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(λ, α)-HOMOMORPHISMS OF INTUITIONISTIC FUZZY GROUPS[§]

Xiao-Ping Li[*†] and Gui-Jun Wang[‡]

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Abstract

In this paper, we present (λ, α)-homomorphisms and (λ, α)-isomorphisms between two intuitionistic fuzzy groups by means of the concept of cut-set of an intuitionistic fuzzy set. Furthermore, we discuss in detail a series of homomorphic properties of intuitionistic fuzzy groups by taking advantage of an intuitionistic L_*-nested set. Consequently, we obtain some important results.

Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy groups, (λ, α)-homomorphisms, (λ, α)-isomorphisms, L_*-nested sets

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1. Introduction

The notion of intuitionistic fuzzy set was put forward by K. Atanassov [1] in 1986. After a development period of more than 20 years, their theory as well as their applications have become rather diverse. In 1994, the pioneering work in fuzzy groups was completed by R. Biswas [2]. Afterwards, there were a number of researches on L-fuzzy groups [3] and interval-valued fuzzy groups [4]. In [5], an intuitionistic fuzzy group was defined for the first time, and gave not only a series of operations and extension principles, but also their homomorphic properties as well as structural characteristics. In [6,7] the authors studied and discussed systematically various intuitionistic fuzzy subgroups, namely intuitionistic fuzzy normal subgroups, intuitionistic fuzzy projective subgroups, intuitionistic fuzzy characteristic subgroups, intuitionistic fuzzy standard subgroups, intuitionistic fuzzy fully invariant subgroups, and so on. On the basis of this, they obtained

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[*]School of Management, Tianjin Normal University, Tianjin 300387, China.
E-mail: lxpmath@126.com

[†]Corresponding Author.

[‡]Department of Mathematics, Tianjin Normal University, Tianjin 300387, China.
E-mail: tjwgj@126.com

several class of intuitionistic fuzzy subgroups based on direct product groups and their projections. In [8-10] the authors generalized intuitionistic fuzzy subgroups by invoking the T-S norm operator, and in the meanwhile, suggested the direct product characteristics of intuitionistic fuzzy subgroups with respect to the T-S norm. Especially, they obtained a few important results for the homomorphisms and isomorphisms of hyper-intuitionistic fuzzy subgroups in the sense of homomorphisms and isomorphisms between two classical groups. All of these enrich and perfect the theoretical aspects of fuzzy groups.

In this paper, we shall present (λ, α)-homomorphisms and (λ, α)-isomorphisms between two intuitionistic fuzzy groups in the light of the concept of the cut set of intuitionistic fuzzy sets introduced in [3] from another angle. Furthermore, we discuss in detail a series of their homomorphic properties by means of an intuitionistic L_*-nested set. Consequently, these results lay the foundations for the wide applications of intuitionistic fuzzy sets.

2. Preliminaries

In this section, as a preparation, we first introduce some notations which will be used in the paper, and meanwhile we recall some conclusions with respect to intuitionistic fuzzy sets and intuitionistic fuzzy groups.

2.1. Definition. . Let X be a classical set. A set of triads of the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X,\ 0 \leq \mu_A(x) + \nu_A(x) \leq 1\},$$

is called an *intuitionistic fuzzy set* on X, where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each element $x \in X$ in the set A, respectively.

2.2. Remark. The condition "$0 \leq \mu_A(x) + \nu_A(x) \leq 1$" satisfied by an intuitionistic fuzzy set A is usually omitted for convenience, and we write $A = \langle x, \mu_A(x), \nu_A(x)\rangle$. The family of all the intuitionistic fuzzy sets on X is written as $IFS[X]$.

2.3. Definition. Let $L_* = \{(\lambda, \alpha) :$ for arbitrary $\lambda, \alpha \in [0, 1],\ \lambda + \alpha \leq 1\}$. For any $(\lambda_1, \alpha_1), (\lambda_2, \alpha_2) \in L_*$, the orders $\leq$ and $<$ on L_* are defined as

$$(\lambda_1, \alpha_1) \leq (\lambda_2, \alpha_2) \iff \lambda_1 \leq \lambda_2 \text{ and } \alpha_1 \geq \alpha_2;$$
$$(\lambda_1, \alpha_1) < (\lambda_2, \alpha_2) \iff (\lambda_1, \alpha_1) \leq (\lambda_2, \alpha_2) \text{ and } \lambda_1 < \lambda_2 \text{ or } \alpha_1 > \alpha_2.$$

By Definition 2.3, clearly, $(L_*, \leq)$ constitutes a complete lattice with maximum element $(1, 0)$ and minimum element $(0, 1)$.

2.4. Definition. Let $A \in IFS[X]$. For every $(\lambda, \alpha) \in L_*$, we define

(1) $A_{[\lambda,\alpha]} = \{x \in X : \mu_A(x) \geq \lambda,\ \nu_A(x) \leq \alpha\}$;
(2) $A_{(\lambda,\alpha)} = \{x \in X : \mu_A(x) > \lambda,\ \nu_A(x) < \alpha\}$;
(3) $A_{[\lambda,\alpha)} = \{x \in X : \mu_A(x) \geq \lambda,\ \nu_A(x) < \alpha\}$;
(4) $A_{(\lambda,\alpha]} = \{x \in X : \mu_A(x) > \lambda,\ \nu_A(x) \leq \alpha\}$.

