by

$$D(L_0) = \big\{Y = (y(t), y_1)/y_1 = y(\pi),$$
$$- y''(t) + Ay(t) \in L_2((0,\pi), H),\ y(0) = 0\big\},$$
$$L_0 Y = \big(- y''(t) + Ay(t),\ y'(\pi)\big).$$

Let us denote by L the perturbed operator: $L = L_0 + Q$, where

$$Q(y(t), y(\pi)) = (q(t)\, y(t), 0).$$

It is known [16] that the operators L_0 and L have a discrete spectrum. Denote their eigenvalues by $\mu_1 \leq \mu_2 \leq \cdots$ and $\lambda_1 \leq \lambda_2 \leq \cdots$, respectively.

The main goal of the paper is to establish a formula for the second regularized trace of operator L. A formula for the first regularized trace of operator L is obtained in [2].

The formula of the regularized trace of the Sturm – Liouville operator was first obtained by I. M. Gelfand and B. M.Levitan (see [8]).

After this work, numerous investigations on the calculation of the regularized trace of concrete operators, as well as of differential operator equations and discrete abstract operators appeared (see, for example, [2]–[8], [11], [13]–[15], [17]–[19]). One can find additional references on the subject in [19].

An individual approach to concrete problems sometimes gives stronger results in comparison with general theorems. Results for operators generated by differential operator equations have applications to concrete problems of mathematical physics.

It should be noted that one of the applications of trace formulas is in approximate calculations of the first eingenvalues of differential operators ([4], [5]), and inverse problems ([14]).

2. Preliminaries

Let us denote the eigenvectors and eigenvalues of operator A by $\varphi_1, \varphi_2, \ldots$ and $\gamma_1 \leq \gamma_2 \leq \cdots$, respectively. It is known, (see [16]), that if $\gamma_i \sim a \cdot i^\alpha,\ a > 0,\ \alpha > 2$ then

$$\mu_k \sim \lambda_k \sim k^{\frac{2\alpha}{2+\alpha}}. \tag{2.1}$$

Let R_λ^0 be the resolvent of operator L_0^2. In view of the asymptotics for μ_k, it follows that R_λ^0 is from $\sigma_1(H)$. In [18] the following theorem was proved.

2.1. Theorem. *Let $D(A_0) \subset D(B)$, where A_0 is a selfadjoint positive discrete operator in the separable Hilbert space H, such that $A_0^{-1} \in \sigma_1(H)$ and let B be a perturbation operator. Assume that there exists a number $\delta \in [0,1)$ such that $BA_0^{-\delta}$ is continuable to a bounded operator, and some number $\omega \in [0,1)$, $\omega + \delta < 1$, such that $A_0^{-(1-\delta-\omega)}$ is a* *trace class operator. Then there exist a subsequence of natural numbers* $\{n_m\}_{m=1}^{\infty}$ *and a sequence of closed contours* $\Gamma_m \in \mathbb{C}$ *such that for* $N \geq \frac{\delta}{\omega}$,

$$\lim_{m\to\infty}\left(\sum_{j=1}^{n_m}(\mu_j-\lambda_j)+\frac{1}{2\pi i}\int\limits_{\Gamma_m}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k}T_r\left(BR_0(\lambda)\right)^k\,d\lambda\right)=0.$$

(here $\{\mu_n\}$ *and* $\{\lambda_n\}$ *are eigenvalues of* A_0+B *and* A_0*, respectively, arranged in ascending order of their real parts, and* $R_0(\lambda)$ *is a resolvent of* A_0*).*

In particular, for $\omega \geq \delta$ *it holds that*

$$\lim_{m\to\infty}\sum_{j=1}^{n_m}\left(\mu_j-\lambda_j-(B\varphi_j,\varphi_j)\right)=0,$$

where $\{\varphi_j\}_{j=1}^{\infty}$ *is a basis formed by eigenvectors of* A_0. □

