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INTRODUCTION TO GENERALIZED SPATIAL LOCALES

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Abstract

We review the notion of generalized topological space and introduce generalized spatial locales (*gs-locales*) and their density, describe homomorphisms and isomorphisms of gs-locales, provide representation theorems for generalized topological spaces and gs-locales and show the categorical relations between gs-locales and T_0 topological spaces.

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1. Motivation

The concept of *topological space modulo ideal* was introduced in [4]. Later we developed the notion of *generalized topological space* (*gt-space* for short) [5]. The generalized topology of a gt-space is a frame. The join and meet operations of a generalized topology may not coincide with the usual union and intersection operations for sets, in fact the join and meet operations are union and intersection operations modulo *small sets*. The family of small sets of a gt-space possesses the structure of an *ideal* and contains no open sets, the only exception being the empty set which is both open and small.

In order to better understand the nature of generalized topological spaces we decided to look at them from the point of view of locale theory.

The second motivation for our research follows from the following observation. Locale theory study the isomorphism between the category of *sober spaces* and the category of *spatial locales* [3]. But there exist topological spaces that are not sober and frames that are not spatial. In Section 2, we provide the right arrow generalized topological space. Obviously, this space has a nontrivial topological structure. On the other hand, the generalized topology of this space is a frame but not a spatial frame and, moreover, the family of all its principal prime ideals [3] is empty, which means this frame is not

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interesting from the point of view of locale theory. Thus, the notion of *generalized locale* is needed to describe this and other gt-spaces in localic form.

In this paper, we review the notion of generalized topological space and introduce generalized spatial locales (gs-locales) and its density, describe homomorphisms and isomorphisms of gs-locales, provide representation theorems for generalized topological spaces and gs-locales and show the categorical relations between gs-locales and T_0 topological spaces.

2. Generalized topological spaces (gt-spaces)

All notions and results from this section are introduced in [5].

2.1. Proposition. *Let a nonempty set X and an ideal $I \subseteq 2^X$ be given. Then the relation $\preceq$, defined as follows, is a preorder on 2^X, that means it is reflexive and transitive:*

$$A \preceq B \text{ iff } A \setminus B \in I.$$

The relation $\approx$, defined as follows, is an equivalence on 2^X, that means it is reflexive, transitive and symmetric:

$$A \approx B \text{ iff } A \setminus B \in I \text{ and } B \setminus A \in I.$$

The relations $\preceq$ and $\approx$ are called the preorder generated by the ideal I *and* the equivalence generated by the ideal I, *respectively.*

2.2. Theorem. *Let X be a nonempty set. Assume that $T \subseteq 2^X$ forms a frame with respect to $\subseteq$ and $\varnothing, X \in T$. Then there exists the least ideal $I \subseteq 2^X$ such that:*

(1) $\bigvee \mathfrak{U} \setminus \bigcup \mathfrak{U} \in I$ *for every* $\mathfrak{U} \subseteq T$;
(2) $(V \cap W) \setminus (V \wedge W) \in I$ *for every* $V, W \in T$;
(3) $T \cap I = \{\varnothing\}$;
(4) $U \preceq V$ *implies* $U \subseteq V$ *for every* $U, V \in T$;
(5) $U \approx V$ *implies* $U = V$ *for every* $U, V \in T$;
(6) *the ideal I is compatible with T, write $I \sim T$, i.e. $A \subseteq X$ and $\mathfrak{U} \subseteq T$ with $A \subseteq \bigvee \mathfrak{U}$ and $A \cap U \in I$, for all $U \in \mathfrak{U}$, imply that $A \in I$.*

2.3. Definition. Let X be a nonempty set. A family $T \subseteq 2^X$ is called a *generalized topology* (or *topology modulo ideal*) and the pair (X, T) is called a *generalized topological space* (*gt-space* for short, or *topological space modulo ideal*) provided that:

(GT1) $\varnothing, X \in T$;
(GT2) $(T, \subseteq)$ is a frame.