Then, $A_{[\lambda,\alpha]}, A_{(\lambda,\alpha)}, A_{[\lambda,\alpha)}$ and $A_{(\lambda,\alpha]}$ are called the (λ, α)-*cut-set*, *strong* (λ, α)-*cut-set*, $[\lambda, \alpha)$-*cut-set* and $(\lambda, \alpha]$-*cut-set* of A, respectively.

2.5. Remark. Throughout this paper, we apply only the (λ, α)-cut-set of an intuitionist fuzzy set to discuss problems, the others three cut-sets will not be mentioned.

2.6. Definition. (Extension Principle 1). Let X and Y be nonempty classical sets, $f : X \to Y$ a mapping and $A = \langle x, \mu_A(x), \nu_A(x)\rangle \in IFS[X]$. We define a mapping

$$F_f : IFS[X] \to IFS[Y].$$

induced by f in the form

$$F_f(A) = \{\langle y, F_f(\mu_A)(y), \hat{F}_f(\nu_A)(y)\rangle : y \in Y\},$$

where

$$F_f(\mu_A)(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \mu_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & f^{-1}(y) = \emptyset \end{cases}$$

and

$$\hat{F}_f(\nu_A)(y) = \begin{cases} \inf\limits_{x\in f^{-1}(y)} \nu_A(x), & f^{-1}(y) \neq \emptyset \\ 1, & f^{-1}(y) = \emptyset \end{cases}$$

2.7. Definition. (Extension Principle 2). Let X and Y be nonempty classical sets and $f : X \to Y$ a mapping. We define a converse mapping $F_f^{-1} : IFS[Y] \to IFS[X]$ induced by f, where for all $B \in IFS[Y]$, we let

$$F_f^{-1}(B) = \{\langle x, F_f^{-1}(\mu_B)(x), F_f^{-1}(\nu_B)(x)\rangle : x \in X\},$$

where $F_f^{-1}(\mu_B)(x) = \mu_B(f(x))$ and $F_f^{-1}(\nu_B)(x) = \nu_B(f(x))$.

The above two definitions together are called the *extension principle for intuitionistic fuzzy sets.*

2.8. Definition. Let G be a classical group. Then $A = \langle x, \mu_A(x), \nu_A(x)\rangle \in IFS[G]$ is called an *intuitionistic fuzzy subgroup on* G if the following conditions (1)–(2) are satisfied for all $x, y \in G$,

(1) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y),\ \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$;
(2) $\mu_A(x^{-1}) \geq \mu_A(x),\ \nu_A(x^{-1}) \leq \nu_A(x)$.

The set of intuitionistic fuzzy subgroups on G is denoted by $IFG[G]$ for short.

2.9. Property. [3]. *Let $A, B \in IFS[X]$. For all $(\lambda,\alpha) \in L_*$, if $A \subseteq B$, then $A_{[\lambda,\alpha]} \subseteq B_{[\lambda,\alpha]}$ and $A_{(\lambda,\alpha)} \subseteq B_{(\lambda,\alpha)}$.* □

2.10. Property. [3] *Let $A, B \in IFS[X]$. For all $(\lambda_1,\alpha_1), (\lambda_2,\alpha_2) \in L_*$ and $(\lambda_1,\alpha_1) \leq (\lambda_2,\alpha_2)$, then $A_{(\lambda_2,\alpha_2)} \subseteq A_{[\lambda_2,\alpha_2]} \subseteq A_{(\lambda_1,\alpha_1)}$.* □

2.11. Theorem. [3] *Let G be a classical group. For any $(\lambda,\alpha) \in L_*$, $A \in IFG[G]$ if and only if $A_{[\lambda,\alpha]}$ is a subgroup of G.* □

2.12. Theorem. [14] *Let $f : X \to Y$ be a mapping. For ordinary fuzzy sets $\tilde{A}, \tilde{B} \in F(X)$, if $\tilde{A} \subseteq \tilde{B}$, then $f(\tilde{A}) \subseteq f(\tilde{B})$.* □

2.13. Theorem. [14] *Let $f : X \to Y$ be a mapping. For ordinary fuzzy sets $\tilde{A}, \tilde{B} \in F(Y)$, if $\tilde{A} \subseteq \tilde{B}$, then $f^{-1}(\tilde{A}) \subseteq f^{-1}(\tilde{B})$.*

3. (λ,α)-homomorphism and (λ,α)-isomorphism

In this section, the cut-set of an intuitionistic fuzzy set will play an important role in our discussion. Taking advantage of the definition of cut-set, we give a succession of concepts, namely (λ,α)-homomorphism, surjective (λ,α)-homomorphism and (λ,α)-isomorphism of intuitionistic fuzzy groups. For each notion, we aim to give an equivalent characterization. In the sense of the so-called (surjective) (λ,α)-homomorphism and (λ,α)-isomorphism, the relationships among the cut-set of the image of the mapping induced by f, the image of the cut-set of an intuitionistic fuzzy set and the cut-set of the field of values of the mapping f will be revealed.