The conditions of this theorem are satisfied for L_0^2 and L^2. Really, if we take $A_0 = L_0^2$, $B = L_0Q+QL_0+Q^2$ ($L^2 = A_0+B$) and $\delta = \dfrac{1}{2}$, then provided $L_0QL_0^{-1}$ is bounded, BA_0^{-1} is also bounded and for $\omega \in [0,1)$, $\omega < \dfrac{1}{2}-\dfrac{2+\alpha}{4\alpha}$,

$$A_0^{-(1-\delta-\omega)} = L_0^{-2(1-\delta-\omega)}$$

is an operator of the trace class because of the asymptotic relations (2.1). Thus by the statement of Theorem 2.1, for $N > \dfrac{1}{2\omega}$,

$$\begin{aligned}\lim_{m\to\infty}&\left(\sum_{n=1}^{n_m}\left(\lambda_n^2-\mu_n^2\right)\right.\\ &\left.+\frac{1}{2\pi i}\int\limits_{\Gamma_m}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k}\mathrm{tr}\left[\left(L_0Q+QL_0+Q^2\right)R_0(\lambda)\right]^k d\lambda\right)=0.\end{aligned}\tag{2.2}$$

3. Regularized trace

Let us call

$$\begin{aligned}\lim_{m\to\infty}&\left\{\sum_{n=1}^{n_m}\left(\lambda_n^2-\mu_n^2-\frac{1}{\pi}\int\limits_0^{\pi}trq^2(t)\,dt\right)\right.\\ &\left.+\frac{1}{2\pi i}\int\limits_{\Gamma_m}\sum_{k=2}^{N}\frac{(-1)^{k-1}}{k}\mathrm{tr}\left[\left(L_0Q+QL_0+Q^2\right)R_0(\lambda)\right]^k d\lambda\right\}\end{aligned}\tag{3.1}$$

the second regularized trace of L, and denote it by $\sum\limits_{n=1}^{\infty}\left(\lambda_n^{(2)}-\mu_n^{(2)}\right)$. Further, we will show that it has finite value which does not depend on the choice of $\{n_m\}$.

By virtue of [18, lemma 3], for large m the number of eigenvalues of L_0^2 and L^2 inside the contour Γ_m is the same and equal to n_m.

In view of (2.2)

$$\lim_{m\to\infty}\left\{\sum_{n=1}^{n_m}\left(\lambda_n^2-\mu_n^2-\frac{1}{\pi}\int_0^\pi trq^2(t)\,dt\right)+\frac{1}{2\pi i}\int_{\Gamma_m}\sum_{k=2}^{N}\frac{(-1)^{k-1}}{k}\left(\operatorname{tr}\left[(L_0Q+QL_0+Q^2)R_0(\lambda)\right]^k\right)d\lambda\right\}$$
$$=\lim_{m\to\infty}\left(-\frac{1}{2\pi\, i}\int_{\Gamma_m}\operatorname{tr}\left[\left(L_0Q+QL_0+Q^2\right)R_0\left(\lambda\right)\right]d\lambda-\sum_{n=1}^{n_m}\frac{1}{\pi}\int_0^\pi trq^2\left(t\right)\,dt\right). \tag{3.2}$$

Denote the eigenvectors of L_0 by $\psi_1,\psi_2,\ldots$. By our assumption, the operator $L_0QL_0^{-1}$ is bounded so $\left[L_0Q+QL_0+Q^2\right]R_\lambda^0$ is an operator of the trace class, and since the eigenvectors of L_0 form a basis in $\mathcal{L}_2$, we can change the first term on the right - hand side of (3.2) in the following way:

$$\frac{1}{2\pi\, i}\int_{\Gamma_m}\operatorname{tr}\left[\left(QL_0+L_0Q+Q^2\right)R_0\left(\lambda\right)\right]d\lambda$$
$$=\frac{1}{2\pi\, i}\int_{\Gamma_m}\sum_{n=1}^{\infty}\left(\left(QL_0+L_0Q+Q^2\right)R_0\left(\lambda\right)\psi_n,\psi_n\right)_{\mathcal{L}_2}d\lambda$$
$$=\frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{\Gamma_m}\frac{1}{\mu_n^2-\lambda}\left(\left(QL_0+L_0Q+Q^2\right)R_0\left(\lambda\right)\psi_n,\psi_n\right)_{\mathcal{L}_2}d\lambda$$
$$=-\sum_{n=1}^{n_m}\left(\left[QL_0+L_0Q+Q^2\right]\psi_n,\psi_n\right)_{\mathcal{L}_2}. \tag{3.3}$$