An ideal $J \subseteq 2^X$ satisfying (1)-(6) of Theorem 2.2 is called *suitable.* If there is no chance for confusion, we keep the notation I, sometimes with an appropriate index, to denote *the least suitable ideal.* If the ideal is not specified in a definition or construction then it is always the least suitable ideal.

If there is no specification or index, we use the symbols $\preceq$ and $\approx$ to denote the preorder and equivalence, respectively, generated by the least suitable ideal. We keep the notations $\vee$ and $\wedge$ for the frame operations of the generalized topology.

2.4. Definition. A gt-space (X, T) is called T_0 iff for every distinct $x, y \in X$ there is $U \in T$ such that $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.

A topological space is a trivial example of gt-space, where the least suitable ideal consists only of the empty set. In order to distinguish between topological spaces and gt-spaces that are not topological spaces, we provide the following classification.

2.5. Definition. A gt-space is called

(1) *crisp gt-space* (*crisp space* for short) iff its least suitable ideal is $\{\varnothing\}$;
(2) *proper gt-space* iff it is not crisp.

2.6. Example (Right arrow gt-space)**.** Consider the real line $\mathbb{R}$ and the family of "right arrows"

$$\mathcal{A} = \{\, [a,b) \mid a,b \in \mathbb{R} \cup \{-\infty, +\infty\} \text{ and } a < b \,\}\,.$$

We say that a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ is *well separated* iff for all $[a,b), [c,d) \in \mathcal{A}'$ it holds that $b < c$ or $d < a$. Construct the family T_{ra} as follows:

$$T_{ra} = \{\, \varnothing, \mathbb{R} \,\} \cup \left\{ \bigcup \mathcal{A}' \mid \mathcal{A}' \subseteq \mathcal{A} \text{ and } \mathcal{A}' \text{ is well separated} \right\}.$$

Then the pair $(\mathbb{R}, T_{ra})$ forms a gt-space. The respective least ideal for this gt-space is the family D of nowhere dense subsets of the real line.

2.7. Definition. Let (X, T_X) and (Y, T_Y) be gt-spaces. A mapping $f\colon X \to Y$ is called a *generalized continuous mapping* (or *g-continuous mapping* for short) provided that there exists a frame homomorphism $h\colon T_Y \to T_X$ such that $h(U) \approx f^{-1}(U)$ holds for every $U \in T_Y$.

The g-continuous mapping f is called a *generalized homeomorphism* (or *g-homeomorphism* for short) iff f is a bijection and f^{-1} is g-continuous.

2.8. Theorem. *Given gt-spaces (X, T_X) and (Y, T_Y) and a g-continuous mapping $f\colon X \to Y$, the following hold:*

(i) *the corresponding frame homomorphism $h\colon T_Y \to T_X$ is unique;*
(ii) *$f^{-1}(B) \in I_X$ holds for all $B \in I_Y$.*

2.9. Proposition. *Given gt-spaces (X, T_X), (Y, T_Y) and (Z, T_Z) and g-continuous mappings $f\colon X \to Y$ and $g\colon Y \to Z$, the composition $g \circ f\colon X \to Z$ is also a g-continuous mapping.*

3. Definition of gs-locale. Density

Let us recall some definitions from [1]. The subset P of a lattice L is called:

- *proper* iff $P \neq L$;
- *lower* iff $a \leq b$ and $b \in P$ imply $a \in P$ for all $a, b \in L$;
- *prime* iff P is lower and $a \wedge b \in P$ implies $a \in P$ or $b \in P$ for all $a, b \in L$;
- *ideal* iff P is lower and $a, b \in P$ implies $a \vee b \in P$ for all $a, b \in L$;
- *principal* iff P is lower and $\bigvee P \in P$.

In what follows, we will use the following notations for a given frame T:

(1) $\mathbf{L}(T)$ – the family of all nonempty proper lower subsets of T;
(2) $\mathbf{Pr}(T)$ – the family of all prime subsets of T;
(3) $\mathbf{pt}(T)$ – the family of all principal prime ideals of T.