3.1. Definition. Let G_1, G_2 be ordinary groups, $A \in IFG[G_1]$, $B \in IFG[G_2]$ and $f : G_1 \to G_2$ a mapping. Then for any $(\lambda, \alpha) \in L_*$, if f is an homomorphic mapping from $A_{[\lambda,\alpha]}$ to $B_{[\lambda,\alpha]}$, f is called a (λ, α)-*homomorphic mapping* from the intuitionistic fuzzy group A to the intuitionistic fuzzy group B.

3.2. Lemma. *Let X, Y be classical sets, $A \in IFS[X]$, $B \in IFS[Y]$, $f : X \to Y$. For any $(\lambda, \alpha) \in L_*$, we have*

(1) $f(A_{(\lambda,\alpha)}) \subseteq (F_f(A))_{(\lambda,\alpha)}$ *and* $f(A_{[\lambda,\alpha]}) \subseteq (F_f(A))_{[\lambda,\alpha]}$*;*
(2) $(F_f^{-1}(B))_{[\lambda,\alpha]} = f^{-1}(B_{[\lambda,\alpha]})$ *and* $(F_f^{-1}(B))_{(\lambda,\alpha)} = f^{-1}(B_{(\lambda,\alpha)})$*.*

Proof. (1) For arbitrary $y \in f(A_{(\lambda,\alpha)})$, we know that there exists $x \in A_{(\lambda,\alpha)}$ such that $f(x) = y$, which means $x \in f^{-1}(y), \mu_A(x) > \lambda$ and $\nu_A(x) < \alpha$. By Definition 2.6, we have

$$F_f(\mu_A)(y) = \sup_{t \in f^{-1}(y)} \mu_A(t) \geq \mu_A(x) > \lambda$$

and

$$\hat{F}_f(\nu_A)(y) = \inf_{t \in f^{-1}(y)} \nu_A(t) \leq \nu_A(x) < \alpha.$$

This shows that $y \in (F_f(A))_{(\lambda,\alpha)}$, i.e., $f(A_{(\lambda,\alpha)}) \subseteq (F_f(A))_{(\lambda,\alpha)}$. Similarly, we can prove that $f(A_{[\lambda,\alpha]}) \subseteq (F_f(A))_{[\lambda,\alpha]}$.

(2) For every x, $x \in f^{-1}(B_{[\lambda,\alpha]})$ if and only if $f(x) \in B_{[\lambda,\alpha]}$ if and only if $\mu_B(f(x)) \geq \lambda$ and $\nu_B(f(x)) \leq \alpha$. By Definition 2.7, it is easy to see that

$$F_f^{-1}(\mu_B)(x) = \mu_B(f(x)) \geq \lambda, \ F_f^{-1}(\nu_B)(x) = \nu_B(f(x)) \leq \alpha$$

i.e., $x \in (F_f^{-1}(B))_{[\lambda,\alpha]}$. Therefore, $(F_f^{-1}(B))_{[\lambda,\alpha]} = f^{-1}(B_{[\lambda,\alpha]})$. Using a similar method to the above, we can obtain $(F_f^{-1}(B))_{(\lambda,\alpha)} = f^{-1}(B_{(\lambda,\alpha)})$, so we omit the proof. □

3.3. Lemma. *Let X, Y be classical sets, $A \in IFS[X]$, $B \in IFS[Y]$, $f : X \to Y$. Then $(F_f(A))_{[\lambda,\alpha]} = f(A_{[\lambda,\alpha]})$ for any $(\lambda, \alpha) \in L_*$ if and only if for each $y \in Y$ there exists $x_0 \in f^{-1}(y)$ such that $F_f(\mu_A)(y) = \mu_A(x_0)$ and $\hat{F}_f(\nu_A)(y) = \nu_A(x_0)$.*

Proof. Necessity. For arbitrary $y \in Y$, let $F_f(\mu_A)(y) = \lambda$ and $\hat{F}_f(\nu_A)(y) = \alpha$. Then $(\lambda, \alpha) \in L_*$, and $y \in (F_f(A))_{[\lambda,\alpha]} = f(A_{[\lambda,\alpha]})$. It follows that there exists $x_0 \in A_{[\lambda,\alpha]}$ such that $y = f(x_0)$. Hence, we have $x_0 \in f^{-1}(y)$ which satisfies $\mu_A(x_0) \geq \lambda$ and $\nu_A(x_0) \leq \alpha$. Consequently, we have

$$\mu_A(x_0) \geq F_f(\mu_A)(y) = \sup_{x \in f^{-1}(y)} \mu_A(x) \geq \mu_A(x_0)$$

and

$$\nu_A(x_0) \leq \hat{F}_f(\nu_A)(y) = \inf_{x \in f^{-1}(y)} \nu_A(x) \leq \nu_A(x_0).$$

It is straightforward to verify that $F_f(\mu_A)(y) = \mu_A(x_0)$ and $\hat{F}_f(\nu_A)(y) = \nu_A(x_0)$.

