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IMAGES AND PREIMAGES OF (L, M)-GRILLBASES

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Abstract

We introduce the notion of (L, M)-grillbases, where L is a complete lattice and M is a strictly two-sided, commutative quantale lattice. We define two types of image and preimage of (L, M)-grillbases using the Zadeh image and preimage operators. We study the images and preimages of (L, M)-grillbases induced by mappings. We investigate their properties.

Keywords: Strictly two-sided, Commutative quantales, (L, M)-grillbases.

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1. Introduction

The convergence theory of grills provides a good tool for interpreting topological structures, and plays an important role in topology [11, 12]. Azad [1], Srivastava and Gupta [16] introduced the notion of L-grill on a complete quasi-monoidal lattice (including GL-monoid [2, 3]). Importance of L-grills can be seen in the papers of Khare and Singh [7, 8], Srivastava and Khare [17, 18, 19], where L-grills are used to study the order structure of various families. The present paper arose as a result of such studies, as it gives a structure closely related to L-grills.

Let L be a complete lattice and $\phi : X \to Y$ a mapping. The Zadeh image and preimage operators $\phi_L^{\rightarrow} : L^X \to L^Y$ and $\phi_L^{\leftarrow} : L^X \leftarrow L^Y$ are defined by

$$\phi_L^{\rightarrow}(f)(y) = \bigvee\{f(x) \mid \phi(x) = y\}, \quad \phi_L^{\leftarrow}(g) = g \circ \phi.$$

Rodabaugh [13, 14, 15] gives two different proofs for all cqml's (complete lattices) L vindicating Zadeh's definitions, first, using the AFT (adjoint functor theorem) to lift the Zadeh operators from traditional operators, and second, classes of naturality diagrams indexed by L to generate Zadeh operators directly from the original mapping.

In this paper, we define (L, M)-grillbases, where L is a complete lattice and M is a strictly two-sided, commutative quantale lattice. We consider the Zadeh image operator

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$\phi_1^{\Rightarrow}$ of the Zadeh image operator and the Zadeh preimage operator $\phi_2^{\Rightarrow}$ of the Zadeh preimage operator. Also we consider the Zadeh preimage operator $\phi_1^{\Leftarrow}$ of the Zadeh image operator.

2. Preliminaries

Throughout this paper, let X be a nonempty set and $L = (L, \leq, \vee, \wedge, \bot, \top)$ a complete lattice where $\bot$ and $\top$ denote the least and the greatest elements in L. For $x \in X$, $1_X(x) = \top$ and $1_\emptyset(x) = \bot$.

2.1. Definition. [4, 9, 10] A complete lattice $(L, \leq, \odot)$ is called a *strictly two-sided, commutative quantale lattice* (scq-lattice, for short) iff it satisfies the following properties:

- (L1) $(L, \odot)$ is a commutative semigroup.
- (L2) $x = x \odot \top$, for each $x \in L$.
- (L3) $\odot$ is distributive over arbitrary joins, i.e., for $\{s, r_i\}_{i\in\Gamma} \subset L$,

$$\Big(\bigvee_{i\in\Gamma} r_i\Big) \odot s = \bigvee_{i\in\Gamma} (r_i \odot s).$$

2.2. Definition. [5, 9, 10] Let $(L, \leq, \odot)$ be a scq-lattice. A mapping $n : L \to L$ is called a *strong negation*, denoted by $n(a) = a^*$, if it satisfies the following conditions:

- (N1) $n(n(a)) = a$ for each $a \in L$.
- (N2) If $a \leq b$ for each $a, b \in L$, then $n(a) \geq n(b)$.

In this paper, we always assume $(M, \leq, \odot, \oplus)$ is a scq-lattice with a strong negation $*$ defined by

$$x \oplus y = (x^* \odot y^*)^*.$$

2.3. Lemma. [3, 4, 5, 6, 9, 10] *For each $x, y, z \in M$, $\{y_i \mid i \in \Gamma\} \subset M$, we have the following properties:*

(1) *If $y \leq z$, then $(x \odot y) \leq (x \odot z)$ and $(x \oplus y) \leq (x \oplus z)$.*
(2) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$.
(3) $\bot^* = \top$ *and* $\top^* = \bot$.
(4) $\bigwedge_{i\in\Gamma} y_i^* = \big(\bigvee_{i\in\Gamma} y_i\big)^*$ *and* $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.
(5) $x \oplus \big(\bigwedge_{i\in\Gamma} y_i\big) = \bigwedge_{i\in\Gamma} (x \oplus y_i)$. □

2.4. Definition. [3, 4, 5, 6, 9, 10] A morphism between $(M_1, \leq_1, \odot_1)$ and $(M_2, \leq_2, \odot_2)$ is a map $\phi : M_1 \to M_2$ provided with the properties:

(1) ϕ commutes with arbitrary joins,
(2) $\phi(a \odot_1 b) = \phi(a) \odot_2 \phi(b)$,
(3) ϕ preserves the universal upper bound, i.e. $\phi(\top) = \top$.

3. Structue of (L, M)-grillbases

3.1. Definition. A mapping $\mathcal{G} : L^X \to M$ is called an (L, M)-*grill on* X if it satisfies the following conditions:

(LG1) $\mathcal{G}(1_\emptyset) = \bot$ and $\mathcal{G}(1_X) = \top$,
(LG2) $\mathcal{G}(f \vee g) \leq \mathcal{G}(f) \oplus \mathcal{G}(g)$, for each $f, g \in L^X$,
(LG3) If $f \leq g$, then $\mathcal{G}(f) \leq \mathcal{G}(g)$.

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be (L, M)-grills on X. We say $\mathcal{G}_1$ *is finer than* $\mathcal{G}_2$ ($\mathcal{G}_2$ *is coarser than* $\mathcal{G}_1$) if $\mathcal{G}_1(f) \leq \mathcal{G}_2(f)$ for all $f \in L^X$.

3.2. Note. Let $\mathcal{B} : L^X \to M$ be a mapping and $f \in L^X$. We set

$$\langle \mathcal{B} \rangle (f) = \bigwedge_{f \le g} \mathcal{B}(g).$$

3.3. Definition. A mapping $\mathcal{B} : L^X \to M$ is called an (L, M)-*grillbase on* X if it satisfies the following conditions:

(LB1) $\mathcal{B}(1_\emptyset) = \bot$ and $\mathcal{B}(1_X) = \top$,

(LB2) $\langle \mathcal{B} \rangle (f \vee g) \le \mathcal{B}(f) \oplus \mathcal{B}(g)$, for each $f, g \in L^X$,

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be (L, M)-grillbases on X. We say $\mathcal{B}_1$ *is finer than* $\mathcal{B}_2$ ($\mathcal{B}_2$ *is coarser than* $\mathcal{B}_1$) if $\langle \mathcal{B}_1 \rangle (f) \le \langle \mathcal{B}_2 \rangle (f)$ for all $f \in L^X$.