Note that the eigenvectors $\{\psi_n\}_{n=1}^{\infty}$ are of the form (see [2])

$$\sqrt{\frac{4x_{j,k}}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}}\left\{\sin\left(x_{j,k}t\right)\varphi_j,\sin\left(x_{j,k}\pi\right)\varphi_j\right\},$$
$$\begin{cases}k=\overline{1,\infty},\ j=\overline{1,\infty}\\ k=0,\ j=\overline{N,\infty},\end{cases} \tag{3.4}$$

where $x_{j,k}$ are the roots (see [16]) of the equation

$$ctg\,x\pi=\frac{\gamma_j+x^2}{x},\ x=\sqrt{\lambda-\gamma_j}. \tag{3.5}$$

It is known that eigenvalues of L_0 form two sequences: $\mu_{j,0}\sim\sqrt{\gamma_j}$, as $j\to\infty$, which correspond to the imaginary roots of (3.5), and $\mu_{j,k}=\gamma_j+x_{j,k}^2=\gamma_j+\eta_k,\ \eta_k\sim k^2$, which correspond to the real roots of (3.5). To calculate the regularized trace, the following lemma will be required.

3.1. Lemma. *If properties 1.1,1.2 hold, and $\gamma_j \sim aj^{\alpha}, a > 0, \alpha > 2$, then the following series is absolutely convergent*

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\left(\gamma_j + x_{j,k}^2\right)\frac{2x_{j,k}\int\limits_0^{\pi}\cos 2x_{j,k}tf_j(t)\,dt}{2x_{j,k}\pi - \sin 2x_{j,k}\pi + 4x_{j,k}\sin^2 x_{j,k}\pi}\right|$$
$$+\sum_{j=N}^{\infty}\left|\frac{\left(\gamma_j + x_{j,0}^2\right)2x_{j,0}\int\limits_0^{\pi}\cos 2x_{j,0}tf_j(t)\,dt}{2x_{j,0}\pi - \sin 2x_{j,0}\pi + 4x_{j,0}\sin^2 x_{j,0}\pi}\right|$$
$$(3.6)\qquad +\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(\left|\frac{4x_{j,k}\int\limits_0^{\pi}\sin^2 x_{j,k}tg_j(t)\,dt}{2x_{j,k}\pi - \sin 2x_{j,k}\pi + 4x_{j,k}\sin^2 x_{j,k}\pi} - \frac{1}{\pi}\int\limits_0^{\pi} g_j(t)dt\right|\right)$$
$$+\sum_{j=N}^{\infty}\left(\left|\int\limits_0^{\pi}\frac{4x_{j,0}\int\limits_0^{\pi}\sin^2 x_{j,0}tg_j(t)dt}{2x_{j,0}\pi - \sin 2x_{j,0}\pi + 4x_{j,0}\sin^2 x_{j,0}\pi} - \frac{1}{\pi}\int\limits_0^{\pi} g_j(t)dt\right|\right) < \infty,$$

where $f_j(t) = (q(t)\varphi_j, \varphi_j)\,, g_j(t) = \left(q^2(t)\varphi_j, \varphi_j\right)$.

Proof. Let us denote the sums on the left of (3.6) by s_1, s_2, s_3, s_4 according to their order. By virtue of property (1.2), and integrating by parts at first twice, then four times, we have

$$(3.7)\qquad \int\limits_0^{\pi}\cos 2x_{j,k}tf_j\left(t\right)\,dt = -\frac{1}{\left(2x_{j,k}\right)^2}\int\limits_0^{\pi}\cos 2x_{j,k}t\,f_j''\left(t\right)\,dt,$$

$$(3.8)\qquad \int\limits_0^{\pi}\cos 2x_{j,k}tf_j\left(t\right)\,dt = -\frac{1}{\left(2x_{j,k}\right)^3}f_j''\left(\pi\right)\sin 2x_{j,k}\pi - \frac{1}{\left(2x_{j,k}\right)^4}\cos 2x_{j,k}tf_j'''\left(t\right)\Big|_0^{\pi} + \frac{1}{\left(2x_{j,k}\right)^4}\int\limits_0^{\pi}\cos 2x_{j,k}t\,f_j^{(IV)}\left(t\right)\,dt.$$

