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ON THE VALUES OF SOME GENERALIZED LACUNARY POWER SERIES WITH ALGEBRAIC COEFFICIENTS FOR LIOUVILLE NUMBER ARGUMENTS[†]

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Abstract

In this work, it is shown that under certain conditions, the values of some generalized lacunary power series with algebraic coefficients from a certain algebraic number field K of degree m for Liouville number arguments belong to either the algebraic number field K or $\bigcup_{i=1}^{m} U_i$ in Mahler's classification of the complex numbers.

Keywords: Generalized lacunary power series, U-numbers in Mahler's classification.

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1. Introduction

A power series $F(z) = \sum_{h=0}^{\infty} c_h z^h$ $(c_h \in \mathbb{C},\ h = 0, 1, 2, \dots)$ with a positive radius of convergence, satisfying the following conditions

$$\begin{cases} c_h = 0, & r_n < h < s_n\ (n = 1, 2, 3, \dots), \\ c_h \neq 0, & h = r_n\ (n = 1, 2, 3, \dots), \\ c_h \neq 0, & h = s_n\ (n = 0, 1, 2, \dots), \end{cases}$$

where $\{s_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are two infinite sequences of non-negative rational integers with

$$0 = s_0 \leq r_1 < s_1 \leq r_2 < s_2 \leq r_3 < s_3 \leq \dots,\ \lim_{n\to\infty} \frac{s_n}{r_n} = \infty,$$

is called a generalized lacunary power series.

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First, Mahler [9], in 1965, investigated a class of generalized lacunary power series with rational integral coefficients and gave a necessary and sufficient condition that these series take transcendental values for non-zero algebraic number arguments. Later, Braune [1], in 1977, obtained further results for some generalized lacunary power series with algebraic coefficients.

Zeren [15], in 1988, considered certain generalized lacunary power series with algebraic coefficients from a certain algebraic number field and showed that under some conditions these series take values belonging to the subclass U_t in Mahler's classification of complex numbers, where t denotes a natural number (recall that natural number means positive rational integer) dependent on the given series and the argument, for non-zero algebraic number arguments.

In the present work, we show that the generalized lacunary power series with algebraic coefficients treated by Zeren [15], under certain conditions, take values belonging to either a certain algebraic number field or $\bigcup_{i=1}^{m} U_i$ in Mahler's classification of the complex numbers, where m denotes the degree of the algebraic number field to which the coefficients of the given series belong, for some Liouville number arguments.

In [4] we considered some non-generalized lacunary power series with algebraic coefficients from a certain algebraic number field K of degree m, and showed that under certain conditions these series take values belonging to either the algebraic number field K or $\bigcup_{i=1}^{m} U_i$ in Mahler's classification of the complex numbers for some Liouville number arguments. Hence, Theorem 3.1 can be regarded as an extension of [4] to generalized lacunary power series.

2. Background

Mahler [8], in 1932, divided the complex numbers into four classes and called numbers in these classes A-numbers, S-numbers, T-numbers, and U-numbers as follows.

We shall be concerned with polynomials $P(z) = a_n z^n + \cdots + a_0$ with rational integral coefficients. The height $H(P)$ of P is defined by $H(P) = \max(|a_n|, \ldots, |a_0|)$, and we shall denote the degree of P by $\deg(P)$.

Given a complex number ξ and natural numbers n and H, Mahler [8] puts

$$w_n(H, \xi) = \min_{\substack{deg(P) \leq n \\ H(P) \leq H \\ P(\xi) \neq 0}} |P(\xi)|.$$

The polynomial $P(z) \equiv 1$ is one of the polynomials which lie in the minimum, and so we have $0 < w_n(H, \xi) \leq 1$. $w_n(H, \xi)$ is a non-increasing function of both n and H. Next, Mahler [8] puts

$$w_n(\xi) = \limsup_{H \to \infty} \frac{-\log w_n(H, \xi)}{\log H} \text{ and } w(\xi) = \limsup_{n \to \infty} \frac{w_n(\xi)}{n}.$$

$w_n(\xi)$ is a non-decreasing function of n. Furthermore, the inequalities $0 \leq w_n(\xi) \leq \infty$ and $0 \leq w(\xi) \leq \infty$ hold. If $w_n(\xi) = \infty$ for some integer n, let $\mu(\xi)$ be the smallest of such integers. In this case, we have $w_n(\xi) < \infty$ for $n < \mu(\xi)$ and $w_n(\xi) = \infty$ for $n \geq \mu(\xi)$. If $w_n(\xi) < \infty$ for every n, put $\mu(\xi) = \infty$. So $\mu(\xi)$ and $w(\xi)$ are uniquely determined and are never finite simultaneously, for the finiteness of $\mu(\xi)$ implies that there is an $n < \infty$ such that $w_n(\xi) = \infty$, whence $w(\xi) = \infty$. Therefore there are the following four possibilities for ξ, and ξ is called

An A-number if $w(\xi) = 0,\ \mu(\xi) = \infty$,

An S-number if $0 < w(\xi) < \infty,\ \mu(\xi) = \infty$,

A T-number if $w(\xi) = \infty,\ \mu(\xi) = \infty$,

A U-number if $w(\xi) = \infty,\ \mu(\xi) < \infty$.