3.4. Remark. (1) If $\mathcal{G}$ is an (L, M)-grill, then $\mathcal{G}$ is an (L, M)-grillbase with $\langle \mathcal{G} \rangle = \mathcal{G}$.

(2) If a map $\mathcal{B} : L^X \to M$ is an (L, M)-grillbase, then by (LB2), $f \vee g = 1_X$ implies $\mathcal{B}(f) \oplus \mathcal{B}(g) = \top$.

3.5. Proposition. *If* $\mathcal{B} : L^X \to M$ *is an* (L, M)*-grillbase, then* $\langle \mathcal{B} \rangle$ *is the coarsest* (L, M)*-grill satisfying* $\mathcal{B} \ge \langle \mathcal{B} \rangle$.

Proof. The conditions (LG1) and (LG3) are easily checked. For each $f_1 \ge f$ and $g_1 \ge g$, since $f \vee g \le f_1 \vee g_1$,

$$\langle \mathcal{B} \rangle (f \vee g) \le \langle \mathcal{B} \rangle (f_1 \vee g_1) \le \mathcal{B}(f_1) \oplus \mathcal{B}(g_1).$$

By Lemma2.3 (5), $\langle \mathcal{B} \rangle (f \vee g) \le \langle \mathcal{B} \rangle (f) \oplus \langle \mathcal{B} \rangle (g)$. Hence $\langle \mathcal{B} \rangle$ is a (L, M)-grill.

Let $\mathcal{G}$ be an (L, M)-grill satisfying $\mathcal{B} \ge \mathcal{G}$. We have

$$\langle \mathcal{B} \rangle (f) = \bigwedge_{f \le g} \mathcal{B}(g) \ge \bigwedge_{f \le g} \mathcal{G}(g) = \mathcal{G}(f). \qquad \square$$

3.6. Theorem. *If* $\mathcal{H} : L^X \to M$ *is a map satisfying the following condition:*

(C) $\mathcal{H}(1_\emptyset) = \bot$,

and for every finite index set K*, if* $\bigvee_{i \in K} g_i = 1_X$*, then* $\oplus_{i \in K} \mathcal{H}(g_i) = \top$.

We define a map $\mathcal{B}_\mathcal{H} : L^X \to M$ *as*

$$\mathcal{B}_\mathcal{H}(f) = \bigwedge \Big\{ \oplus_{i \in K} \mathcal{H}(g_i) \mid f = \bigvee_{i \in K} g_i \Big\},$$

where the $\bigwedge$ *is taken for every finite set* K *such that* $f = \bigvee_{i \in K} g_i$*. Then*

(1) $\mathcal{B}_\mathcal{H}$ *is an* (L, M)*-grillbase on* X.

(2) *If* $\mathcal{H} \ge \mathcal{B}$ *and* $\mathcal{B}$ *is an* (L, M)*-grillbase on* X*, then* $\langle \mathcal{B}_\mathcal{H} \rangle \ge \langle \mathcal{B} \rangle$.

Proof. (1) (LB1) By the condition (C), $\mathcal{B}_\mathcal{H}(1_X) = \top$ and $\mathcal{B}_\mathcal{H}(1_\emptyset) = \bot$.

(LB2) For each two finite index sets K and J with $f_1 = \bigvee_{k \in K} g_k$ and $f_2 = \bigvee_{j \in J} h_j$, since $f_1 \vee f_2 = (\bigvee_{k \in K} g_k) \vee (\bigvee_{j \in J} h_j)$, by the definition of $\mathcal{B}_\mathcal{H}$, we have

$$\langle \mathcal{B}_\mathcal{H} \rangle (f_1 \vee f_2) \le (\oplus_{k \in K} \mathcal{H}(g_k)) \oplus (\oplus_{j \in J} \mathcal{H}(h_j)).$$

By Lemma 2.3 (5), $\langle \mathcal{B}_\mathcal{H} \rangle (f_1 \vee f_2) \le \mathcal{B}_\mathcal{H}(f_1) \oplus \mathcal{B}_\mathcal{H}(f_2)$, for all $f_1, f_2 \in L^X$. Thus, $\mathcal{B}_\mathcal{H}$ is an (L, M)-grillbase on X.

(2) For each finite family $\{g_i \mid \bigvee_{i \in K} g_i \ge f\}$, we have

$$\langle \mathcal{B} \rangle (f) \le \langle \mathcal{B} \rangle \Big(\bigvee_{i \in K} g_i \Big) \le \oplus_{i \in K} \mathcal{B}(g_i) \le \oplus_{i \in K} \mathcal{H}(g_i).$$

Thus, $\langle \mathcal{B}_\mathcal{H} \rangle \ge \langle \mathcal{B} \rangle$. $\square$

3.7. Definition. Let $(X, \mathcal{G})$ and $(Y, \mathcal{G}^{'})$ be two (L, M)-grill spaces. Then a map $\phi : X \to Y$ is said to be:

(1) An (L, M)-*grill map* iff $\mathcal{G}^{'} \geq \mathcal{G} \circ \phi_L^{\leftarrow}$.
(2) An (L, M)-*grill preserving map* iff $\mathcal{G} \geq \mathcal{G}^{'} \circ \phi_L^{\rightarrow}$.

Naturally, the composition of (L, M)-grill maps (resp., (L, M)-grill preserving maps) is an (L, M)-grill map (resp., (L, M)-grill preserving map).