In virtue of the estimate

$$(3.9)\qquad \frac{2x_{j,0}}{2x_{j,0}\pi - \sin 2x_{j,0}\pi + 4x_{j,0}\sin^2 x_{j,0}\pi} = \frac{1}{\pi} + O\left(\frac{1}{x_{j,0}}\right),$$

and using property (1.1) and relation (3.7) we have

$$\sum_{j=N}^{\infty}\left|\frac{2x_{j,0}\gamma_j\int\limits_0^{\pi}\cos 2x_{j,0}tf_j\left(t\right)\,dt}{2x_{j,0}\pi - \sin 2x_{j,0}\pi + 4x_{j,0}\sin^2 x_{j,0}\pi}\right| \le \sum_{j=N}^{\infty}\gamma_j\left(\frac{1}{\pi} + O\left(\frac{1}{x_{j,0}}\right)\right)\int\limits_0^{\pi}\left|f_j\left(t\right)\right|dt < \infty,$$

$$\sum_{j=N}^{\infty}\left|\frac{2x_{j,0}^{3}\int\limits_{0}^{\pi}\cos 2x_{j,0}tf_{j}\left(t\right)\,dt}{2x_{j,0}\pi-\sin 2x_{j,0}\pi+4x_{j,0}\sin^{2}x_{j,0}\pi}\right| \le \sum_{j=N}^{\infty}\left(\frac{1}{2\pi}+O\left(\frac{1}{x_{j,0}}\right)\right)\int\limits_{0}^{\pi}\left|f_{j}''\left(t\right)\right|\,dt.$$

So, we get that the series denoted by s_2 is absolutely convergent.

Then by virtue of (3.8), the asymptotic relation $x_{j,k}\sim k+\dfrac{k}{\gamma_j+k^2}$, the property $\left\|Aq''\left(t\right)\right\|_1\le$ const (the norm in $\sigma_1(H)$ we denote simply by $\|\cdot\|_1$) and (3.7), the following estimate holds

$$\begin{aligned}&\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\frac{2\gamma_j x_{j,k}\int\limits_{0}^{\pi}\cos 2x_{j,k}tf_{j}\left(t\right)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^{2}x_{j,k}\pi}\right|\\ &=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\gamma_j\left(\frac{1}{\pi}+O\left(\frac{1}{k^2}\right)\right)O\left(\frac{1}{k^2}\right)\int\limits_{0}^{\pi}\left|f_{j}''\left(t\right)\right|\,dt\\ &=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}O\left(\frac{1}{k^2}\right)\int\limits_{0}^{\pi}\left|\left(Aq''\left(t\right)\varphi_j,\varphi_j\right)\right|\,dt<\text{const}.\end{aligned}\tag{3.10}$$

Since $\left\|q^{(l)}\left(t\right)\right\|_1\le const\,(l=2,4)$, again by using the asymptotic relations for $x_{j,k}$, and (3.8), we obtain

$$\begin{aligned}&\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\frac{2x_{j,k}^{3}\int\limits_{0}^{\pi}\cos 2x_{j,k}tf_{j}\left(t\right)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^{2}x_{j,k}\pi}\right|=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{\pi}+O\left(\frac{1}{k^2}\right)\right)\\ &\times\left[\frac{1}{x_{j,k}}\left|f_{j}''\left(\pi\right)\sin 2x_{j,k}\pi\right|+\frac{1}{\left(2x_{j,k}\right)^2}\left(\left|f_{j}''\left(\pi\right)\right|+\left|f_{j}''\left(0\right)\right|\right)\right.\\ &\left.+\frac{1}{\left(2x_{j,k}\right)^2}\int\limits_{0}^{\pi}\left|f_{j}^{(IV)}\left(t\right)\right|dt\right]<\infty.\end{aligned}\tag{3.11}$$

Here it was also used that $\sin(2x_{j,k}\pi)\sim\dfrac{1}{k}$.

From (3.10) and (3.11) it follows that series denoted by s_1 is also convergent.