Sufficiency. For arbitrary $(\lambda, \alpha) \in L_*$, by the hypothesis we can derive that $y \in (F_f(A))_{[\lambda,\alpha]}$ if and only if there exists $x_0 \in f^{-1}(y)$ such that $\mu_A(x_0) = F_f(\mu_A)(y) = \sup_{x \in f^{-1}(y)} \mu_A(x) \geq \lambda$ and $\nu_A(x_0) = \hat{F}_f(\nu_A)(y) = \inf_{x \in f^{-1}(y)} \nu_A(x) \leq \alpha$. Conversely, $f(x_0) = y$ and $x_0 \in A_{[\lambda,\alpha]}$, i.e., $y \in f(A_{[\lambda,\alpha]})$. □

3.4. Definition. Let X, Y be classical sets, $f : X \to Y$, $A \in IFS[X]$. Then for every $y \in Y$, if there exists $x_0 \in f^{-1}(y)$ such that $F_f(\mu_A)(y) = \mu_A(x_0)$ and $\hat{F}_f(\nu_A)(y) = \nu_A(x_0)$, then f is said to be *quasi-surjective*.

By Definition 3.4 and Lemma 3.3, we can easily get the following Lemma.

3.5. Lemma. *Let X, Y be classical sets, $f : X \to Y$, $A \in IFS[X]$. Then for any $(\lambda,\alpha) \in L_*$ we have $(F_f(A))_{[\lambda,\alpha]} = f(A_{[\lambda,\alpha]})$ if and only if f is quasi-surjective.*

Based on the above preparation, we give the theorem as follows.

3.6. Theorem. *Let G_1, G_2 be ordinary groups, $A \in IFG[G_1]$, $B \in IFG[G_2]$ and $f : G_1 \to G_2$ quasi-surjective. Then f is a (λ,α)-homomorphic mapping from A to B if and only if f is an homomorphic mapping from G_1 to G_2, and $(F_f(A))_{[\lambda,\alpha]} \subseteq B_{[\lambda,\alpha]}$ for arbitrary $(\lambda,\alpha) \in L_*$.*

Proof. Necessity. Since f is a (λ,α)-homomorphic mapping from A to B, for every $(\lambda,\alpha) \in L_*$, then using Theorem 3.6, we can infer that f is an homomorphic mapping from $A_{[\lambda,\alpha]}$ to $B_{[\lambda,\alpha]}$. Actually, $G_1 = A_{[0,1]}, G_2 = B_{[0,1]}$, thus f is an homomorphic mapping from G_1 to G_2.

As f is quasi-surjective, in the light of Lemma 3.5 and Definition 3.1, we know that

$$(F_f(A))_{[\lambda,\alpha]} = f(A_{[\lambda,\alpha]}) \subseteq B_{[\lambda,\alpha]}.$$

Sufficiency. For all $(\lambda,\alpha) \in L_*$, because f is quasi-surjective, for any $x \in A_{[\lambda,\alpha]} \subseteq G_1$, we have

$$f(x) \in f(A_{[\lambda,\alpha]}) = (F_f(A))_{[\lambda,\alpha]} \subseteq B_{[\lambda,\alpha]}.$$

Therefore, f is a mapping from $A_{[\lambda,\alpha]}$ to $B_{[\lambda,\alpha]}$. Since f is an homomorphic mapping from G_1 to G_2, then $f(xy) = f(x)f(y)$ holds for arbitrary $x, y \in A_{[\lambda,\alpha]} \subseteq G_1$, where $f(x), f(y) \in B_{[\lambda,\alpha]}$. According to Theorem 2.11, this indicates that $B_{[\lambda,\alpha]}$ is a subgroup of G_2. Therefore, $f(x)f(y) \in B_{[\lambda,\alpha]}$, that is to say, f preserves the operation.

Synthesizing the above discussion, we can prove that f is an homomorphic mapping from $A_{[\lambda,\alpha]}$ to $B_{[\lambda,\alpha]}$. Hence, by Definition 3.1, we obtain that f is a (λ,α)-homomorphic mapping from A to B. □

3.7. Definition. Let G_1, G_2 be ordinary groups, $A \in IFG[G_1]$, $B \in IFG[G_2]$, $f : X \to Y$. Then for any $(\lambda,\alpha) \in L_*$, if f is a surjective homomorphic mapping from $A_{[\lambda,\alpha]}$ to $B_{[\lambda,\alpha]}$, then f is called a *surjective (λ,α)-homomorphic mapping* from the intuitionistic fuzzy group A the to intuitionistic fuzzy group B, moreover A and B are said to be (λ,α)-*homomorphic with respect to f*.

3.8. Theorem. *Let G_1, G_2 be ordinary groups, $A \in IFG[G_1]$, $B \in IFG[G_2]$, $f : G_1 \to G_2$ with f quasi-surjective. Then f is a surjective (λ,α)-homomorphic mapping from A to B if and only if f is a surjective homomorphic mapping from G_1 to G_2 with $f(A_{[\lambda,\alpha]}) = B_{[\lambda,\alpha]}$ for arbitrary $(\lambda,\alpha) \in L_*$.*

Proof. Necessity. By hypothesis and Definition 3.7, for all $(\lambda,\alpha) \in L_*$ we get that f is a surjective homomorphic mapping from $A_{[\lambda,\alpha]}$ to $B_{[\lambda,\alpha]}$. Noting that $G_1 = A_{[0,1]}$ and $G_2 = B_{[0,1]}$, evidently, f is a surjective homomorphic mapping from G_1 to G_2. Of course, we have $f(A_{[\lambda,\alpha]}) \subseteq B_{[\lambda,\alpha]}$.