Every complex number ξ is of precisely one of these four types. The A-numbers are precisely the algebraic numbers (see Schneider [11, pp. 68-69]). So the transcendental numbers are distributed into the three disjoint classes S, T, U. Let ξ be a U-number such that $\mu(\xi) = m$, and let U_m denote the set of all such numbers, i.e. $U_m = \{\xi \in U : \mu(\xi) = m\}$. Obviously, the set U_m $(m = 1, 2, 3, \ldots)$ is a subclass of U, and U is the union of all the disjoint sets U_m. LeVeque [6] showed that U_m is not empty for any $m \geq 1$.

Koksma [5], in 1939, set up another classification of the complex numbers. He divided the complex numbers into four classes A^*, S^*, T^*, U^*, as follows.

Suppose that α is an algebraic number and $P(z)$ is the minimal defining polynomial of α such that its coefficients are rational integers, relatively prime, and its highest coefficient is positive. Then the height $H(\alpha)$ of α is defined by $H(\alpha) = H(P)$, and the degree $\deg(\alpha)$ of α is defined as the degree of P.

Given a complex number ξ and natural numbers n and H, Koksma [5] puts

$$w_n^*(H, \xi) = \min_{\substack{\alpha \text{ is algebraic} \\ \deg(\alpha) \leq n \\ H(\alpha) \leq H \\ \alpha \neq \xi}} |\xi - \alpha|,$$

$$w_n^*(\xi) = \limsup_{H \to \infty} \frac{-\log(H w_n^*(H, \xi))}{\log H}, \text{ and } w^*(\xi) = \limsup_{n \to \infty} \frac{w_n^*(\xi)}{n}.$$

$w_n^*(H, \xi)$ is a non-increasing function of both n and H, and so $w_n^*(\xi)$ is a non-decreasing function of n. The functions $w_n^*(\xi)$ and $w^*(\xi)$ satisfy the respective inequalities $0 \leq w_n^*(\xi) \leq \infty$ and $0 \leq w^*(\xi) \leq \infty$. If $w_n^*(\xi) = \infty$ for some integer n, let $\mu^*(\xi)$ be the smallest of such integers. In this case, we have $w_n^*(\xi) < \infty$ for $n < \mu^*(\xi)$ and $w_n^*(\xi) = \infty$ for $n \geq \mu^*(\xi)$. If $w_n^*(\xi) < \infty$ for every n, put $\mu^*(\xi) = \infty$. So $\mu^*(\xi)$ and $w^*(\xi)$ are uniquely determined and are never finite simultaneously. Therefore there are the following four possibilities for ξ. Then, ξ is called

An A^*-number if $w^*(\xi) = 0,\ \mu^*(\xi) = \infty$,

An S^*-number if $0 < w^*(\xi) < \infty,\ \mu^*(\xi) = \infty$,

A T^*-number if $w^*(\xi) = \infty,\ \mu^*(\xi) = \infty$,

A U^*-number if $w^*(\xi) = \infty,\ \mu^*(\xi) < \infty$.

Every complex number ξ is of precisely one of these four types. Hence, the complex numbers are distributed into the four disjoint classes A^*, S^*, T^*, U^*. Let ξ be a U^*-number such that $\mu^*(\xi) = m$, and let U_m^* denote the set of all such numbers, i.e. $U_m^* = \{\xi \in U^* : \mu^*(\xi) = m\}$. Obviously, the set U_m^* $(m = 1, 2, 3, \ldots)$ is a subclass of U^*, and U^* is the union of all the disjoint sets U_m^*.

Koksma's classification of the complex numbers is equivalent to Mahler's, i.e. A^*-, S^*-, T^*-, U^*-numbers are the same as A-, S-, T-, U-numbers, respectively. Moreover, $U_m = U_m^*$ $(m = 1, 2, 3, \ldots)$ holds (see Schneider [11] and Wirsing [12]).

A real number ξ is called a Liouville number if to each natural number n there exists a rational number p_n/q_n $(p_n, q_n \in \mathbb{Z})$ such that the inequalities

$$q_n > 1,\ 0 < \left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{q_n^n}$$

hold. We deduce from the definition that a Liouville number is an irrational number. The set of Liouville numbers is identical with the subclass U_1 in Mahler's classification (for more information about Liouville numbers see Perron [10, pp. 178-190] and Schneider [11, Kapitel I]).

We need the following lemma in order to prove the main result of this paper.

2.1. Lemma. (İçen [3]) *Let $\alpha_1, \ldots, \alpha_k$ $(k \geq 1)$ be algebraic numbers which belong to an algebraic number field K of degree m, and let $F(y, x_1, \ldots, x_k)$ be a polynomial with rational integral coefficients and with degree at least 1 in y. If η is any algebraic number such that $F(\eta, \alpha_1, \ldots, \alpha_k) = 0$, then*

$$\deg(\eta) \leq dm$$

and

$$H(\eta) \leq 3^{2dm+(l_1+\cdots+l_k)m} H^m H(\alpha_1)^{l_1 m} \cdots H(\alpha_k)^{l_k m},$$

where H is the height of the polynomial F, d is the degree of F in y, and l_i $(i = 1, \ldots, k)$ is the degree of F in x_i $(i = 1, \ldots, k)$.