Then

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\frac{4x_{j,k}\int\limits_0^\pi \sin^2(x_{j,k}t)g_j\,(t)\;dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^\pi g_j\,(t)\;dt\right|$$

$$=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\frac{2x_{j,k}\int\limits_0^\pi (1-\cos 2x_{j,k}t)\,g_j\,(t)\;dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^\pi g_j\,(t)\;dt\right|$$

$$=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\left(\frac{1}{\pi}+O\left(\frac{1}{k^2}\right)\right)\int\limits_0^\pi (1-cos2x_{j,k}t)g_j(t)dt-\frac{1}{\pi}\int\limits_0^\pi g_j(t)dt\right|$$

$$=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\left(\frac{1}{\pi}+O\left(\frac{1}{k^2}\right)\right)\int\limits_0^\pi \cos 2x_{j,k}tg_j\,(t)\;dt+O\left(\frac{1}{k^2}\right)\int\limits_0^\pi g_j\,(t)\;dt\right|.$$

The last equality, by virtue of (3.7) and the properties $g_j''(t)\in\sigma_1(H)$, $g_j(t)\in\sigma_1(H)$, gives that the series denoted by s_3 converges. Similarly it can be shown that s_4 also converges and this completes the proof of the lemma. □

Now let us calculate the value of the series called the second regularized trace. This is done in the following theorem, where we assume that

$$\text{(3.12)}\qquad \int\limits_{\pi-\delta}^{\pi}\frac{g_j(t)}{\pi-t}dt<\infty$$

for small $\delta>0$.

3.2. Theorem. *Let $q\,(t)$ be an operator-function with properties 1.1–1.3, $L_0^{-1}QL_0$ a bounded operator in $\mathbb{L}_2$, and $\gamma_j\sim a\cdot j^\alpha$, $a>0$, $\alpha>2$. Then provided that (3.12) holds*

$$\text{(3.13)}\qquad \sum_{n=0}^{\infty}\left(\lambda_n^{(2)}-\mu_n^{(2)}\right)=-\frac{trq^2\,(0)}{4}--\frac{\mathrm{tr}Aq\,(0)+\mathrm{tr}Aq\,(\pi)}{2}+\frac{\mathrm{tr}q''\,(0)+\mathrm{tr}q''\,(\pi)}{8}.$$

Proof. It follows from Lemma 3.1 and relations (3.2) and (3.3) that

$$\lim_{m\to\infty}\left\{\sum_{n=1}^{n_m}\left(\lambda_n^2-\mu_n^2-\frac{1}{\pi}\int\limits_0^\pi \mathrm{tr}q^2\,(t)\;dt\right)\right.$$
$$\left.+\frac{1}{2\pi i}\int\limits_{\Gamma_m}\sum_{k=2}^{N}\frac{(-1)^{k-1}}{k}\mathrm{tr}\left[(L_0Q+QL_0+Q^2)R_0(\lambda)\right]^k d\lambda\right\}$$

$$=\sum_{j=1}^{N-1}\sum_{k=1}^{\infty}2\left(\gamma_j+x_{j,k}^2\right)\frac{4x_{j,k}\int\limits_0^\pi \frac{1-\cos 2x_{j,k}t}{2}f_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}$$

$$+\sum_{j=N}^{\infty}\sum_{k=0}^{\infty}2\left(\gamma_j+x_{j,k}^2\right)\frac{4x_{j,k}\int\limits_0^\pi \frac{1-\cos 2x_{j,k}t}{2}f_j(t)dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}$$

$$+\sum_{j=1}^{N-1}\sum_{k=1}^{\infty}\left[\frac{2x_{j,k}\int\limits_0^\pi (1-\cos 2x_{j,k}t)\, g_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^\pi g_j(t)\,dt\right]$$

$$+\sum_{j=N}^{\infty}\sum_{k=0}^{\infty}\left[\frac{2x_{j,k}\int\limits_0^\pi (1-\cos 2x_{j,k}t)\, g_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^\pi g_j(t)dt\right]$$

$$\begin{aligned}&=-\sum_{j=1}^{N-1}\sum_{k=1}^{\infty}\left(\gamma_j+x_{j,k}^2\right)\frac{4x_{j,k}\int\limits_0^\pi \cos 2x_{j,k}t f_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}\\&-\sum_{j=N}^{\infty}\sum_{k=0}^{\infty}\left(\gamma_j+x_{j,k}^2\right)\frac{4x_{j,k}\int\limits_0^\pi \cos 2x_{j,k}t f_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}\\&+\sum_{j=1}^{N-1}\sum_{k=1}^{\infty}\left[\frac{2x_{j,k}\int\limits_0^\pi (1-\cos 2x_{j,k}t)\, g_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^\pi g_j(t)\,dt\right]\\&+\sum_{j=N}^{\infty}\sum_{k=0}^{\infty}\left[\frac{2x_{j,k}\int\limits_0^\pi (1-\cos 2x_{j,k}t)\, g_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^\pi g_j(t)\,dt\right].\end{aligned}\tag{3.14}$$