On the other hand, $B_{[\lambda,\alpha]} \subseteq f(A_{[\lambda,\alpha]})$ is obvious. And so $f(A_{[\lambda,\alpha]}) = B_{[\lambda,\alpha]}$.

Sufficiency. For any $(\lambda,\alpha) \in L_*$ and $y \in B_{[\lambda,\alpha]}$ then $f(A_{[\lambda,\alpha]}) = B_{[\lambda,\alpha]}$ implies that there exists $x \in A_{[\lambda,\alpha]}$ such that $f(x) = y$, i.e., f is a surjection from $A_{[\lambda,\alpha]}$ to $B_{[\lambda,\alpha]}$.

The proof that f preserves the operation is parallel to that in the proof of sufficiency in Theorem 3.1, so we omit it. Hence, For any $(\lambda,\alpha) \in L_*$, it follows that f is a surjective homomorphic mapping from $A_{[\lambda,\alpha]}$ to $B_{[\lambda,\alpha]}$. □

3.9. Definition. Let G_1, G_2 be ordinary groups, $A \in IFG[G_1]$, $B \in IFG[G_2]$ and $f : X \to Y$. Then for arbitrary $(\lambda, \alpha) \in L_*$, if f is an isomorphic mapping from $A_{[\lambda,\alpha]}$ to $B_{[\lambda,\alpha]}$, then f is called a (λ, α)-*isomorphic mapping* from the intuitionistic fuzzy group A to the intuitionistic fuzzy group B, moreover A and B are said to be (λ, α)-*isomorphic* with respect to f.

3.10. Theorem. *Let G_1, G_2 be ordinary groups, $A \in IFG[G_1]$, $B \in IFG[G_2]$, $f : G_1 \to G_2$ with f be quasi-surjective. Then f is a (λ, α)-isomorphic mapping from A to B if and only if f is an isomorphic mapping from G_1 to G_2 with $f(A_{[\lambda,\alpha]}) = B_{[\lambda,\alpha]}$ for each $(\lambda, \alpha) \in L_*$.*

Proof. The proof follows by combining Theorem 3.6 and Theorem 3.8. We omit the details. □

4. (λ, α)-homomorphic properties

This section focuses on (λ, α)-homomorphic properties of intuitionistic fuzzy group. Firstly, we give the definition of a L_*-pre-surjective mapping f and present a characterization. We then introduce the concept of an intuitionistic L_*-nested set. Finally, we summarize some homomorphic properties with respect to an intuitionistic fuzzy group by taking advantage of these concepts.

4.1. Definition. Let X, Y be classical sets, $f : X \to Y$ a mapping, and $A \in IFS[X]$. If for every $(\lambda_1, \alpha_1), (\lambda_2, \alpha_2) \in L_*$ with $(\lambda_1, \alpha_1) < (\lambda_2, \alpha_2)$ we have $(F_f(A))_{[\lambda_2,\alpha_2]} \subseteq f(A_{[\lambda_1,\alpha_1]})$, then f is called L_*-*pre-surjective*, or it is said that f *posses the L_*-pre-surjective property.*

4.2. Theorem. *Let $f : X \to Y$ be a mapping, $A \in IFS[X]$, $(\lambda_1, \alpha_1), (\lambda_2, \alpha_2) \in L_*$ satisfy $(\lambda_1, \alpha_1) < (\lambda_2, \alpha_2)$. Then f is L_*-pre-surjective if and only if for every $y \in (F_f(A))_{[\lambda_2,\alpha_2]}$ there exists $x \in A_{[\lambda_1,\alpha_1]}$ such that $f(x) = y$.*

Proof. By the hypothesis and Definition 4.1, it is not hard to see that f is L_*-pre-surjective iff $(F_f(A))_{[\lambda_2,\alpha_2]} \subseteq f(A_{[\lambda_1,\alpha_1]})$ iff $y \in (F_f(A))_{[\lambda_2,\alpha_2]}$ implies $y \in f(A_{[\lambda_1,\alpha_1]})$ iff there exists $x \in A_{[\lambda_1,\alpha_1]}$ such that $f(x) = y$. □

4.3. Definition. Let $H : L_* \to P(X)$, $(\lambda, \alpha) \mapsto H(\lambda, \alpha) \in P(X)$ be a mapping, T an index set. Then the mapping H is called an *intuitionistic L_*-nested set on X*, if the following conditions are satisfied

(1) $(\lambda_1, \alpha_1) < (\lambda_2, \alpha_2)$ implies $H(\lambda_2, \alpha_2) \subseteq H(\lambda_1, \alpha_1)$;
(2) $\bigcap_{t\in T} H(\lambda_t, \alpha_t) \subseteq \bigcap\{H(\lambda, \alpha) : (\lambda, \alpha) < (\sup_{t\in T} \lambda_t,\ \inf_{t\in T} \alpha_t)\}$.

For simplicity, the set of all intuitionistic L_*-nested sets on X is written as $N_{L_*}(X)$.