3.1. Definition. Let T be a frame. We say that a subfamily $L \subseteq \mathbf{L}(T)$ *strongly separates* the elements of T iff for every $u, v \in T$ with $v \not\leq u$ there exists $A \in L$ such that $u \in A$ and $v \notin A$.

3.2. Proposition. *Let T be a frame and $L \subseteq \mathbf{L}(T)$ a subfamily that strongly separates the elements of T. Define a mapping $p\colon T \to 2^L$ as follows, for all $u \in T$:*

$$p(u) = \{A \in L \mid u \notin A\}.$$

Then the pair $(L, p(T))$ forms a T_0 gt-space and the mapping $p\colon T \to p(T)$ is a frame isomorphism.

Proof. The mapping $p\colon T \to p(T)$ is one-to-one mapping due to its definition, and injective since L strongly separates the elements of T. Indeed, consider different $u, v \in T$ and, without loss of generality, assume that $v \not\leq u$. Then there is $A \in L$ such that $u \in A$ and $v \notin A$. Hence, $A \notin p(u)$ and $A \in p(v)$ and we conclude that $p(u) \neq p(v)$. We have shown that p is bijective.

Let us show that the mapping p is monotone. Consider $u, v \in T$ such that $u \leq v$ and $A \in p(u)$. Then $u \notin A$ and we conclude that $v \notin A$, since A is a lower set. Hence, $A \in p(v)$ and $p(u) \subseteq p(v)$.

Since p is bijective, we can define the followings operations for $p(T)$, where $U \subseteq T$ and $v, w \in T$:

$$\bigvee_{u \in U} p(u) = p\left(\bigvee U\right) \text{ and } p(v) \wedge p(w) = p(v \wedge w).$$

Since p is monotone, it is easy to see that $p(T)$ together with the defined operations forms a frame and $(L, p(T))$ is a gt-space.

Let us show that $(L, p(T))$ is T_0. Consider different $A, B \in L$. Without loss of generality, assume that there is $u \in T$ such that $u \in B$ and $u \notin A$. Hence, $A \in p(u)$ and $B \notin p(u)$, i.e. the *points* A and B are separated by the *open set* $p(u)$.

We already showed that p is bijective. The fact that p preserves arbitrary joins and finite meets is due to the definition of join and meet for $p(T)$. Hence, p is a frame homomorphism. The proof is complete. □

3.3. Definition. Given a frame T and a subfamily $L \subseteq \mathbf{L}(T)$, we call the pair (T, L) a *generalized spatial locale* (*gs-locale* for short) provided that L strongly separates the elements of T.

3.4. Example. It is known that in a distributive lattice two elements can be strongly separated by a prime ideal [1]. Hence, given frame T, the pair $(T, \mathbf{Pr}(T))$ forms a gs-locale.

3.5. Proposition. *Given a gs-locale (T, L) where T is finite, it holds that* $\mathbf{Pr}(T) = \mathbf{pt}(T) \subseteq L$.

Proof. Consider a prime ideal $P \in \mathbf{Pr}(T)$. Since T is finite, $u = \bigvee P \in P$, and, hence, the prime ideal P is principal, that is $P \in \mathbf{pt}(T)$. Take $v = \bigwedge(T \setminus P)$. Then $v \not\leq u$ and there exists $A \in L$ such that $u \in A$ and $v \notin A$. We conclude that $A = P$ and, hence, $P \in L$. □

3.6. Definition. Let T be a frame. The least cardinal number of the form $|S|$ where $S \subseteq \mathbf{L}(T)$ strongly separates the elements of T is called the *density of* T, and is denoted by $d(T)$.

We know already (Example 3.4) that $d(T) \leq |\mathbf{Pr}(T)|$ for a given frame T, and if T is finite then $d(T) = |\mathbf{Pr}(T)|$ by Proposition 3.5. The following example demonstrates that the later equality does not necessarily hold in an arbitrary frame.