We first derive a formula for the fourth term on the right of (3.14). For that consider

$$\sum_{j=N}^{\infty}\sum_{k=0}^{\infty}\left[\frac{2x_{j,k}\int\limits_0^\pi g_j(t)\,dt}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^\pi g_j(t)\,dt\right].$$

Let us calculate the value of the inner series for each fixed j:

$$\begin{aligned}&\sum_{k=0}^{\infty}\left[\frac{2x_{j,k}}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\right]\\&\quad=\lim_{N\to\infty}\sum_{k=0}^{N-1}\left[\frac{2x_{j,k}}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\right]\end{aligned}\tag{3.15}$$

We denote the partial sums of the above series by T_N and investigate its behavior as $N\to\infty$. Let us express the k-th term of the sum T_N as a residue at a pole $x_{j,k}$ of some function of the complex variable z for which $x_{j,0},\dots,x_{j,N}$ are poles. Thus, consider the following complex-valued function

$$g(z)=\frac{-z}{(z\operatorname{ctg} z\pi-z^2-\gamma_j)\sin^2 z\pi},\tag{3.16}$$

for which as it is easy to see that $x_{j,k}$ and k are simple poles. The residue at the point $x_{j,k}$ is

$$\underset{z=x_{j,k}}{res}\, g(z) = \frac{-x_{j,k}}{\left(\operatorname{ctg} x_{j,k}\pi - \dfrac{\pi x_{j,k}}{\sin^2 x_{j,k}\pi} - 2x_{j,k}\right)\sin^2 x_{j,k}\pi} = \frac{2x_{j,k}}{2x_{j,k}\pi - \sin 2x_{j,k}\pi + 4x_{j,k}\sin^2 x_{j,k}\pi},$$

and at the point k is

$$\underset{z=k}{res}\, g(z) = \frac{-k}{(k\cos k\pi - k^2 \sin k\pi - \gamma_j \sin k\pi)\,\pi\cos \pi k} = -\frac{1}{\pi}.$$

Now take a rectangular contour of integration with vertices at the points $\pm iB$, $A_N \pm iB$, which has a cut at $ix_{j,0}$ and will pass it by on the left, and the points $-ix_{j,0}$ and 0 on the right. Take also $B > x_{j,0}$. Then B will go to infinity and $A_N = N + \frac{1}{2}$. For this choice of A_N we have $x_{j,N-1} < A_N < x_{j,N}$, and the number of points $x_{j,k}$ inside of the contour of integration equals N, $\left(k = \overline{0, N-1}\right)$.

One can easily show that inside this contour the function $z\operatorname{ctg} z\pi - z^2 - \gamma_j$ has exactly N roots, so $x_{j,N-1} < A_N < x_{j,N}$.

The function (3.16) is an odd function of z, that is why the integral along the part of the contour on the imaginary axis, as well as along the semicircles centered at $\pm x_{j,0}$, vanishes.

If $z = u + iv$ then for large v and $u \geq 0$, the order of $g(z)$ is $O\left(e^{-2\pi|v|}\right)$, and for the chosen A_N the integrals along the upper and lower sides of the contour go to zero as $B \to \infty$.

So, we arrive at the following equality

$$\begin{aligned} T_N &= \frac{1}{2\pi i}\lim_{B\to\infty}\int_{A_N-iB}^{A_N+iB} \frac{-z\,dz}{(z\operatorname{ctg} z\pi - z^2 - \gamma_j)\sin^2 z\pi} \\ &\quad + \frac{1}{2\pi i}\lim_{r\to 0}\int_{\substack{|z|=r \\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}} \frac{-z\,dz}{(z\operatorname{ctg} z\pi - z^2 - \gamma_j)\sin^2 z\pi}, \end{aligned} \tag{3.17}$$

where in the second integral $z = re^{i\varphi}$.