3. The main result

3.1. Theorem. *Let $K = \mathbb{Q}(\theta)$ be an algebraic number field of degree m, and let*

$$F(z) = \sum_{h=0}^{\infty} c_h z^h \ (c_h \in K,\ h = 0, 1, 2, \ldots)$$

be a power series which satisfies the following conditions:

$$(3.1)\qquad \begin{cases} c_h = 0, & r_n < h < s_n\ (n = 1, 2, 3, \ldots), \\ c_h \neq 0, & h = r_n\ (n = 1, 2, 3, \ldots), \\ c_h \neq 0, & h = s_n\ (n = 0, 1, 2, \ldots), \end{cases} .$$

where $\{s_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are two infinite sequences of non-negative rational integers with

$$(3.2)\qquad 0 = s_0 < r_1 < s_1 \leq r_2 < s_2 \leq r_3 < s_3 \leq \cdots,$$

$$(3.3)\qquad \lim_{n\to\infty} \frac{s_n}{r_n} = \infty.$$

Suppose that the radius of convergence R of the series $\sum_{h=0}^{\infty} |\overline{c_h}|\, z^h$ † is positive (R may be finite or infinite), and

$$(3.4)\qquad \limsup_{h\to\infty} \frac{\log A_h}{h} < \infty \ (A_h = [a_0, a_1, \ldots, a_h],\ h = 1, 2, 3, \ldots),\ ‡$$

where a_h $(h = 0, 1, 2, \ldots)$ is a suitable natural number such that $a_h c_h$ $(h = 0, 1, 2, \ldots)$ is an algebraic integer. Moreover, let ξ be a Liouville number such that for $n = 1, 2, 3, \ldots,$

† $|\overline{c_h}|$ denotes the maximum of the absolute values of the conjugates of the algebraic number c_h over $\mathbb{Q}$

‡ $[a_0, a_1, \ldots, a_h]$ denotes the least common multiple of the rational integers $a_0, a_1, \ldots, a_h$.

there are rational integers p_n, q_n *with* $q_n > 1$ *and real numbers* $\omega_n = \frac{s_n}{r_n \log q_n}$ *with* $\lim_{n\to\infty} \omega_n = \infty$ *satisfying the following inequality*

$$\left|\xi - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^{r_n \omega_n}}, \tag{3.5}$$

and let

$$|\xi| < R. \tag{3.6}$$

Then either $F(\xi)$ *is an algebraic number in* K*, or* $F(\xi) \in \bigcup_{i=1}^{m} U_i$.

Proof. By (3.1), the series $F(z)$ can be written, for the complex numbers z at which $F(z)$ converges, as

$$F(z) = \sum_{h=0}^{\infty} c_h z^h = \sum_{k=0}^{\infty} P_k(z), \tag{3.7}$$

where $P_k(z) = \sum_{h=s_k}^{r_{k+1}} c_h z^h$ $(k = 0, 1, 2, \ldots)$.

We shall prove the theorem in four steps.

1) The radius of convergence of the series $F(z) = \sum_{h=0}^{\infty} c_h z^h$ is $\geq R$. For since $|c_h| \leq |\overline{c_h}|$ $(h = 0, 1, 2, \ldots)$, $F(z)$ converges for all the complex numbers z for which the series $\sum_{h=0}^{\infty} |\overline{c_h}|\, z^h$ converges. Then $F(z)$ converges for $z = \xi$.

2) We shall consider the polynomials

$$F_n(z) = \sum_{k=0}^{n-1} P_k(z) \ (n = 1, 2, 3, \ldots). \tag{3.8}$$

Define the algebraic numbers

$$\eta_n = F_n\left(\frac{p_n}{q_n}\right) = \sum_{h=s_0}^{r_n} c_h \left(\frac{p_n}{q_n}\right)^h \in K \ (n = 1, 2, 3, \ldots). \tag{3.9}$$

Since $\eta_n \in K$ $(n = 1, 2, 3, \ldots)$, $\deg(\eta_n) \leq m$ $(n = 1, 2, 3, \ldots)$. By multiplying both sides of the equality

$$\eta_n = \sum_{h=s_0}^{r_n} c_h \left(\frac{p_n}{q_n}\right)^h \ (n = 1, 2, 3, \ldots)$$

by A_{r_n}, we obtain

$$A_{r_n}\eta_n - \sum_{h=s_0}^{r_n} A_{r_n} c_h \left(\frac{p_n}{q_n}\right)^h = 0. \tag{3.10}$$