3.7. Example. Consider the frame T_{ra} (Example 2.6) and the subfamilies $s(\mathbb{Q})$ and $s(\mathbb{R})$ of $\mathbf{L}(T_{ra})$ defined as follows:

$$s(\mathbb{Q}) = \{\, \{\, U \in T_{ra} \mid q \notin U \,\} \mid q \in \mathbb{Q} \,\},$$
$$s(\mathbb{R}) = \{\, \{\, U \in T_{ra} \mid r \notin U \,\} \mid r \in \mathbb{R} \,\}.$$

We claim that the family $s(\mathbb{Q})$ strongly separates the elements of T_{ra}. Let us prove it. Consider $U, V \in T_{ra}$, and assume that $V \nsubseteq U$. Then, since $(\mathbb{R}, T_{ra})$ is a gt-space and the least suitable ideal is the family of nowhere dense subsets of $\mathbb{R}$, it holds that $V \not\precsim U$

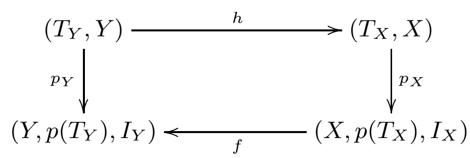
and the subset $V \setminus U$ cannot be nowhere dense in $\mathbb{R}$. Hence, there exists $[a,b) \subseteq V \setminus U$. Fix a point $q \in [a,b) \cap \mathbb{Q}$. The family $s(\mathbb{Q})$ separates U and V. Indeed, $U \in s(\mathbb{Q})$ and $V \notin s(\mathbb{Q})$.

On the other hand, it is not difficult to check that $\mathbf{Pr}(T_{ra}) = s(\mathbb{R})$. Hence, we obtain the following estimation

$$d(T_{ra}) \leq |\mathbb{Q}| < |\mathbf{Pr}(T_{ra})|.$$

4. Morphisms

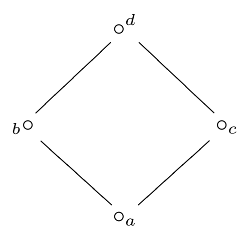
4.1. Definition. Let (T_Y, Y) and (T_X, X) be gs-locales. A pair of mappings (h, f) is called a *homomorphism* provided that $h\colon T_Y \to T_X$ is a frame homomorphism, and $f\colon X \to Y$ is such that $p_X(h(U)) \Delta f^{-1}(p_Y(U)) \in I_X$ for every $U \in T_Y$.



The homomorphism (h, f) is called an *isomorphism* provided that f is a bijection and h is a frame isomorphism.

Note that for a given frame homomorphism h the corresponding mapping f (if it exists) is not necessarily unique. This is illustrated by the following example.

4.2. Example. Consider the four element frame $T = \{a, b, c, d\}$

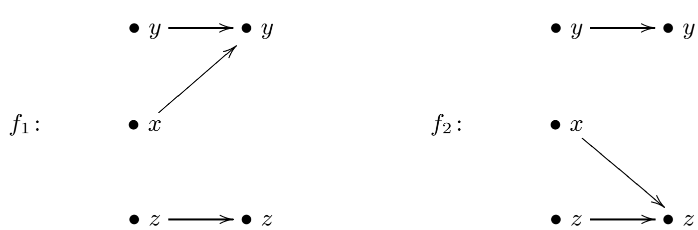


Let $x = \{a\}$, $y = \{a, b\}$, and $z = \{a, c\}$. Take two subfamilies of lower sets $X = \{x, y, z\}$ and $Y = \{y, z\}$. Then (T, X) and (T, Y) form gs-locales and the corresponding g-topological spaces look like this:

$$(X, \{\varnothing, X, \{x,y\}, \{x,z\}\}),\ I_X = \{\varnothing, \{x\}\},$$

$$\text{and } (Y, \{\varnothing, Y, \{y\}, \{z\}\}),\ I_Y = \{\varnothing\}.$$

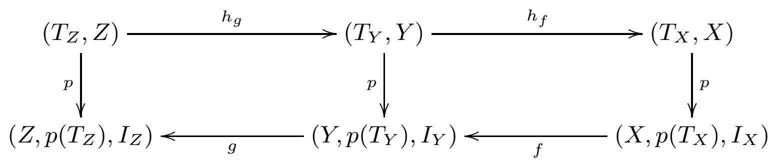
Set $h\colon T \to T$ to be the identity mapping and define mappings f_1 and f_2 as follows:



Then (h, f_1) and (h, f_2) are homomorphisms of the gs-locales (T, X) and (T, Y).