3.8. Proposition. *Let $\mathcal{B}$ and $\mathcal{B}^{'}$ be two (L, M)-grillbases on X and Y, respectively, and $\phi : X \to Y$ a map. Then we have the following properties:*

(1) *$\phi : (X, \langle \mathcal{B} \rangle) \to (Y, \langle \mathcal{B}^{'} \rangle)$ is an (L, M)-grill map iff $\mathcal{B}^{'} \geq \langle \mathcal{B} \rangle \circ \phi_L^{\leftarrow}$.*
(2) *$\phi : (X, \langle \mathcal{B} \rangle) \to (Y, \langle \mathcal{B}^{'} \rangle)$ is an (L, M)-grill preserving map iff $\mathcal{B} \geq \langle \mathcal{B}^{'} \rangle \circ \phi_L^{\rightarrow}$.*
(3) *If $\mathcal{B}^{'} \geq \mathcal{B} \circ \phi_L^{\leftarrow}$, then $\phi : (X, \langle \mathcal{B} \rangle) \to (Y, \langle \mathcal{B}^{'} \rangle)$ is an (L, M)-grill map.*
(4) *If $\mathcal{B} \geq \mathcal{B}^{'} \circ \phi_L^{\rightarrow}$, then $\phi : (X, \langle \mathcal{B} \rangle) \to (Y, \langle \mathcal{B}^{'} \rangle)$ is an (L, M)-grill preserving map.*

Proof. Straightforward. □

4. The type $(\phi_1^{\Leftarrow}, \phi_2^{\Rightarrow})$ of the preimages and images of (L, M)-grillbases

Let L be a complete lattice. The basic scheme for second order image operators is as follows. Let $\phi : X \to Y$ be a map. Then:

Case 1. Consider $[\phi_L^{\rightarrow}]_L^{\rightarrow} : L^{L^X} \to L^{L^Y}$. This is the Zadeh image operator of the Zadeh image operator. We denote it by $\phi_1^{\Rightarrow}$, that is, for all $\mathcal{U} \in L^{L^X}$, $\forall g \in L^Y$,

$$\phi_1^{\Rightarrow}(\mathcal{U})(g) = [\phi_L^{\rightarrow}]_L^{\rightarrow}(\mathcal{U})(g) = \bigvee \{\mathcal{U}(f) \mid g = \phi_L^{\rightarrow}(f)\}.$$

Case 2. Consider $[\phi_L^{\leftarrow}]_L^{\leftarrow} : L^{L^X} \to L^{L^Y}$. This is the Zadeh preimage operator of the Zadeh preimage operator. We denote it by $\phi_2^{\Rightarrow}$, that is, for all $\mathcal{U} \in L^{L^X}$, $\forall g \in L^Y$,

$$\phi_2^{\Rightarrow}(\mathcal{U})(g) = [\phi_L^{\leftarrow}]_L^{\leftarrow}(\mathcal{U})(g) = \mathcal{U} \circ \phi_L^{\leftarrow}(g).$$

The basic scheme for second order preimage operators is as follows. Let $\phi : X \to Y$ be a map. Then:

Case 1. Consider $[\phi_L^{\leftarrow}]_L^{\rightarrow} : L^{L^X} \leftarrow L^{L^Y}$. This is the Zadeh image operator of the Zadeh preimage operator. We denote it by $\phi_1^{\Leftarrow}$, that is, for all $\mathcal{V} \in L^{L^Y}$, $\forall f \in L^X$,

$$\phi_1^{\Leftarrow}(\mathcal{V})(f) = [\phi_L^{\leftarrow}]_L^{\rightarrow}(\mathcal{V})(f) = \bigvee \{\mathcal{V}(g) \mid f = \phi_L^{\leftarrow}(g)\}.$$

Case 2. Consider $[\phi_L^{\rightarrow}]_L^{\leftarrow} : L^{L^X} \leftarrow L^{L^Y}$. This is the Zadeh preimage operator of the Zadeh image operator. We denote it by $\phi_2^{\Leftarrow}$, that is, for all $\mathcal{V} \in L^{L^Y}$, $\forall f \in L^X$,

$$\phi_2^{\Leftarrow}(\mathcal{V})(f) = [\phi_L^{\rightarrow}]_L^{\leftarrow}(\mathcal{V})(f) = \mathcal{V} \circ \phi_L^{\rightarrow}(f).$$

In this section we consider the preimages and images of (L, M)-grillbases with respect to the pair $(\phi_1^{\Leftarrow}, \phi_2^{\Rightarrow})$.

4.1. Theorem. *Let $\phi : X \to Y$ be a map and $\mathcal{B}$ an (L, M)-grillbase on Y. Then we have the following properties:*

(1) *If $\phi_L^{\leftarrow}(h) = 1_\emptyset$ implies $\mathcal{B}(h) = \bot$, then $\phi_1^{\Leftarrow}(\mathcal{B})$ is an (L, M)-grillbase on X and $\langle \phi_1^{\Leftarrow}(\mathcal{B}) \rangle$ is the coarsest (L, M)-grill for which $\phi : (X, \langle \phi_1^{\Leftarrow}(\mathcal{B}) \rangle) \to (Y, \langle \mathcal{B} \rangle)$ is an (L, M)-grill map.*
(2) *If ϕ is surjective, $\phi_1^{\Leftarrow}(\mathcal{B})$ is an (L, M)-grillbase.*

(3) *If* $\phi_L^{\leftarrow}(h) = 1_\emptyset$ *implies* $\mathcal{B}(h) = \bot$, ϕ *is injective and* $\mathcal{B}$ *is an* (L, M)*-grill, then* $\phi_1^{\Leftarrow}(\mathcal{B})$ *is an* (L, M)*-grill.*

Proof. (1) (LB1) Since $\phi_L^{\leftarrow}(1_X) = 1_X$, $\phi_1^{\Leftarrow}(\mathcal{B})(1_X) = \top$. By the assumption, $\phi_1^{\Leftarrow}(\mathcal{B})(1_\emptyset) = \bot$.

(LB2) Suppose there exist $f_1, f_2 \in L^X$ such that

$$\langle\phi_1^{\Leftarrow}(\mathcal{B})\rangle(f_1 \vee f_2) \not\leq \phi_1^{\Leftarrow}(\mathcal{B})(f_1) \oplus \phi_1^{\Leftarrow}(\mathcal{B})(f_2).$$

By the definition of $\phi_1^{\Leftarrow}(\mathcal{B})(f_i)$, for $i \in \{1, 2\}$ there exist $h_i \in L^Y$ with $f_i = \phi_L^{\leftarrow}(h_i)$ such that

$$\langle\phi_1^{\Leftarrow}(\mathcal{B})\rangle(f_1 \vee f_2) \not\leq \mathcal{B}(h_1) \oplus \mathcal{B}(h_2).$$

Since $\mathcal{B}$ is an (L, M)-grillbase, i.e., $\langle\mathcal{B}\rangle(h_1 \vee h_2) \leq \mathcal{B}(h_1) \oplus \mathcal{B}(h_2)$, we have

$$\langle\phi_1^{\Leftarrow}(\mathcal{B})\rangle(f_1 \vee f_2) \not\leq \langle\mathcal{B}\rangle(h_1 \vee h_2).$$

By the definition of $\langle\mathcal{B}\rangle$, there exists $h \in L^Y$ with $h \geq h_1 \vee h_2$ such that

$$\langle\phi_1^{\Leftarrow}(\mathcal{B})\rangle(f_1 \vee f_2) \not\leq \mathcal{B}(h).$$

On the other hand, $f_1 \vee f_2 = \phi_L^{\leftarrow}(h_1) \vee \phi_L^{\leftarrow}(h_2) = \phi_L^{\leftarrow}(h_1 \vee h_2) \leq \phi_L^{\leftarrow}(h)$,

$$\langle\phi_1^{\Leftarrow}(\mathcal{B})\rangle(f_1 \vee f_2) \leq \mathcal{B}(h).$$

It is a contradiction. Hence $\phi_1^{\Leftarrow}(\mathcal{B})$ is an (L, M)-grillbase on X.