$A_{r_n} c_h$ $(h = s_0, s_0 + 1, \ldots, r_n)$ is an algebraic integer in the algebraic number field $K = \mathbb{Q}(\theta)$. Moreover, we can assume that the algebraic number $\theta \in K$ given in the hypothesis of the theorem is an algebraic integer and shall do so. Then we have

$$A_{r_n} c_h = \frac{\xi_0^{(h)}}{D} + \frac{\xi_1^{(h)}}{D}\theta + \cdots + \frac{\xi_{m-1}^{(h)}}{D}\theta^{m-1} \ (h = s_0, s_0 + 1, \ldots, r_n), \tag{3.11}$$

where $\xi_0^{(h)}, \xi_1^{(h)}, \ldots, \xi_{m-1}^{(h)}$, and $D = \left|\Delta^2(1, \theta, \ldots, \theta^{m-1})\right| > 0$ are rational integers. Here,

$$\Delta = \Delta(1, \theta, \ldots, \theta^{m-1}) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \theta^{\{1\}} & \theta^{\{2\}} & \ldots & \theta^{\{m\}} \\ \vdots & \vdots & \vdots & \vdots \\ (\theta^{m-1})^{\{1\}} & (\theta^{m-1})^{\{2\}} & \ldots & (\theta^{m-1})^{\{m\}} \end{vmatrix},$$

and $(\theta^i)^{\{1\}}, \ldots, (\theta^i)^{\{m\}}$ $(i = 1, 2, \ldots, m-1)$ denote the field conjugates of θ^i $(i = 1, 2, \ldots, m-1)$ for $K = \mathbb{Q}(\theta)$. Obviously Δ, and so D depend only on θ and the conjugates of θ. From (3.10) and (3.11) we obtain,

$$DA_{r_n}\eta_n - \sum_{h=s_0}^{r_n}\sum_{\mu=0}^{m-1}\xi_\mu^{(h)}\theta^\mu\left(\frac{p_n}{q_n}\right)^h = 0. \tag{3.12}$$

By multiplying both sides of (3.12) by $q_n^{r_n}$, we obtain

$$q_n^{r_n}DA_{r_n}\eta_n - \sum_{h=s_0}^{r_n}\sum_{\mu=0}^{m-1}\xi_\mu^{(h)}q_n^{r_n-h}p_n^h\theta^\mu = 0. \tag{3.13}$$

Then we have

$$L(\eta_n, \theta) = 0, \tag{3.14}$$

where

$$L(y, x) = q_n^{r_n}DA_{r_n}y - \sum_{h=s_0}^{r_n}\sum_{\mu=0}^{m-1}\xi_\mu^{(h)}q_n^{r_n-h}p_n^h x^\mu \tag{3.15}$$

is a polynomial in y, x with rational integral coefficients. Since $q_n^{r_n}DA_{r_n} \neq 0$, the polynomial $L(y, x)$ is of degree 1 in y. The degree of $L(y, x)$ in x is $\leq m-1$. Denote the height of the polynomial $L(y, x)$ by H. Then, by Lemma 2.1, we obtain

$$H(\eta_n) \leq 3^{2m+(m-1)m}H^m H(\theta)^{(m-1)m} = 3^{m(m+1)}H^m H(\theta)^{(m-1)m}. \tag{3.16}$$

Now let us determine an upper bound for the height H of the polynomial $L(y, x)$. Since ξ is a Liouville number, we can assume that $p_n \neq 0$ $(n = 1, 2, 3, \ldots)$, and shall do so. Hence $|p_n| \geq 1$, for p_n is a non-zero rational integer. Also we have $q_n > 1$ $(n = 1, 2, 3, \ldots)$ by the hypothesis of the theorem. Thus it follows from (3.15) that

$$\begin{aligned} H &= \max_{\substack{h=s_0,\ldots,r_n \\ \mu=0,\ldots,m-1}}\left(q_n^{r_n}DA_{r_n}, |\xi_\mu^{(h)}|q_n^{r_n-h}|p_n^h|\right) \\ &\leq q_n^{r_n}|p_n|^{r_n}\max_{\substack{h=s_0,\ldots,r_n \\ \mu=0,\ldots,m-1}}\left(DA_{r_n}, |\xi_\mu^{(h)}|\right). \end{aligned} \tag{3.17}$$

Now we shall determine an upper bound for $|\xi_\mu^{(h)}|$ $(\mu = 0, 1, \ldots, m-1;\ h = s_0, s_0+1, \ldots, r_n)$. Put

$$\delta = DA_{r_n}c_h. \tag{3.18}$$

Note that δ is an algebraic integer in K, since $A_{r_n}c_h$ is an algebraic integer in K and D is a natural number. By (3.11) and (3.18), we have

$$\delta = \xi_0^{(h)} + \xi_1^{(h)}\theta + \cdots + \xi_{m-1}^{(h)}\theta^{m-1}\ \ (h = s_0, s_0+1, \ldots, r_n). \tag{3.19}$$