4.3. Proposition. *Given gs-locales* (T_Z, Z)*,* (T_Y, Y) *and* (T_X, X)*; and homomorphisms* $(h_g, g)\colon (T_Z, Z) \to (T_Y, Y)$ *and* $(h_f, f)\colon (T_Y, Y) \to (T_X, X)$*; the pair* $(h_f \circ h_g, g \circ f)$ *is a homomorphism of the gs-locales* (T_Z, Z) *and* (T_X, X)*.*

Proof. Let us consider the following diagram:



Clearly, $h_f \circ h_g$ is a frame homomorphism. We have to check that

$$p(h_f(h_g(U)))\Delta f^{-1}(g^{-1}(p(U))) \in I_X$$

holds for every $U \in T_Z$. This holds by Theorem 2.9. The symbol Δ denotes here the set-theoretic difference, that means $A\Delta B = (A \setminus B) \cup (B \setminus A)$ for sets A and B. □

5. Representation theorems

5.1. Proposition. *Consider a* T_0 *gt-space* (X, T) *and a mapping* $s\colon X \to 2^T$ *defined as*

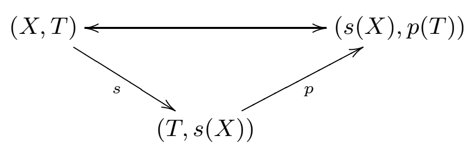
$$s(x) = \{U \in T \mid x \notin U\}$$

for all $x \in X$*. Then the pair* $(T, s(X))$ *forms a gs-locale and the mapping* $s\colon X \to s(X)$ *is a bijection.*

Proof. It follows from the definition of gt-space that T is a frame. The family $s(X)$ separates the elements of T since (X, T) is T_0. □

5.2. Theorem. *A* T_0 *gt-space* (X, T) *is characterized up to g-homeomorphism by the gs-locale* $(T, s(X))$*.*

Proof. We start with the T_0 gt-space (X, T), and construct the gs-locale $(T, s(X))$ according to Proposition 5.1. Then we construct the gt-space $(s(X), p(T))$ according to Proposition 3.2. We have to verify that the gt-spaces (X, T) and $(s(X), p(T))$ are g-homeomorphic. This is shown in the following diagram:



The mapping $s\colon X \to s(X)$ is a bijection and the mapping $p\colon T \to p(T)$ is a frame isomorphism. Hence, to prove that s is a g-homeomorphism, it is enough to show that $s(U) = p(U)$ holds for all $U \in T$:

$$\begin{aligned} s(U) &= \{s(x) \in s(X) \mid x \in U\} \\ &= \{s(x) \in s(X) \mid U \notin s(x)\} \\ &= \{s(x) \in s(X) \mid s(x) \in p(U)\} \\ &= p(U). \end{aligned}$$

The proof is complete. □

5.3. Theorem. *A gs-locale* (T, L) *is characterized up to isomorphism by the gt-space* $(L, p(T))$*.*

Proof. The proof is similar to the proof of the previous theorem. □

5.4. Proposition. *Consider a gs-locale* (T, L)*, the corresponding gt-space* $(L, p(T))$*, and* $Y \in L$*. Then*

(i) Y *is prime iff* $Y \in U \cap V$ *implies* $Y \in U \wedge V$ *for every* $U, V \in p(T)$*;*
(ii) Y *is ideal iff* $Y \in U \vee V$ *implies* $Y \in U \cup V$ *for every* $U, V \in p(T)$*;*
(iii) Y *is principal iff* $Y \in \bigvee \mathcal{U}$ *implies* $Y \in \bigcup \mathcal{U}$ *for every* $\mathcal{U} \subseteq p(T)$*.* □

5.5. Proposition. *Let* (T, L) *be a gs-locale. The following are equivalent:*

(i) $L \subseteq \mathbf{pt}(T)$*;*
(ii) *The mapping* $p\colon T \to 2^L$ *is a frame homomorphism;*
(iii) *The pair* $(L, p(T))$ *is a* T_0 *topological space.*

5.6. Definition. A pair (T, L) is called a

(1) *crisp gs-locale* iff $L \subseteq \mathbf{pt}(T)$,
(2) *spatial locale* iff $L = \mathbf{pt}(T)$,
(3) *proper gs-locale* iff $L \setminus \mathbf{pt}(T) \neq \varnothing$.