Let $\phi : (X, \mathcal{G}) \to (Y, \langle\mathcal{B}\rangle)$ be an (L, M)-grill map. For each $f \in L^X$, we have

$$\begin{aligned}\langle\phi_1^{\Leftarrow}(\mathcal{B})\rangle(f) &= \bigwedge\{\mathcal{B}(g) \mid f \leq \phi_L^{\leftarrow}(g)\}\\ &\geq \bigwedge\{\mathcal{G}(\phi_L^{\leftarrow}(g)) \mid f \leq \phi_L^{\leftarrow}(g)\}\\ &\geq \mathcal{G}(f).\end{aligned}$$

(2) Since ϕ is surjective, $\phi_L^{\leftarrow}(h) = 1_\emptyset$ implies $h = 1_\emptyset$. So, $\mathcal{B}(1_\emptyset) = \bot$. By (1), it is easy.

(3) (LG1) and (LG2) are obvious.

(LG3) Let $f \leq g$ for $f, g \in L^X$. Since ϕ is surjective, then $h \in L^Y$ exists with $h \circ \phi = f$ and $g = \phi_L^{\leftarrow}(h \vee \phi_L^{\rightarrow}(g))$. It implies

$$\phi_1^{\Leftarrow}(\mathcal{B})(g) \geq \mathcal{B}(h \vee \phi_L^{\rightarrow}(g)) \geq \mathcal{B}(h).$$

Hence $\phi_1^{\Leftarrow}(\mathcal{B})(g) \geq \phi_1^{\Leftarrow}(\mathcal{B})(f)$. □

4.2. Theorem. *Let* $\phi_i : X \to X_i$ *be maps, for all* $i \in \Gamma$. *Let* $\{\mathcal{B}_i\}_{i\in\Gamma}$ *be a family of* (L, M)*-grillbases on* X_i *satisfying the following condition:*

(C) *For every finite subset* K *of* Γ, *if* $\bigvee_{i\in K}(h_i \circ \phi_i) = 1_X$, *then* $\oplus_{i\in K}\mathcal{B}_i(h_i) = \top$.

We define a map $\bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\mathcal{B}_i) : L^X \to M$ *as*

$$\bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\mathcal{B}_i)(f) = \bigwedge\Big\{\oplus_{i\in K}\,\mathcal{B}_i(h_i) \mid f = \bigvee_{i\in K}(h_i \circ \phi_i)\Big\},$$

where the $\bigwedge$ *is taken for every finite subset* K *of* Γ *such that* $f = \bigvee_{i\in K}(h_i \circ \phi_i)$. *Let* $\mathcal{B} = \bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\mathcal{B}_i)$ *be given. Then,*

(1) $\mathcal{B}$ *is* (L, M)*-grillbase on* X *and* $\langle\mathcal{B}\rangle$ *is the coarsest* (L, M)*-grill for which* $\phi_i : (X, \langle\mathcal{B}\rangle) \to (X_i, \langle\mathcal{B}_i\rangle)$ *is an* (L, M)*-grill map, for all* $i \in \Gamma$.

(2) *A map* $\phi : (Y, \mathcal{G}') \to (X, \langle\mathcal{B}\rangle)$ *is an* (L, M)*-grill map iff for each* $i \in \Gamma$, $\phi_i \circ \phi : (Y, \mathcal{G}') \to (X_i, \langle\mathcal{B}_i\rangle)$ *is an* (L, M)*-grill map.*

(3) $\big\langle\bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\mathcal{B}_i)\big\rangle = \big\langle\bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\langle\mathcal{B}_i\rangle)\big\rangle$.

Proof. (1) (LB1) By the condition (C) and $1_X \circ \phi_i = 1_X$, $\mathcal{B}(1_\emptyset) = \bot$ and $\mathcal{B}(1_X) = \top$, respectively.

(LB2) Suppose there exist $f_1, f_2 \in L^X$ such that

$$\langle \mathcal{B} \rangle (f_1 \vee f_2) \not\leq \mathcal{B}(f_1) \oplus \mathcal{B}(f_2).$$

By the definition of $\mathcal{B}(f_1)$, there exists a finite subset K of Γ with $f_1 = \bigvee_{k\in K}(h_k \circ \phi_k)$ such that

$$\langle \mathcal{B} \rangle (f_1 \vee f_2) \not\leq (\oplus_{k\in K}\mathcal{B}_k(h_k)) \oplus \mathcal{B}(f_2).$$

Again, by the definition of $\mathcal{B}(f_2)$, there exists a finite subset J of Γ with $f_2 = \bigvee_{j\in J}(g_j \circ \phi_j)$ such that

$$\langle \mathcal{B} \rangle (f_1 \vee f_2) \not\leq (\oplus_{k\in K}\mathcal{B}_k(h_k)) \oplus (\oplus_{j\in J}\mathcal{B}_j(g_j)).$$

Put for $m \in K \cup J$,

$$p_m = \begin{cases} h_m, & \text{if } m \in K - (K \cap J) \\ g_m, & \text{if } m \in J - (K \cap J) \\ h_m \vee g_m, & \text{if } m \in (K \cap J). \end{cases}$$

Since for each $m \in K \cap J$, $\langle \mathcal{B}_m \rangle (h_m \vee g_m) \leq \mathcal{B}_m(h_m) \oplus \mathcal{B}_m(g_m)$, then we have

$$\langle \mathcal{B} \rangle (f_1 \vee f_2) \not\leq (\oplus_{m\in(K\cup J)-(K\cap J)}\mathcal{B}_m(p_m)) \oplus (\oplus_{m\in(K\cap J)}\langle \mathcal{B}_m \rangle (h_m \vee g_m)).$$

From the definition of $\langle \mathcal{B}_m \rangle$, there exists $q_m \in L^{X_m}$ with $q_m \geq h_m \vee g_m$ such that

$$\langle \mathcal{B} \rangle (f_1 \vee f_2) \not\leq (\oplus_{m\in(K\cup J)-(K\cap J)}\mathcal{B}_m(p_m)) \oplus (\oplus_{m\in(K\cap J)}\mathcal{B}_m(q_m)).$$