By using the field conjugates of θ for K in (3.19), we obtain the system of linear equations

$$\begin{cases} \delta^{\{1\}} = \xi_0^{(h)} + \xi_1^{(h)}\theta^{\{1\}} + \cdots + \xi_{m-1}^{(h)}(\theta^{m-1})^{\{1\}} \\ \delta^{\{2\}} = \xi_0^{(h)} + \xi_1^{(h)}\theta^{\{2\}} + \cdots + \xi_{m-1}^{(h)}(\theta^{m-1})^{\{2\}} \\ \vdots \\ \delta^{\{m\}} = \xi_0^{(h)} + \xi_1^{(h)}\theta^{\{m\}} + \cdots + \xi_{m-1}^{(h)}(\theta^{m-1})^{\{m\}} \end{cases} \tag{3.20}$$

in the unknowns $\xi_0^{(h)}, \xi_1^{(h)}, \ldots, \xi_{m-1}^{(h)}$. The coefficient matrix of (3.20) is different from zero, since $\Delta^2(1, \theta, \ldots, \theta^{m-1}) \neq 0$. Thus, the system of linear equations (3.20) has a unique solution which is

$$\xi_\mu^{(h)} = \sum_{j=1}^{m} \frac{\Delta_{\mu j}}{\Delta} \delta^{\{j\}} \ (\mu = 0, 1, \ldots, m-1), \tag{3.21}$$

where $\Delta_{\mu j}$ ($\mu = 0, 1, \ldots, m-1$; $j = 1, 2, \ldots, m$) are complex constants which depend only on θ and the conjugates of θ, are independent of δ, n, and h. It follows from (3.21) that

$$|\xi_\mu^{(h)}| \le \sum_{j=1}^{m} \frac{|\Delta_{\mu j}|}{|\Delta|} |\delta^{\{j\}}| \le \sum_{j=1}^{m} \frac{|\Delta_{\mu j}|}{|\Delta|} |\overline{\delta}| \le |\overline{\delta}| \sum_{\mu=0}^{m-1} \sum_{j=1}^{m} \frac{|\Delta_{\mu j}|}{|\Delta|}. \tag{3.22}$$

However, since, by (3.18), $\delta = DA_{r_n} c_h$, we have

$$|\overline{\delta}| \le DA_{r_n} |\overline{c_h}|. \tag{3.23}$$

By (3.22) and (3.23),

$$\begin{aligned} |\xi_\mu^{(h)}| &\le DA_{r_n} |\overline{c_h}| \sum_{\mu=0}^{m-1} \sum_{j=1}^{m} \frac{|\Delta_{\mu j}|}{|\Delta|} \\ &= \overline{C}(K) A_{r_n} |\overline{c_h}| \quad (\mu = 0, 1, \ldots, m-1;\ h = s_0, \ldots, r_n), \end{aligned} \tag{3.24}$$

where $\overline{C}(K) = D \sum_{\mu=0}^{m-1} \sum_{j=1}^{m} \frac{|\Delta_{\mu j}|}{|\Delta|}$ is a positive real number which depends only on θ and the conjugates of θ, is independent of n, h, and μ. From (3.17) and (3.24) follows

$$\begin{aligned} H &\le q_n^{r_n} |p_n|^{r_n} \max_{h=s_0,\ldots,r_n} \left(DA_{r_n}, \overline{C}(K) A_{r_n} |\overline{c_h}| \right) \\ &\le q_n^{r_n} |p_n|^{r_n} C(K) A_{r_n} \max_{h=s_0,\ldots,r_n} \left(1, |\overline{c_h}|\right), \end{aligned} \tag{3.25}$$

where $C(K) = \max\left(D, \overline{C}(K)\right) \ge 1$ is a real constant which depends only on θ and the conjugates of θ.

Let us choose a real number ρ satisfying the inequality

$$0 < |\xi| < \rho < R. \tag{3.26}$$

$\big($If $R = \infty$, then ρ is chosen as $\rho > |\xi|\big)$. By (3.26), the series $\sum_{h=0}^{\infty} |\overline{c_h}|\, \rho^h$ is convergent. Thus, $\lim_{h\to\infty} |\overline{c_h}|\, \rho^h = 0$, so the sequence $\left\{|\overline{c_h}|\, \rho^h\right\}_{h=0}^{\infty}$ is bounded, and therefore there is a real number $M > 0$ such that

$$|\overline{c_h}| \le \frac{M}{\rho^h} \ (h = 0, 1, 2, \ldots). \tag{3.27}$$

Then

$$\begin{aligned} \max_{h=s_0,\ldots,r_n} \left(1, |\overline{c_h}|\right) &\le \max_{h=s_0,\ldots,r_n} \left(1, \frac{M}{\rho^h}\right) \\ &\le \max_{h=s_0,\ldots,r_n} \left(M_1, \frac{M_1}{\rho^h}\right) = M_1 \left(\max\left(1, \frac{1}{\rho}\right)\right)^{r_n}, \end{aligned} \tag{3.28}$$

where $M_1 = \max(1, M) \ge 1$.