As $N \to \infty$, the first term on the right of (3.17) is equivalent to

$$\begin{aligned} &\frac{1}{\pi i}\int_{A_N-i\infty}^{A_N+i\infty} \frac{dz}{2z\sin^2 z\pi - \sin 2z\pi} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dv}{2\,(A_N+iv)}\,(1+\operatorname{ch}2v\pi) - \frac{\operatorname{sh}2v\pi}{i}, \end{aligned} \tag{3.18}$$

whose absolute value is less then

$$\begin{aligned}
&\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\upsilon}{2\left|A_N+i\upsilon\right|(1+\mathrm{ch}2\upsilon\pi)-\left|\mathrm{sh}2\upsilon\pi\right|}\\
&\qquad=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\sqrt{A_N^2+\upsilon^2}\,(1+\mathrm{ch}2\upsilon\pi)-\dfrac{\mathrm{sh}2\upsilon\pi}{2}}\,d\upsilon\\
&\qquad<\frac{1}{2A_N\pi}\int_{-\infty}^{\infty}\frac{d\upsilon}{(1+\mathrm{ch}2\upsilon\pi)-\dfrac{\mathrm{sh}2\upsilon\pi}{2\sqrt{A_N^2+\upsilon^2}}}\\
&\qquad<\frac{1}{2A_N\pi}\int_{-\infty}^{\infty}\frac{d\upsilon}{1+\mathrm{ch}2\upsilon\pi-\dfrac{1+\mathrm{ch}2\upsilon\pi}{2}}=\frac{\mathrm{const}}{A_N}.
\end{aligned}\tag{3.19}$$

Therefore,

$$\begin{aligned}
&\int_0^{\pi}T_Ng_j(t)\,dt\\
&\qquad=-\frac{1}{2\pi i}\int_0^{\pi}g_j\left(t\right)\int_{A_N-i\infty}^{A_N+i\infty}\frac{z\,dz\,dt}{(z\mathrm{ctg}\,z\pi-z^2-\gamma_j)\sin^2 z\pi}\\
&\qquad\quad-\frac{1}{2\pi i}\lim_{r\to 0}\int_0^{\pi}g_j\left(t\right)\,dt\int_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}\frac{z\,dz}{(z\mathrm{ctg}\,z\pi-z^2-\gamma_j)\sin^2 z\pi}.
\end{aligned}\tag{3.20}$$

But, as $r\to 0$,

$$\begin{aligned}
&-\frac{1}{2\pi i}\int_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}\frac{z\,dz}{(z\mathrm{ctg}\,z\pi-z^2-\gamma_j)\sin^2 z\pi}\\
&\qquad\sim-\frac{1}{2\pi i}\int_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}\frac{z\,dz}{z\sin z\pi-\gamma_j\sin^2 z\pi}\\
&\qquad=-\frac{1}{2\pi i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{ir^2e^{2i\varphi}\,d\varphi}{r^2e^{2i\varphi}\pi-\gamma_j\pi^2r^2e^{2i\varphi}}\\
&\qquad=-\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\varphi}{\pi-\gamma_j\pi^2}=-\frac{1}{2\pi}\frac{1}{1-\gamma_j\pi}.
\end{aligned}\tag{3.21}$$

So, using (3.17), (3.18), (3.19) and (3.21) in (3.20) we have

$$\lim_{N\to\infty}\int_0^{\pi}T_Ng_j\left(t\right)\,dt=-\frac{1}{2\pi\left(1-\gamma_j\pi\right)}\int_0^{\pi}g_j\left(t\right)\,dt.\tag{3.22}$$

Now let us derive calculations for

$$S_N(t) = -\sum_{k=0}^{N-1} \frac{2x_{jk}\int_0^\pi \cos 2x_{jk}t g_j(t)\,dt}{2x_{j,k}\pi - \sin 2x_{j,k}\pi + 4x_{j,k}\sin^2 x_{j,k}\pi}.$$

Consider the complex valued function

$$G(z) = \frac{z\cos 2zt}{(z\operatorname{ctg} z\pi - z^2 - \gamma_j)\sin^2 z\pi},$$

whose residues at the poles k and $x_{j,k}$ are equal to $\dfrac{2x_{j,k}\cos 2x_{j,k}t}{\sin 2x_{j,k}\pi - 2x_{j,k}\pi - 4x_{j,k}\sin^2 2x_{j,k}\pi}$ and $\dfrac{\cos 2kt}{\pi}$, respectively. Again take as a contour of integration the above considered contour. One can show that as $N \to \infty$,

$$\frac{1}{2\pi i}\int_{A_N - i\infty}^{A_N + i\infty} G(z)\,dz \sim \frac{\text{const}}{A_N \cos\frac{t}{2}}. \tag{3.23}$$