On the other hand,

$$\begin{aligned} f_1 \vee f_2 &= \Big(\bigvee_{k\in K}(h_k \circ \phi_k)\Big) \vee \Big(\bigvee_{j\in J}(g_j \circ \phi_j)\Big) \\ &\leq \Big(\bigvee_{m\in(K\cup J)-(K\cap J)}(p_m \circ \phi_m)\Big) \vee \Big(\bigvee_{m\in K\cap J}(q_m \circ \phi_m)\Big), \end{aligned}$$

and since $K \cup J$ is finite,

$$\langle \mathcal{B} \rangle (f_1 \vee f_2) \leq \big(\oplus_{m\in(K\cup J)-(K\cap J)}\,\mathcal{B}_m(p_m)\big) \oplus (\oplus_{m\in(K\cap J)}\mathcal{B}_m(q_m)).$$

It is a contradiction. Hence $\mathcal{B}$ is an (L, M)-grillbase on X.

From Proposition 3.8 (3), since $\mathcal{B}_i(f_i) \geq \mathcal{B}(f_i \circ \phi_i)$ for each $i \in \Gamma$, ϕ_i is an (L, M)-grill map.

Let $\mathcal{G}(f_i \circ \phi_i) \leq \langle \mathcal{B}_i \rangle (f_i)$ be given for each $i \in \Gamma$. For each finite subset K of Γ with $f \leq \bigvee_{k\in K} h_k \circ \phi_k$, since $\mathcal{G}(h_k \circ \phi_k) \leq \langle \mathcal{B}_k \rangle (h_k)$ for all $k \in K$, we have

$$\mathcal{G}(f) \leq \mathcal{G}\Big(\bigvee_{k\in K} h_k \circ \phi_k\Big) \leq \oplus_{k\in K}\mathcal{G}(h_k \circ \phi_k) \leq \oplus_{k\in K}\langle \mathcal{B}_k \rangle (h_k) \leq \oplus_{k\in K}\mathcal{B}_k(h_k).$$

Hence, by the definition of $\langle \mathcal{B} \rangle$, $\mathcal{G} \leq \langle \mathcal{B} \rangle$.

(2) Necessity of the composition condition is clear since the composition of (L, M)-grill maps is an (L, M)-grill map.

Conversely, for each finite index set K with $g \leq \bigvee_{k\in K} h_k \circ \phi_k$, since for each $k \in K$, $\phi_k \circ \phi : (Y, \mathcal{G}') \to (X_k, \langle \mathcal{B}_k \rangle)$ is an (L, M)-grill map,

$$\langle \mathcal{B}_k \rangle (h_k) \geq \mathcal{G}'(h_k \circ (\phi_k \circ \phi)).$$

Since $g \circ \phi \leq \bigvee_{k\in K}((h_k \circ \phi_k) \circ \phi)$, we have

$$\mathcal{G}'(g \circ \phi) \leq \oplus_{k\in K}\mathcal{G}'(h_k \circ (\phi_k \circ \phi)) \leq \oplus_{k\in K}\langle \mathcal{B}_k \rangle (h_k) \leq \oplus_{k\in K}\mathcal{B}_k(h_k).$$

By the definition of $\langle\mathcal{B}\rangle$, $\langle\mathcal{B}\rangle(g) \geq \mathcal{G}'(g \circ \phi)$.

(3) Put $\mathcal{G} = \bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\langle\mathcal{B}_i\rangle)$, by applying (1) to both $\langle\mathcal{B}\rangle$ and $\langle\mathcal{G}\rangle$ we get the related equality. □

From Theorem 4.2, we can obtain the following corollaries:

4.3. Corollary. *Let $\{\mathcal{B}_i\}_{i\in\Gamma}$ be a family of (L, M)-grillbases on X satisfying the following condition:*

(C) *For every finite subset K of Γ, if $\bigvee_{i\in K} f_i = 1_X$, then $\oplus_{i\in K}\mathcal{B}_i(f_i) = \top$.*

We define a map $\bigsqcup_{i\in\Gamma}\mathcal{B}_i : L^X \to M$ as

$$\bigsqcup_{i\in\Gamma}\mathcal{B}_i(g) = \bigwedge\Big\{ \oplus_{i\in K}\,\mathcal{B}_i(g_i) \mid g = \bigvee_{i\in K} g_i\Big\},$$

where the $\bigwedge$ is taken for every finite subset K of Γ such that $g = \bigvee_{i\in K} g_i$. Then, $\bigsqcup_{i\in\Gamma}\mathcal{B}_i$ is an (L, M)-grillbase on X and $\langle\bigsqcup_{i\in\Gamma}\mathcal{B}_i\rangle$ is the coarsest (L, M)-grill finer than $\langle\mathcal{B}_i\rangle$ for each $i \in \Gamma$. □

4.4. Corollary. *Let $X = \prod_{i\in\Gamma}$ be a product set and $\pi_i : X \to X_i$ projection maps, for all $i \in \Gamma$. Let $\{\mathcal{B}_i\}_{i\in\Gamma}$ be a family of (L, M)-grillbases on X_i satisfying the following condition:*

(C) *For every finite subset K of Γ, if $\bigvee_{i\in K}(h_i \circ \pi_i) = 1_X$, then $\oplus_{i\in K}\mathcal{B}_i(h_i) = \top$.*

We define a map $\bigsqcup_{i\in\Gamma}(\pi_i)_1^{\Leftarrow}(\mathcal{B}_i) : L^X \to M$ as

$$\bigsqcup_{i\in\Gamma}(\pi_i)_1^{\Leftarrow}(\mathcal{B}_i)(g) = \bigwedge\Big\{ \oplus_{i\in K}\,\mathcal{B}_i(g_i) \mid g = \bigvee_{i\in K}(g_i \circ \pi_i)\Big\},$$

where the $\bigwedge$ is taken for every finite subset K of Γ such that $g = \bigvee_{i\in K}(g_i \circ \pi_i)$. Let $\mathcal{B} = \bigsqcup_{i\in\Gamma}(\pi_i)_1^{\Leftarrow}(\mathcal{B}_i)$ be given. Then,

1. *$\mathcal{B}$ is (L, M)-grillbase on X and $\langle\mathcal{B}\rangle$ is the coarsest (L, M)-grill on X for which $\pi_i : (X, \langle\mathcal{B}\rangle) \to (X_i, \langle\mathcal{B}_i\rangle)$ is an (L, M)-grill map.*
2. *A map $\phi : (Y, \mathcal{G}') \to (X, \langle\mathcal{B}\rangle)$ is an (L, M)-grill map iff for each $i \in \Gamma$, $\pi_i \circ \phi : (Y, \mathcal{G}') \to (X_i, \langle\mathcal{B}_i\rangle)$ is an (L, M)-grill map.* □

In Corollary 4.4, the structure $\langle\bigsqcup_{i\in\Gamma}(\pi_i)_1^{\Leftarrow}(\mathcal{B}_i)\rangle$ is called a *product (L, M)-grill* on X.