4.4. Theorem. *Let $f : X \to Y$ be a mapping, $A \in IFS[X]$, and for all $(\lambda, \alpha) \in L_*$, let $H(\lambda, \alpha) = f(A_{[\lambda,\alpha]})$. Then $H \in N_{L_*}(Y)$ if and only if f is L_*-pre-surjective.*

Proof. Necessity. In order to prove f is L_*-pre-surjective, we need only prove $(F_f(A))_{[\lambda_2,\alpha_2]} \subseteq f(A_{[\lambda_1,\alpha_1]})$, where $(\lambda_1, \alpha_1), (\lambda_2, \alpha_2) \in L_*$ and $(\lambda_1, \alpha_1) < (\lambda_2, \alpha_2)$.

In fact, for any $y \in (F_f(A))_{[\lambda_2,\alpha_2]}$, we easily find

$$F_f(\mu_A)(y) = \sup_{x\in f^{-1}(y)} \mu_A(x) \geq \lambda_2, \quad \hat{F}_f(\nu_A)(y) = \inf_{x\in f^{-1}(y)} \nu_A(x) \leq \alpha_2.$$

Putting $T = \{t \in X : f(t) = y\}$ and $\mu_A(t) = \lambda_t, \nu_A(t) = \alpha_t$, we have

$$\sup_{t\in T} \lambda_t = F_f(\mu_A)(y) \geq \lambda_2, \quad \inf_{t\in T} \alpha_t = \hat{F}_f(\nu_A)(y) \leq \alpha_2.$$

For $t \in T$, we have $t \in A_{[\lambda_t,\alpha_t]}$ with $y = f(t)$, thus $y \in f(A_{[\lambda_t,\alpha_t]})$.

Since the mapping H is an intuitionistic L_*-nested set on Y, by Definition 4.3, it is straightforward to get that

$$y \in \bigcap_{t\in T} f(A_{[\lambda_t,\alpha_t]}) \subseteq \{f(A_{[\lambda,\alpha]}) : \sup_{t\in T}\lambda_t > \lambda,\ \inf_{t\in T}\alpha_t < \alpha\}.$$

Considering $(\lambda_1,\alpha_1) < (\lambda_2,\alpha_2) \le (\sup_{t\in T}\lambda_t, \inf_{t\in T}\alpha_t)$, we infer that $y \in f(A_{[\lambda_1,\alpha_1]})$, which means that $(F_f(A))_{[\lambda_2,\alpha_2]} \subseteq f(A_{[\lambda_1,\alpha_1]})$. In the light of Definition 4.1, that is to say that f posses L_*-pre-surjective property.

Sufficiency. On the one hand, whenever $(\lambda_1,\alpha_1) < (\lambda_2,\alpha_2)$, by using Property 2.10 and Theorem 2.12, $f(A_{[\lambda_2,\alpha_2]}) \subseteq f(A_{[\lambda_1,\alpha_1]})$ is clear.

On the other hand, for any $y \in \bigcap_{t\in T} f(A_{[\lambda_t,\alpha_t]})$ there exists $x_t \in A_{[\lambda_t,\alpha_t]}$ such that $f(x_t) = y$. Consequently, for arbitrary $t \in T$, we have

$$F_f(\mu_A)(y) \ge \sup_{t\in T}\mu_A(x_t) \ge \sup_{t\in T}\lambda_t, \quad \hat{F}_f(\nu_A)(y) \le \inf_{t\in T}\nu_A(x_t) \le \inf_{t\in T}\alpha_t.$$

Therefore, $y \in (F_f(A))_{[\sup_{t\in T}\lambda_t, \inf_{t\in T}\alpha_t]}$.

As f is L_*-pre-surjective, for $(\lambda,\alpha) < (\sup_{t\in T}\lambda_t, \inf_{t\in T}\alpha_t)$, in accordance with Theorem 4.2, we can deduce that there exists $x \in A_{[\lambda,\alpha]}$ such that $f(x) = y$. This implies $y \in f(A_{[\lambda,\alpha]})$, thus $y \in \bigcap\{f(A_{[\lambda,\alpha]}) : (\lambda,\alpha) < (\sup_{t\in T}\lambda_t, \inf_{t\in T}\alpha_t)\}$. It is not hard to see that

$$\bigcap_{t\in T} f(A_{[\lambda_t,\alpha_t]}) \subseteq \bigcap\{f(A_{[\lambda,\alpha]}) : (\lambda,\alpha) < (\sup_{t\in T}\lambda_t, \inf_{t\in T}\alpha_t)\}.$$

And so, making use of Definition 4.3, this indicates that $H \in N_{L_*}(X)$. □

4.5. Corollary. *Let $f : X \to Y$ be a mapping, $A \in IFS[X]$. For every $(\lambda,\alpha) \in L_*$, we define $H(\lambda,\alpha) = f(A_{[\lambda,\alpha]})$, then $H \in N_{L_*}(X)$ iff $(F_f(A))_{[\lambda,\alpha]} \subseteq f(A_{[\lambda,\alpha]})$.*

Proof. Take any $(\lambda_1,\alpha_1), (\lambda_2,\alpha_2) \in L_*$ with $(\lambda_1,\alpha_1) < (\lambda_2,\alpha_2) \in L_*$. By Property 2.10 and Theorem 2.12, we obtain $f(A_{[\lambda_2,\alpha_2]}) \subseteq f(A_{[\lambda_1,\alpha_1]})$.