Thus, if $g_j(t)$ has the property (3.12), then

$$\lim_{N\to\infty}\int_0^\pi g_j(t)\int_{A_N - i\infty}^{A_N + i\infty} G(z)\,dz\,dt = 0. \tag{3.24}$$

By virtue of (3.24),

$$\begin{aligned}\lim_{N\to\infty}\int_0^\pi & S_N(t)\,g_j(t)\,dt \\ &= -\lim_{N\to\infty}\int_0^\pi M_N(t)\,g_j(t)\,dt \\ &\quad + \frac{1}{2\pi i}\lim_{r\to 0}\int_0^\pi g_j(t)\int\limits_{\substack{|z|=r \\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}} \frac{z\cos 2zt}{(z\operatorname{ctg} z\pi - z^2 - \gamma_j)\sin^2 z\pi}\,dt,\end{aligned} \tag{3.25}$$

where

$$M_N(t) = \sum_{k=1}^{N} \frac{\cos 2kt}{\pi}.$$

Since

$$\lim_{N\to\infty}\int_0^\pi M_N(t)\,g_j(t)\,dt = \frac{1}{\pi}\sum_{k=0}^{\infty}\int_0^\pi g_j(t)\cos 2kt\,dt = \frac{g_j(\pi) + g_j(0)}{4},$$

and the second term in (3.25) as $r \to 0$ goes to $\dfrac{1}{2\pi(1-\gamma_j\pi)}\int_0^\pi g_j(t)\,dt$, then

$$\lim_{N\to\infty}\int_0^\pi S_N(t)\,g_j(t)\,dt = -\frac{g_j(\pi) + g_j(0)}{4} + \frac{1}{2\pi(1-\gamma_j\pi)}\int_0^\pi g_j(t)\,dt. \tag{3.26}$$

Combining (3.22) and (3.26), we get

$$\sum_{j=N}^{\infty}\sum_{k=0}^{\infty}\left(\frac{2x_{jk}\int\limits_0^{\pi}(1-\cos 2x_{jk}t)\,g_j(t)\,dt}{2x_{jk}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^{\pi}g_j(t)\,dt\right)$$

$$=\sum_{j=N}^{\infty}-\frac{g_j(\pi)+g_j(0)}{4}+\int\limits_0^{\pi}\left(\frac{g_j(t)}{2\pi(1-\gamma_j\pi)}-\frac{g_j(t)}{2\pi(1-\gamma_j\pi)}\right)dt \tag{3.27}$$

$$=-\sum_{j=N}^{\infty}\frac{g_j(\pi)+g_j(0)}{4}=-\sum_{j=N}^{\infty}\frac{g_j(0)}{4}.$$

Here the condition

$$g_j(\pi)=\left(q^2(\pi)\varphi_j,\varphi_j\right)=(q(\pi)\varphi_j,q(\pi)\varphi_j)=0$$

has been used.

By similar computations (this time the contour of integration bypasses only the origin along a small semicircle, since this time the chosen complex function has no imaginary roots), we will have

$$\sum_{j=1}^{N-1}\sum_{k=1}^{\infty}\left(\frac{2x_{jk}\int\limits_0^{\pi}(1-\cos 2x_{jk}t)\,g_j(t)\,dt}{2x_{jk}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}-\frac{1}{\pi}\int\limits_0^{\pi}g_j(t)\,dt\right) \tag{3.28}$$

$$=-\sum_{j=1}^{N-1}\frac{g_j(0)}{4}$$

From (3.27) and (3.28), the sum of values of two last series in (3.14) gives

$$-\sum_{j=1}^{N-1}\frac{g_j(0)}{4}-\sum_{j=N}^{\infty}\frac{g_j(0)}{4}=-\frac{\mathrm{tr}q^2(0)}{4}. \tag{3.29}$$

By the method used above the required calculations may also be made for the first two series in (3.14), and we arrive finally at formula (3.13). □

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