4.5. Proposition. *Let $\phi : X \to Y$ be a bijective map and $\mathcal{B}$ an (L, M)-grillbase on X. Then*

1. *$\phi_2^{\Rightarrow}(\mathcal{B})$ is an (L, M)-grillbase on Y and $\langle\phi_2^{\Rightarrow}(\mathcal{B})\rangle$ is the coarsest (L, M)-grill for which $\phi : (X, \langle\mathcal{B}\rangle) \to (Y, \langle\phi_2^{\Rightarrow}(\mathcal{B})\rangle)$ is an (L, M)-grill preserving map.*
2. *$\phi_1^{\Leftarrow}(\phi_2^{\Rightarrow}(\mathcal{B}))$ is an (L, M)-grillbase on X with $\phi_1^{\Leftarrow}(\phi_2^{\Rightarrow}(\mathcal{B})) = \mathcal{B}$.*

Proof. (1) (LB1) is obvious.

(LB2) Suppose there exist $h_1, h_2 \in L^Y$ such that

$$\langle\phi_2^{\Rightarrow}(\mathcal{B})\rangle(h_1 \vee h_2) \not\leq \phi_2^{\Rightarrow}(\mathcal{B})(h_1) \oplus \phi_2^{\Rightarrow}(\mathcal{B})(h_2).$$

By the definition of $\phi_2^{\Rightarrow}(\mathcal{B})$, we have

$$\langle\phi_2^{\Rightarrow}(\mathcal{B})\rangle(h_1 \vee h_2) \not\leq \mathcal{B}(h_1 \circ \phi) \oplus \mathcal{B}(h_2 \circ \phi).$$

Since $\mathcal{B}$ is an (L, M)-grillbase, i.e., $\langle\mathcal{B}\rangle((h_1 \vee h_2) \circ \phi) \leq \mathcal{B}(h_1 \circ \phi) \oplus \mathcal{B}(h_2 \circ \phi)$,

$$\langle\phi_2^{\Rightarrow}(\mathcal{B})\rangle(h_1 \vee h_2) \not\leq \langle\mathcal{B}\rangle((h_1 \vee h_2) \circ \phi).$$

By the definition of $\langle\mathcal{B}\rangle$, there exists $g \in L^X$ with $g \geq (h_1 \vee h_2) \circ \phi$ such that

$$\langle\phi_2^{\Rightarrow}(\mathcal{B})\rangle(h_1 \vee h_2) \not\leq \mathcal{B}(g).$$

Since $\phi_L^{\rightarrow}(g) \geq \phi_L^{\rightarrow}(\phi_L^{\leftarrow}(h_1 \vee h_2)) = h_1 \vee h_2$ and ϕ is injective,

$$\langle \phi_2^{\Rightarrow}(\mathcal{B})\rangle(h_1 \vee h_2) \leq \phi_2^{\Rightarrow}(\mathcal{B})(\phi_L^{\rightarrow}(g)) = \mathcal{B}(\phi_L^{\leftarrow}(\phi_L^{\rightarrow}(g))) = \mathcal{B}(g).$$

It is a contradiction. Hence $\phi_2^{\Rightarrow}(\mathcal{B})$ is an (L, M)-grillbase on Y. Other cases are similarly proved as in Theorem 4.1 (1).

(2) From the condition of Theorem 4.1 (1), we have $h \circ \phi = 1_\emptyset$ implies $\phi_2^{\Rightarrow}(\mathcal{B})(h) = \mathcal{B}(\phi_L^{\leftarrow}(h)) = \bot$. Thus, $\phi_1^{\Leftarrow}(\phi_2^{\Rightarrow}(\mathcal{B}))$ is an (L, M)-grillbase on X. By an easy computation, $\phi_1^{\Leftarrow}(\phi_2^{\Rightarrow}(\mathcal{B})) = \mathcal{B}$. □

4.6. Theorem. *Let $\phi_i : X_i \to X$ be injective maps, for all $i \in \Gamma$. Let $\{\mathcal{B}_i\}_{i\in\Gamma}$ be a family of (L, M)-grillbases on X_i satisfying the following condition:*

(C) *For every finite subset K of Γ, if $\bigvee_{i\in K} g_i = 1_X$, then $\oplus_{i\in K}(\phi_i)_2^{\Rightarrow}(\mathcal{B}_i)(g_i) = \top$.*

We define a map $\mathcal{B} : L^X \to M$ as

$$\mathcal{B}(g) = \bigwedge \Big\{ \oplus_{i\in K}\, (\phi_i)_2^{\Rightarrow}(\mathcal{B}_i)(g_i) \mid g = \bigvee_{i\in K} g_i \Big\},$$

where the $\bigwedge$ is taken for every finite subset K of Γ. Then,

(1) *$\mathcal{B}$ is (L, M)-grillbase on X and $\langle\mathcal{B}\rangle$ is the coarsest (L, M)-grill for which $\phi_i : (X_i, \langle\mathcal{B}_i\rangle) \to (X, \langle\mathcal{B}\rangle)$ is an (L, M)-grill preserving map.*
(2) *A map $\phi : (X, \langle\mathcal{B}\rangle) \to (Y, \mathcal{G})$ is (L, M)-grill preserving map iff for each $i \in \Gamma$, $\phi \circ \phi_i : (X_i, \langle\mathcal{B}_i\rangle) \to (Y, \mathcal{G})$ is an (L, M)-grill preserving map.*

Proof. (1) From Corollary 4.4 and Proposition 4.5, $\mathcal{B}$ is an (L, M)-grillbase on X. Since ϕ_i is injective, for each $i \in \Gamma$,

$$\mathcal{B}((\phi_i)_L^{\rightarrow}(f_i)) \leq (\phi_i)_2^{\Rightarrow}\mathcal{B}_i((\phi_i)_L^{\rightarrow}(f_i)) \leq \mathcal{B}_i((\phi_i)_L^{\leftarrow}((\phi_i)_L^{\rightarrow}(f_i))) = \mathcal{B}_i(f_i).$$

Hence ϕ_i is an (L, M)-grill preserving map for each $i \in \Gamma$. Other cases are similarly proved as in Theorem 4.2 (1).