Combined with $(F_f(A))_{[\lambda_2,\alpha_2]} \subseteq f(A_{[\lambda_2,\alpha_2]})$, we have $(F_f(A))_{[\lambda_2,\alpha_2]} \subseteq f(A_{[\lambda_1,\alpha_1]})$, i.e., f is L_*-pre-surjective. Thus, utilizing Theorem 4.4, we can infer the mapping H is an intuitionistic L_*-nested set on X, i.e., $H \in N_{L_*}(X)$.

We may prove the converse by adopting a similar method to the above. □

4.6. Corollary. *Let $f : X \to Y$ be a quasi-surjective mapping, $A \in IFS[X]$, and for any $(\lambda,\alpha) \in L_*$ let $H(\lambda,\alpha) = f(A_{[\lambda,\alpha]})$. Then $H \in N_{L_*}(X)$.*

Proof. Easy in the light of Lemma 3.5 and Corollary 4.5, and we omit the details. □

4.7. Lemma. *Let X be a classical set, $A \in IFS[X]$. Then for arbitrary $(\lambda_t,\alpha_t) \in L_*, t \in T$, $\bigcap_{t\in T} A_{[\lambda_t,\alpha_t]} = A_{[\sup_{t\in T}\lambda_t, \inf_{t\in T}\alpha_t]}$.*

Proof. On one hand, for any $x \in \bigcap_{t\in T} A_{[\lambda_t,\alpha_t]}$ we have $x \in A_{[\lambda_t,\alpha_t]}$ for all $t \in T$, furthermore, we get $\mu_A(x) \ge \lambda_t$ and $\nu_A(x) \le \alpha_t$. Consequently,

$$\mu_A(x) = \sup_{t\in T}\mu_A(x) \ge \sup_{t\in T}\lambda_t \text{ and } \nu_A(x) = \inf_{t\in T}\nu_A(x) \le \inf_{t\in T}\alpha_t.$$

Furthermore, $x \in A_{[\sup_{t\in T}\lambda_t, \inf_{t\in T}\alpha_t]}$, i.e.,

$$\bigcap_{t\in T} A_{[\lambda_t,\alpha_t]} \subseteq A_{[\sup_{t\in T}\lambda_t, \inf_{t\in T}\alpha_t]}.$$

On the other hand, for all $x \in A_{[\sup_{t\in T}\lambda_t, \inf_{t\in T}\alpha_t]}$ and $t \in T$, we can derive that

$$\mu_A(x) \geq \sup_{t\in T}\lambda_t \geq \lambda_t \text{ and } \nu_A(x) \leq \inf_{t\in T}\alpha_t \leq \alpha_t.$$

This shows that $\mu_A(x) \geq \lambda_t$ and $\nu_A(x) \geq \alpha_t$, i.e., $x \in \bigcap_{t\in T} A_{[\lambda_t,\alpha_t]}$. Therefore, $\bigcap_{t\in T} A_{[\lambda_t,\alpha_t]} = A_{[\sup_{t\in T}\lambda_t, \inf_{t\in T}\alpha_t]}$. □

4.8. Theorem. *Let G_1, G_2 be ordinary groups, $f : G_1 \to G_2$ a mapping with f quasi-surjective, $A \in IFG[G_1]$, $B \in IFG[G_2]$. Then the following three conclusions hold.*

(1) *If f is a (λ,α)-homomorphic mapping from A to B with the L_*-pre-surjective property, then $F_f(A) \in IFG[G_2]$, and f is also a (λ,α)-homomorphic mapping from A to $F_f(A)$.*

(2) *If f is a (λ,α)-surjective homomorphic mapping from A to B, then $F_f^{-1}(B) \in IFG[G_1]$, and f is also a surjective homomorphic mapping from $F_f^{-1}(B)$ to B.*

(3) *If f is a (λ,α)-isomorphic mapping from A to B, and f is L_*-pre-surjective, then*
   - (i) *$F_f(A) \in IFG[G_2]$, and f is a (λ,α)-isomorphic mapping from A to $F_f(A)$;*
   - (ii) *$F_f^{-1}(B) \in IFG[G_1]$, and f is also a (λ,α)-isomorphic mapping from $F_f^{-1}(B)$ to B.*

Proof. (1) On the one hand, as f is L_*-pre-surjective, it follows that $H \in N_{L_*}(G_2)$, where $H(\lambda,\alpha) = f(A_{[\lambda,\alpha]})$ for every $(\lambda,\alpha) \in L_*$. Since f is quasi-surjective, by Lemma 3.5, for arbitrary $(\lambda,\alpha) \in L_*$, we have $(F_f(A))_{[\lambda,\alpha]} = f(A_{[\lambda,\alpha]})$. And so, $F_f(A) \in IFG[G_2]$.

In addition, for $A \in IFG[G_1]$, according to Theorem 2.11, we infer that $A_{[\lambda,\alpha]}$ is a subgroup of X. Thus, utilizing the homomorphic property of classical groups, we can conclude that $f(A_{[\lambda,\alpha]})$ is a subgroup of G_2. It is easy to verify that $(F_f(A))_{[\lambda,\alpha]}$ is a subgroup of G_2, further, from Theorem 2.11, we obtain $F_f(A) \in IFG[G_2]$.