6. Isomorphism of categories $\mathbf{GTop}_0$ and GSLoc. Classification of subcategories

We use the categorical notions from [2].

6.1. Proposition. *All* T_0 *gt-spaces and g-continuous mappings between* T_0 *gt-spaces form a concrete category. We denote this category by* $\mathbf{GTop}_0$*.*

Proof. The composition law is the natural composition of mappings. It is easy to check that the matching and associativity conditions are satisfied. The identity morphisms are the identity mappings. The smallness of morphism class condition is satisfied, since $hom_{\mathbf{C}}(X, Y) \subseteq Y^X$ and Y^X is a set for all sets X and Y. Hence, $\mathbf{GTop}_0$ forms a category.

Clearly, the the function F that takes each T_0 gt-space (X, T) to the set X defines a forgetful functor. The property (i) from Theorem 2.8 shows that the functor F is faithful. Hence, the category $\mathbf{GTop}_0$ is concrete. □

6.2. Proposition. *All gs-locales and homomorphisms of gs-locales form a category. We denote this category by* **GSLoc**.

Proof. The composition law is the natural composition of mappings. Clearly, the matching and associativity conditions are satisfied. The identity morphisms are identity mappings. The smallness of morphism class condition is also satisfied. Hence, **GSLoc** forms a category. □

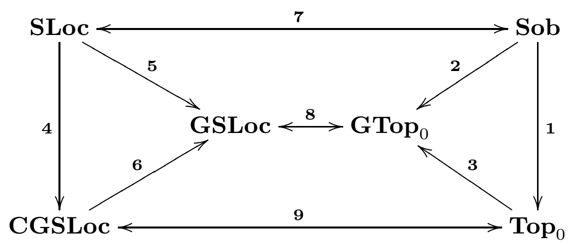
6.3. Theorem. *The categories* $\mathbf{GTop}_0$ *and* **GSLoc** *are isomorphic.*

Proof. In Propositions 5.2 and 5.3, we showed that $F\colon Obj(\mathbf{GTop}_0) \to Obj(\mathbf{GSLoc})$ is a bijection. Applying Theorem 2.8, we conclude that F is a faithful functor. The fact that F is a full functor can be easily obtained using a similar argument to that used in the proofs of Propositions 5.2 and 5.3. □

Consider the following categories:

GTop$_0$	T_0 gt-spaces,
Top$_0$	T_0 topological spaces,
Sob	sober spaces,
GSLoc	gs-locales,
CGSLoc	crisp gs-locales,
SLoc	spatial locales.

The following chart provides the relations between these categories. The arrows of the form $\longrightarrow$ represent the relation "to be a subcategory", e.g. **CGSLoc** is a subcategory of **GSLoc**. The arrows of the form $\longleftrightarrow$ represent the relation "to be isomorphic categories", e.g. the categories **CGSLoc** and **Top**$_0$ are isomorphic.



Let us comment the considered relations.

1,7: These are known results from locale theory – sober spaces are T_0 spaces, and the categories of sober spaces and of spatial locales are isomorphic [3].

2,3: It follows from the definition of gt-space that every T_0 topological space and, hence, every sober space is a T_0 gt-space.

4,5,6: According to our classification (Def. 5.6), every spatial locale is also a crisp gs-locale, and spatial locales and crisp spatial locales are gs-locales.

8,9: The relation 8 is presented in Theorem 6.3. The relation 9 is a corollary from this theorem and Proposition 5.5.

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