(2) It is similarly proved as in Theorem 4.2 (2). □

5. The images of (L, M)-grillbases

5.1. Theorem. *Let $\phi : X \to Y$ be a surjective map and $\mathcal{B}$ an (L, M)-grillbase on X. Then we have the following properties:*

(1) *$\phi_1^{\Rightarrow}(\mathcal{B})$ is an (L, M)-grillbase on Y.*
(2) *$\langle\phi_1^{\Rightarrow}(\mathcal{B})\rangle$ is the coarsest (L, M)-grill on Y for which $\phi : (X, \langle\mathcal{B}\rangle) \to (Y, \langle\phi_1^{\Rightarrow}(\mathcal{B})\rangle)$ is an (L, M)-grill preserving map.*
(3) *If $\mathcal{B}$ is an (L, M)-grill, then $\langle\phi_1^{\Rightarrow}(\mathcal{B})\rangle = \phi_2^{\Rightarrow}(\mathcal{B})$.*

Proof. (1) Similar to the proof of Theorem 4.1 (1).

(2) Easy because $\langle\phi_1^{\Rightarrow}(\mathcal{B})\rangle(1_X) = \mathcal{B}(1_X) = \top$.

(3) Let $\mathcal{B}$ be an (L, M)-grill. Since $\phi_L^{\rightarrow}(f) \geq h$ iff $f \geq h \circ \phi$, for each $h \in L^Y$, we have

$$\begin{aligned}\langle\phi_1^{\Rightarrow}(\mathcal{B})\rangle(h) &= \bigwedge\{\phi_1^{\Rightarrow}(\mathcal{B})(g) \mid g \geq h\} \\ &= \bigwedge\{\mathcal{B}(f) \mid \phi_L^{\rightarrow}(f) = g \geq h\} \\ &= \bigwedge\{\mathcal{B}(f) \mid f \geq h \circ \phi\} \\ &= \mathcal{B}(h \circ \phi) = \phi_2^{\Rightarrow}(\mathcal{B})(h).\end{aligned}$$

□

5.2. Remark. (1) If $\phi : X \to Y$ is a bijective function, then $\phi_1^{\Rightarrow} = \phi_2^{\Rightarrow}$ and $\phi_1^{\Leftarrow} = \phi_2^{\Leftarrow}$.

(2) If $\phi : X \to Y$ is a bijective function and $\mathcal{B}$ an (L, M)-grillbase on Y, then by (1) and Theorem 4.1 we obtain that $\phi_2^{\Leftarrow}(\mathcal{B})$ is an (L, M)-grillbase on X and $\langle\phi_2^{\Leftarrow}(\mathcal{B})\rangle$ is the coarsest (L, M)-grill on X for which $\phi : (X, \langle\phi_2^{\Leftarrow}(\mathcal{B})\rangle) \to (Y, \langle\mathcal{B}\rangle)$ is an (L, M)-grill map. Furthermore, $\phi_2^{\Leftarrow} = \phi_1^{\Leftarrow}$.

5.3. Theorem. *Let $\phi : X \to Y$ be a map and $\{\mathcal{B}_i\}_{i\in\Gamma}$ a family of (L, M)-grillbases on X satisfying the following condition:*

(C) *For every finite subset K of Γ, if $\bigvee_{i\in K} g_i = 1_X$, then $\oplus_{i\in K}\mathcal{B}_i(g_i) = \top$.*

Then,

1. *If $\phi : X \to Y$ is bijective, $\phi_1^{\Rightarrow}(\bigsqcup_{i\in\Gamma} \mathcal{B}_i) = \bigsqcup_{i\in\Gamma} \phi_1^{\Rightarrow}(\mathcal{B}_i)$,*
2. *If $\phi : X \to Y$ is injective, $\langle\phi_1^{\Rightarrow}(\bigsqcup_{i\in\Gamma} \mathcal{B}_i)\rangle = \bigsqcup_{i\in\Gamma}\langle\phi_1^{\Rightarrow}(\mathcal{B}_i)\rangle$.*

Proof. (1) We show that (C) and the following condition (C1) are equivalent:

(C1) For every finite subset K of Γ, if $\bigvee_{i\in K} h_i = 1_X$, then $\oplus_{i\in K}\phi_1^{\Rightarrow}(\mathcal{B}_i)(h_i) = \top$.

(C1)⇒(C) For every finite subset K of Γ with $\bigvee_{i\in K} g_i = 1_X$, since ϕ is injective, $\phi_L^{\rightarrow}(\bigvee_{i\in K} g_i) = \bigvee_{i\in K} \phi_L^{\rightarrow}(g_i) = 1_X$. By (C1),

$$\top = \oplus_{i\in K}\phi_1^{\Rightarrow}(\mathcal{B}_i)(\phi_L^{\rightarrow}(g_i)) \leq \oplus_{i\in K}\mathcal{B}_i(g_i).$$

(C)⇒(C1) If $\oplus_{i\in K}\phi_1^{\Rightarrow}(\mathcal{B}_i)(h_i) \neq \top$, for each $i \in K$, there exists $g_i \in L^X$ with $h_i = \phi_L^{\rightarrow}(g_i)$ such that

$$\oplus_{i\in K}\phi_1^{\Rightarrow}(\mathcal{B}_i)(h_i) \leq \oplus_{i\in K}\mathcal{B}_i(g_i) \neq \top.$$

By (C), $\bigvee_{i\in K} g_i \neq 1_X$. Hence $\bigvee_{i\in K} h_i \neq 1_X$.

Since ϕ is surjective, by Theorem 5.1, $\phi_1^{\Rightarrow}(\mathcal{B}_i)$ exists for $i \in \Gamma$. By Corollary 4.3 and (C1), $\bigsqcup_{i\in\Gamma} \phi_1^{\Rightarrow}(\mathcal{B}_i)$ exists.

For each finite subset K of Γ such that $g = \bigvee_{k\in K} g_k$ with $\phi_L^{\rightarrow}(g) = h$, we have

$$\bigsqcup_{i\in\Gamma} \phi_1^{\Rightarrow}(\mathcal{B}_i)(h) \leq \oplus_{k\in K}\phi_1^{\Rightarrow}(\mathcal{B}_k)(\phi_L^{\rightarrow}(g_k)) \leq \oplus_{k\in K}\mathcal{B}_k(g_k).$$

It implies $\bigsqcup_{i\in\Gamma} \mathcal{B}_i(g) \geq \bigsqcup_{i\in\Gamma} \phi_1^{\Rightarrow}(\mathcal{B}_i)(h)$. So, $\phi_1^{\Rightarrow}(\bigsqcup_{i\in\Gamma} \mathcal{B}_i) \geq \bigsqcup_{i\in\Gamma} \phi_1^{\Rightarrow}(\mathcal{B}_i)$.