Since $\limsup_{h\to\infty} \frac{\log A_h}{h} < \infty$ by (3.4), the sequence $\left\{\frac{\log A_h}{h}\right\}_{h=1}^{\infty}$ is bounded above. So there exists a real number $\sigma > 0$ such that

$$\frac{\log A_h}{h} \le \sigma \quad (h = 1, 2, 3, \ldots). \tag{3.29}$$

From (3.29), we obtain

$$A_{r_n} \le e^{\sigma r_n} \ (n = 1, 2, 3, \ldots). \tag{3.30}$$

By (3.16), (3.25), (3.28), and (3.30), we have

(3.31) $\quad H(\eta_n) \le e_0^{r_n m} q_n^{r_n m} |p_n|^{r_n m} \ (n = 1, 2, 3, \ldots),$

where $e_0 = 3^{m+1} C(K) e^{\sigma} M_1 \max\left(1, \frac{1}{\rho}\right) H(\theta)^{m-1} > 1$ is a real constant independent of n, r_n, η_n, and q_n. On the other hand, since ξ is a Liouville number, we can assume that $\lim_{n\to\infty} q_n = \infty$, and shall do so. So $e_0 \le q_n$ for sufficiently large n. Hence, by (3.31),

(3.32) $\quad H(\eta_n) \le q_n^{2 r_n m} |p_n|^{r_n m}$

for sufficiently large n. It follows from (3.5) that

(3.33) $$\left|\frac{p_n}{q_n}\right| < |\xi| + 1,$$

and so

(3.34) $\quad |p_n| < q_n \left(|\xi| + 1\right).$

From (3.32), (3.34), and the fact that $|\xi| + 1 \le q_n$ for sufficiently large n, we obtain

(3.35) $\quad H(\eta_n) \le q_n^{e_1 r_n}$

for sufficiently large n, where $e_1 = 4m > 0$.

3) We have

(3.36) $\quad |F(\xi) - \eta_n| \le |F(\xi) - F_n(\xi)| + |F_n(\xi) - \eta_n| \ (n = 1, 2, 3, \ldots).$

Now we shall determine an upper bound for $|F(\xi) - F_n(\xi)|$ and $|F_n(\xi) - \eta_n|$. By (3.8), (3.26), and (3.27), we have

$$\begin{aligned} |F(\xi) - F_n(\xi)| &\le \sum_{h=s_n}^{\infty} |\overline{c_h}|\, |\xi|^h \\ &\le \sum_{h=s_n}^{\infty} \frac{M}{\rho^h} |\xi|^h = M \left(\frac{|\xi|}{\rho}\right)^{s_n} \left(1 + \frac{|\xi|}{\rho} + \left(\frac{|\xi|}{\rho}\right)^2 + \cdots\right), \end{aligned}$$

thus,

(3.37) $$|F(\xi) - F_n(\xi)| \le \frac{e_2}{e_3^{s_n}} \qquad (n = 1, 2, 3, \ldots),$$

where $e_2 = \frac{M}{1 - \frac{|\xi|}{\rho}} > 0$ and $e_3 = \frac{\rho}{|\xi|} > 1$ are real constants independent of n, r_n, s_n, η_n, and q_n. By (3.27),

(3.38) $$|\overline{c_h}| \le \frac{M}{\rho^h} \le M B^h \le M_1 B^h \qquad (h = 0, 1, 2, \ldots),$$

where $B = \max\left(1, \frac{1}{\rho}\right) \ge 1$, $M_1 = \max\left(1, M\right) \ge 1$. From (3.5), (3.8), (3.9), (3.33), (3.38), and the fact that $|\xi| < |\xi| + 1$, it follows

(3.39) $$\begin{aligned} |F_n(\xi) - \eta_n| &\le \sum_{h=s_0}^{r_n} |\overline{c_h}| \left|\xi - \frac{p_n}{q_n}\right| \left(|\xi|^{h-1} + |\xi|^{h-2}\left|\frac{p_n}{q_n}\right| + \cdots + \left|\frac{p_n}{q_n}\right|^{h-1}\right) \\ &\le \sum_{h=s_0}^{r_n} M_1^{r_n} B^{r_n} \frac{1}{q_n^{r_n \omega_n}} r_n \left(|\xi| + 1\right)^{r_n} \\ &\le \frac{1}{q_n^{r_n \omega_n}} \left(r_n + 1\right)^2 M_1^{r_n} B^{r_n} \left(|\xi| + 1\right)^{r_n}. \end{aligned}$$

Since $\lim_{n\to\infty} r_n = \infty$, it follows $\lim_{n\to\infty} \sqrt[r_n]{(r_n+1)^2} = 1$, and so there is a real number $e_4 > 1$ such that

$$(3.40)\quad (r_n+1)^2 \le e_4^{r_n}$$

for sufficiently large n. By (3.39) and (3.40), we have for sufficiently large n

$$(3.41)\quad |F_n(\xi) - \eta_n| \le \frac{e_5^{r_n}}{q_n^{r_n\omega_n}},$$

where $e_5 = e_4 M_1 B\,(|\xi|+1) > 1$. From (3.41) and the fact $e_5 \le q_n$ for sufficiently large n, we obtain

$$(3.42)\quad |F_n(\xi) - \eta_n| \le \frac{1}{q_n^{r_n(\omega_n-1)}}$$

for sufficiently large n. Let λ be a real number such that $0 < \lambda < \min(1, \log e_3)$. Then the inequalities

$$(3.43)\quad \frac{e_2}{e_3^{s_n}} \le \frac{1}{q_n^{r_n(\omega_n-1)\lambda}}$$

and

$$(3.44)\quad \frac{1}{q_n^{r_n(\omega_n-1)}} \le \frac{1}{q_n^{r_n(\omega_n-1)\lambda}}$$

hold for sufficiently large n. It follows from (3.36), (3.37), (3.42), (3.43), and (3.44) that