On the other hand, since f is a (λ,α)-homomorphic mapping from A to B, we see that f is an homomorphic mapping from $A_{[\lambda,\alpha]}$ to $B_{[\lambda,\alpha]}$. Evidently, for every $x, y \in A_{[\lambda,\alpha]}$, then $f(x), f(y) \in f(A_{[\lambda,\alpha]})$. Combining with Theorem 3.6, we have

$$f(A_{[\lambda,\alpha]}) = (F_f(A))_{[\lambda,\alpha]} \subseteq B_{[\lambda,\alpha]}.$$

Now, as f is a homomorphism between classical groups, $f(xy) = f(x)f(y)$. Because $(F_f(A))_{[\lambda,\alpha]}$ is a subgroup of G_2, this implies $f(x)f(y) \in (F_f(A))_{[\lambda,\alpha]}$, i.e., f preserves the operation. Consequently, f is a (λ,α)-homomorphic mapping from A to $F_f(A)$.

(2) For any $(\lambda,\alpha) \in L_*$, setting $H(\lambda,\alpha) = f^{-1}(B_{[\lambda,\alpha]})$, we prove $H \in N_{L_*}(X)$.

Firstly, for any $(\lambda_1,\alpha_1), (\lambda_2,\alpha_2) \in L_*$ with $(\lambda_1,\alpha_1) < (\lambda_2,\alpha_2) \in L_*$, by Property 2.10 and Theorem 2.13, we get $f^{-1}(B_{[\lambda_2,\alpha_2]}) \subseteq f^{-1}(B_{[\lambda_1,\alpha_1]})$.

Secondly, applying Lemma 3.2 and the properties of cut-sets of intuitionistic fuzzy sets over and over again, we can summarize that

$$\bigcap_{t\in T} f^{-1}(B_{[\lambda_t,\alpha_t]}) = \bigcap_{t\in T} (F_f^{-1}(B))_{[\lambda_t,\alpha_t]}.$$

By means of Lemma 4.7, it follows that

$$\bigcap_{t\in T} (F_f^{-1}(B))_{[\lambda_t,\alpha_t]} = (F_f^{-1}(B))_{[\sup_{t\in T}\lambda_t, \inf_{t\in T}\alpha_t]}.$$

Since $(F_f^{-1}(B))_{[\sup_{t\in T}\lambda_t, \inf_{t\in T}\alpha_t]} = f^{-1}(B_{[\sup_{t\in T}\lambda_t, \inf_{t\in T}\alpha_t]})$, it is straightforward to see that

$$\bigcap_{t\in T} f^{-1}(B_{[\lambda_t,\alpha_t]}) \subseteq \{f^{-1}(B_{[\lambda,\alpha]}) : (\lambda,\alpha) < (\sup_{t\in T}\lambda_t, \inf_{t\in T}\alpha_t)\}.$$

Consequently, $H \in N_{L_*}(G_1)$. By Lemma 3.2, $(F_f^{-1}(B))_{[\lambda,\alpha]} = f^{-1}(B_{[\lambda,\alpha]})$. Thus, $F_f^{-1}(B) \in IFS[X]$.

Without loss of generality, for every $(\lambda, \alpha) \in L_*$, let $F_f^{-1}(B)_{[\lambda,\alpha]} \neq \emptyset$. Then, for any $x, y \in (F_f^{-1}(B))_{[\lambda,\alpha]} = f^{-1}(B_{[\lambda,\alpha]})$, there exists $x_0, y_0 \in B_{[\lambda,\alpha]}$ such that $f(x) = x_0$ and $f(y) = y_0$.

As f is a (λ, α)-homomorphic mapping from A to B, and $B_{[\lambda,\alpha]}$ is a subgroup of G_2, we infer immediately that

$$f(xy^{-1}) = f(x)(f(y))^{-1} = x_0 y_0^{-1} \in B_{[\lambda,\alpha]}.$$

This means that

$$xy^{-1} \in f^{-1}(B_{[\lambda,\alpha]}) = (F_f^{-1}(B))_{[\lambda,\alpha]},$$

and so, $(F_f^{-1}(B))_{[\lambda,\alpha]}$ is a subgroup of G_1. Furthermore, $F_f^{-1}(B) \in IFG[G_1]$.

Since f is a surjective (λ, α)-homomorphic mapping from A to B, from Definition 3.7, we know that f is a surjective homomorphic mapping from $A_{[\lambda,\alpha]}$ to $B_{[\lambda,\alpha]}$. For all $x, y \in (F_f^{-1}(B))_{[\lambda,\alpha]}$, notice that $B \in IFG[G_2]$, $f(xy) = f(x)f(y) \in B_{[\lambda,\alpha]}$. Hence, f is a surjective (λ, α)-homomorphic mapping from $F_f^{-1}(B)$ to B.

(3) The conditions in (3) bring about two cases, which can be proved using a similar method. We omit the details. □

5. Conclusions

The concepts of (λ, α)-homomorphism and (λ, α)-isomorphism between two intuitionistic fuzzy groups are introduced, and then some important results are obtained. Finally, we summarize some homomorphic properties with respect to intuitionistic fuzzy groups by means of intuitionistic L_*-nested sets. It follows that the explorations into the theory of fuzzy groups are becoming deeper and more extensive. There is no doubt that these achievements have an important effect on enriching the theory of fuzzy groups. Consequently, we may say that (λ, α)-homomorphisms and (λ, α)-isomorphisms could be applied in some computational technical problems in the future.

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