For each finite subset J of Γ with $p = \bigvee_{j\in J} h_j$, there exists $f_j \in L^X$ with $\phi_L^{\rightarrow}(f_j) = h_j$. Thus,

$$\phi_1^{\Rightarrow}(\bigsqcup_{i\in\Gamma} \mathcal{B}_i)(p) \leq (\bigsqcup_{i\in\Gamma} \mathcal{B}_i)(\bigvee_{j\in K} f_j) \leq \oplus_{j\in K}\mathcal{B}_j(f_j).$$

So, $\phi_1^{\Rightarrow}(\bigsqcup_{i\in\Gamma} \mathcal{B}_i) \leq \bigsqcup_{i\in\Gamma} \phi_1^{\Rightarrow}(\mathcal{B}_i)$.

(2) Similarly proved as in (1) and Theorem 5.1 (2). □

5.4. Theorem. *Let $\{\phi_i : X_i \to X \mid i \in \Gamma\}$ be a family of maps. Let $\{\mathcal{B}_i\}_{i\in\Gamma}$ be a family of (L, M)-grillbases on X_i satisfying the following condition:*

(C) *For every finite subset K of Γ, if $\bigvee_{i\in K}(\phi_i)_L^{\rightarrow}(g_i) = 1_X$, then $\oplus_{i\in K}\mathcal{B}_i(g_i) = \top$.*

We define a mapping $\biguplus_{i\in\Gamma}(\phi_i)_1^{\Rightarrow}(\mathcal{B}_i) : L^X \to M$ as

$$\biguplus_{i\in\Gamma}(\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)(h) = \bigwedge\Big\{ \oplus_{i\in K} \mathcal{B}_i(g_i) \mid h = \bigvee_{i\in K} (\phi_i)_L^{\rightarrow}(g_i)\Big\},$$

where the $\bigwedge$ is taken for every finite subset K of Γ. Then,

1. *If ϕ_j is surjective for some $j \in \Gamma$, then $\mathcal{B} = \biguplus_{i\in\Gamma}(\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)$ is an (L, M)-grillbase on X and $\langle\mathcal{B}\rangle$ is the coarsest (L, M)-grill for which $\phi_i : (X_i, \langle\mathcal{B}_i\rangle) \to (X, \langle\mathcal{B}\rangle)$ is an (L, M)-grill preserving map.*

(2) *a map* $\phi : (X, \langle\mathcal{B}\rangle) \to (Y, \langle\mathcal{G}\rangle)$ *is an* (L, M)*-grill preserving map iff for each* $i \in \Gamma$, $\phi \circ \phi_i : (X_i, \langle\mathcal{B}_i\rangle) \to (Y, \mathcal{G})$ *is an* (L, M)*-grill preserving map.*

(3) *If* ϕ_i *are surjective for all* $i \in \Gamma$,

$$\langle \biguplus_{i\in\Gamma} (\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)\rangle = \langle \bigsqcup_{i\in\Gamma} (\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)\rangle.$$

Proof. (1) (LB1) Since ϕ_j is surjective for some $j \in \Gamma$ and (C), $\mathcal{B}(1_X) = \top$ and $\mathcal{B}(1_\emptyset) = \bot$. The other cases are similar to the proof of Theorem 4.2 (1).

(2) Similarly proved as in Theorem 4.2 (2).

(3) We show that the following condition (C1) and (C) are equivalent:

(C1) For every finite subset K of Γ, if $\bigvee_{i\in K} h_i = \underline{1}$, then $\oplus_{i\in K}(\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)(h_i) = 1$.

(C1) $\Longrightarrow$ (C) For every finite subset K of Γ, if $\bigvee_{i\in K}(\phi_i)_L^{\rightarrow}(g_i) = 1_X$, by (C1), $\top = \oplus_{i\in K}(\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)((\phi_i)_L^{\rightarrow}(g_i)) \le \oplus_{i\in K}\mathcal{B}_i(g_i)$.

(C) $\Longrightarrow$ (C1) If $\oplus_{i\in K}(\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)(h_i) \neq \top$, for each $i \in K$, there exists $g_i \in L^{X_i}$ with $h_i = (\phi_i)_L^{\rightarrow}(g_i)$ such that

$$\oplus_{i\in K}(\phi_i)_1^{\Rightarrow}\mathcal{B}_i(h_i) \le \oplus_{i\in K}\mathcal{B}_i(g_i) \neq \top.$$

By (C), $\bigvee_{i\in K}(\phi_i)_L^{\rightarrow}(g_i) = \bigvee_{i\in K} h_i \neq 1_X$.

For each finite index K with $\{g_i \mid \bigvee_{i\in K}(\phi_i)_L^{\rightarrow}(g_i) \ge h\}$, by the definition of $\langle\bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)\rangle$, we have

$$\begin{aligned}\langle \bigsqcup_{i\in\Gamma} (\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)\rangle(h) &\le \bigwedge_{i\in\Gamma}(\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)(\bigvee_{i\in K}(\phi_i)_L^{\rightarrow}(g_i)) \\ &\le \oplus_{i\in K}(\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)((\phi_i)_L^{\rightarrow}(g_i))) \\ &\le \oplus_{i\in K}\mathcal{B}_i(g_i).\end{aligned}$$

Hence $\langle\biguplus_{i\in\Gamma}(\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)\rangle \ge \langle\bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)\rangle$.

For each finite index J with $\{h_i \mid \bigvee_{i\in J} h_i \ge p\}$, since ϕ_i is surjective for each $i \in J$, there exists $p_i \in L^{X_i}$ with $(\phi_i)_L^{\rightarrow}(p_i) = h_i$ such that $p \le \bigvee_{i\in J} h_i = \bigvee_{i\in J}(\phi_i)_L^{\rightarrow}(p_i)$. Thus,

$$\langle \biguplus_{i\in\Gamma} (\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)\rangle(p) \le \oplus_{i\in J}(\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)(h_i) \le \oplus_{i\in J}\mathcal{B}_i(p_i).$$

Hence $\langle\biguplus_{i\in\Gamma}(\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)\rangle \le \langle\bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Rightarrow}(\mathcal{B}_i)\rangle$. □

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