$$(3.45)\quad |F(\xi) - \eta_n| \le \frac{2}{q_n^{r_n(\omega_n-1)\lambda}} \le \frac{1}{q_n^{r_n(\omega_n-2)\lambda}}$$

for sufficiently large n. We deduce from (3.45) that $\lim_{n\to\infty} |F(\xi) - \eta_n| = 0$, and so $\lim_{n\to\infty} \eta_n = F(\xi)$. We obtain from (3.35) and (3.45) that

$$(3.46)\quad |F(\xi) - \eta_n| \le \frac{1}{H(\eta_n)^{\gamma_n}} \ (\lim_{n\to\infty} \gamma_n = \infty)$$

for sufficiently large n, where $\gamma_n = \frac{(\omega_n-2)\lambda}{e_1}$ $(n = 1, 2, 3, \ldots)$.

4) There exist the following two cases for the sequence $\{|F(\xi) - \eta_n|\}$:

a) $|F(\xi) - \eta_n| = 0$ from some n onward:

In this case, $\eta_n = F(\xi)$ from some n onward, that is, $\{\eta_n\}$ is a constant sequence. Since $\eta_n \in K$ $(n = 1, 2, 3, \ldots)$, in case a) it is obtained that $F(\xi)$ is an algebraic number in K.

b) $|F(\xi) - \eta_n| \ne 0$ for infinitely many n:

In this case, the sequence $\{\eta_n\}$ has an infinite number of different terms. For otherwise $\{\eta_n\}$ would have a finite number of different terms, and so the sequence $\{|F(\xi) - \eta_n|\}$ would have a finite number of different terms. Since $|F(\xi) - \eta_n| \ne 0$ for an infinite number of n, there is a non-zero term in the sequence $\{|F(\xi) - \eta_n|\}$. Then $\{|F(\xi) - \eta_n|\}$ would have only a finite number of different terms which are not zero. Hence, let us denote the different and non-zero terms in the sequence $\{|F(\xi) - \eta_n|\}$ by $u_1, u_2, \ldots, u_t$ $(t \ge 1)$. Put $c = \min(u_1, u_2, \ldots, u_t)$. Note that c is a positive real number, since all the u_i $(i = 1, 2, \ldots, t)$ are positive real numbers. Thus, for any natural number n

$$(3.47)\quad \text{either } |F(\xi) - \eta_n| = 0 \text{ or } |F(\xi) - \eta_n| \ge c.$$

Since $\lim_{n\to\infty} |F(\xi) - \eta_n| = 0$, there exists a natural number n_0 such that

$$(3.48)\quad |F(\xi) - \eta_n| < c$$

for $n \geq n_0$. However, since $|F(\xi) - \eta_n| \neq 0$ for an infinite number of n, there exists a natural number $\overline{n} > n_0$ for which $|F(\xi) - \eta_{\overline{n}}| \neq 0$. By (3.47), we have $|F(\xi) - \eta_{\overline{n}}| \geq c$ which contradicts (3.48). Therefore $\{\eta_n\}$ must have an infinite number of different terms.

The sequence $\{H(\eta_n)\}$ of natural numbers, formed by the heights of the algebraic numbers η_n, is not bounded. For otherwise there would be a real number $M_2 > 0$ such that $H(\eta_n) \leq M_2$ for $n = 1, 2, 3, \ldots$. Then since also $\deg(\eta_n) \leq m$ $(n = 1, 2, 3, \ldots)$, the sequence $\{\eta_n\}$ would have a finite number of different terms, contrary to the fact that $\{\eta_n\}$ has an infinite number of different terms. Thus $\limsup_{n\to\infty} H(\eta_n) = \infty$, for $\{H(\eta_n)\}$ is not bounded above. Since $\limsup_{n\to\infty} H(\eta_n) = \infty$, the sequence $\{H(\eta_n)\}$ of natural numbers has a subsequence $\left\{H(\eta_{n_j})\right\}_{j=1}^{\infty}$ such that

$$1 < H(\eta_{n_1}) < H(\eta_{n_2}) < H(\eta_{n_3}) < \ldots, \quad \lim_{j\to\infty} H(\eta_{n_j}) = \infty. \tag{3.49}$$

By (3.49), the terms of the sequence $\{\eta_{n_j}\}_{j=1}^{\infty}$ are all different, i.e. if $i \neq j$, then $\eta_{n_i} \neq \eta_{n_j}$. So the sequence $\{\eta_{n_j}\}_{j=1}^{\infty}$ may have at most one term equal to $F(\xi)$. If there is a term equal to $F(\xi)$ among the terms η_{n_j} $(j = 1, 2, 3, \ldots)$, i.e. if there exists a natural number j_0 for which $\eta_{n_{j_0}} = F(\xi)$, then we throw away the first j_0 terms $\eta_{n_1}, \eta_{n_2}, \ldots, \eta_{n_{j_0}}$ and renumber the terms of the sequence $\left\{\eta_{n_j}\right\}$ $(j_0 + 1 \to 1,\ j_0 + 2 \to 2, \ldots)$, and so all the terms of the sequence $\left\{\eta_{n_j}\right\}$ are now different from $F(\xi)$. To summarize, the sequence $\{\eta_n\}_{n=1}^{\infty}$ has a subsequence $\{\eta_{n_j}\}_{j=1}^{\infty}$ for which the following properties hold:

i) $\eta_{n_j} \neq F(\xi)$ $(j = 1, 2, 3, \ldots)$,
ii) $1 < H(\eta_{n_1}) < H(\eta_{n_2}) < H(\eta_{n_3}) < \ldots$, $\lim_{j\to\infty} H(\eta_{n_j}) = \infty$,
iii) $\deg(\eta_{n_j}) \leq m$ $(j = 1, 2, 3, \ldots)$, for $\eta_{n_j} \in K$ $(j = 1, 2, 3, \ldots)$.

From (3.46) and i), we obtain for sufficiently large j that

$$0 < |F(\xi) - \eta_{n_j}| \leq \frac{1}{H(\eta_{n_j})^{\gamma_{n_j}}} \ (\lim_{j\to\infty} \gamma_{n_j} = \infty). \tag{3.50}$$

Put $H_j = H(\eta_{n_j}) > 1$ $(j = 1, 2, 3, \ldots)$. By ii), $\{H_j\}_{j=1}^{\infty}$ is a strictly increasing subsequence of natural numbers. By i), iii), and (3.50), we have for sufficiently large j

$$w_m^*\left(H_j, F(\xi)\right) = \min_{\substack{\alpha \text{ is algebraic} \\ \deg(\alpha)\leq m \\ H(\alpha)\leq H_j \\ \alpha\neq F(\xi)}} |F(\xi) - \alpha| \leq |F(\xi) - \eta_{n_j}| \leq \frac{1}{H(\eta_{n_j})^{\gamma_{n_j}}} = \frac{1}{H_j^{\gamma_{n_j}}},$$

and so it follows that $0 < w_m^*\left(H_j, F(\xi)\right) \leq \frac{1}{H_j^{\gamma_{n_j}}}$ for sufficiently large j. Consequently, $\frac{\log \frac{1}{H_j w_m^*\left(H_j, F(\xi)\right)}}{\log H_j} \geq \gamma_{n_j} - 1$ for sufficiently large j. Since $\lim_{j\to\infty} \gamma_{n_j} = \infty$, we obtain

$$\lim_{j\to\infty} \frac{\log \frac{1}{H_j w_m^*\left(H_j, F(\xi)\right)}}{\log H_j} = \infty.$$

Hence $w_m^*\left(F(\xi)\right) = \infty$. This implies that $F(\xi) \in U^*$ and $\mu^*\left(F(\xi)\right) \leq m$, in other words, $F(\xi) \in \bigcup_{i=1}^{m} U_i^*$. Since $U_i^* = U_i$ for $i = 1, 2, \ldots$, this implies that in case b) we have $F(\xi) \in \bigcup_{i=1}^{m} U_i$. This completes our proof. □

If we take $m = 1$ in Theorem **??**, we obtain the following corollary.

3.2. Corollary. *Let* $F(z) = \sum_{h=0}^{\infty} c_h z^h$ $\left(c_h \in \mathbb{Q};\ c_h = \frac{b_h}{a_h},\ b_h \in \mathbb{Z},\ a_h \in \mathbb{N};\ h = 0, 1, 2, \ldots\right)$ *be a power series which satisfies the following conditions:*

$$\begin{cases} c_h = 0, & r_n < h < s_n \ (n = 1, 2, 3, \ldots), \\ c_h \neq 0, & h = r_n \ (n = 1, 2, 3, \ldots), \\ c_h \neq 0, & h = s_n \ (n = 0, 1, 2, \ldots), \end{cases}$$

where $\{s_n\}_{n=0}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *are two infinite sequences of non-negative rational integers with*

$$0 = s_0 < r_1 < s_1 \le r_2 < s_2 \le r_3 < s_3 \le \cdots \text{ and } \lim_{n\to\infty} \frac{s_n}{r_n} = \infty.$$

Suppose that the radius of convergence R *of the series* $F(z)$ *is positive (*R *may be finite or infinite), and*

$$\limsup_{h\to\infty} \frac{\log A_h}{h} < \infty \ \ (A_h = [a_0, a_1, \dots, a_h], \ h = 1, 2, 3, \dots)\,.$$

Moreover, let ξ *be a Liouville number such that for* $n = 1, 2, 3, \dots$, *there are rational integers* p_n, q_n *with* $q_n > 1$ *and real numbers* $\omega_n = \frac{s_n}{r_n \log q_n}$ *with* $\lim_{n\to\infty} \omega_n = \infty$ *satisfying the following inequality*

$$\left|\xi - \frac{p_n}{q_n}\right| \le \frac{1}{q_n^{r_n \omega_n}},$$

and let $|\xi| < R$. *Then* $F(\xi)$ *is either a rational number or a Liouville number.*

3.3. Note. Theorem **??** and Corollary 3.2 also hold true for real numbers $\omega_n = \frac{s_n}{r_n \log q_n}$ with $\limsup_{n\to\infty} \omega_n = \infty$.

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