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SOFT LATTICE IMPLICATION ALGEBRAS BASED ON FUZZY SETS

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Abstract

In this paper, we deal with soft lattice implication algebras based on fuzzy sets. Using the concepts of fuzzy (implicative) filters, $(\in, \in\vee q)$-fuzzy (implicative) filters and $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy (implicative) filters, some characterizations for $\in$-soft sets and q-soft sets to be (implicative) filteristic soft lattice implication algebras are established.

Keywords: (Implicative) filter, $(\in, \in\vee q)$-fuzzy (implicative) filter, $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy (implicative) filter, Fuzzy (implicative) filter with thresholds; (Implicative) filteristic soft lattice implication algebra.

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1. Introduction

To solve complicated problems in economics, engineering, and the environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which have been pointed out in [9]. Maji *et al.* [7] and Molodtsov [9] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tools of the theory. To overcome these difficulties, Molodtsov [9] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At

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present, research on soft set theory is progressing rapidly. Maji *et al.* [8] described the application of soft set theory to a decision making problem.

The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [14, 15]. Jun *et al.* [3] applied the notion of the soft sets of Molodtsov to the theory of BCK/BCI-algebras, introduced the notions of soft BCK/BCI-algebras, and then investigated their basic properties [4]. Aktas *et al.* [1] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing some examples to clarify their differences. They also discussed the notion of soft groups.

A logical system whose propositional value is given in a lattice was first studied by Xu in [11] from the semantic viewpoint. He then proposed the concept of lattice implication algebras and discussed some of their properties. Later on, Xu and Qin [12] discussed the properties of implicative filters in a lattice implicative algebra. Since then these logical algebras have been extensively investigated by several researchers, see [5,6, 16-19]. For more details, the reader is refereed to the book [13].

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which was mentioned in [10], played a vital role in generating some different types of fuzzy subsets. It is worth pointing out that Bhakat and Das [2] initiated the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation ($\in$) and the "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee\, \mathrm{q})$-fuzzy subgroup. In fact, the $(\in, \in \vee\, \mathrm{q})$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup.

In this paper, we deal with soft lattice implication algebras based on fuzzy sets. By means of $\in$-soft sets and q-soft sets, we investigate some characterizations of (implicative) filteristic soft lattice implication algebras.

2. Preliminaries

To start with, we first recall that an algebra $(L, \vee, \wedge, ', \rightarrow, 0, 1)$ is a lattice implication algebra [11] if $(L, \vee, \wedge, 0, 1)$ is a bounded lattice with an order-reversing involution " $'$ " and there is a binary operation " $\rightarrow$ " such that the following axioms are satisfied:

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(I2) $x \rightarrow x = 1$,
(I3) $x \rightarrow y = y' \rightarrow x'$,
(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,
(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$, for all $x, y, z \in L$.

Throughout this paper, L is a lattice implication algebra unless otherwise specified.

We cite below some notations, definitions and basic results which will be needed in the sequel.

A subset A of L is said to be a *filter* of L if it satisfies: (i) $1 \in A$, (ii) $x \in A$ and $x \rightarrow y \in A$ imply $y \in A$, for all $x, y \in L$. A subset A of L is called an *implicative filter* of L if it satisfies (i) and (iii) $x \rightarrow (y \rightarrow z) \in A$ and $x \rightarrow y \in A$ imply $x \rightarrow z \in A$, for all $x, y, z \in L$.

We now review some fuzzy logic concepts. A fuzzy set of L is a function $\mu : L \rightarrow [0, 1]$.

Now, we recall some the following concepts and results in [13].

2.1. Definition. A fuzzy set μ of L is called a *fuzzy filter* of L if it satisfies:

(F1) $\forall x \in L, \mu(1) \geq \mu(x)$;
(F2) $\forall x, y \in L, \mu(y) \geq \min\{\mu(x \rightarrow y), \mu(x)\}$.

2.2. Definition. A fuzzy filter μ of L is called a *fuzzy implicative filter* of L if it satisfies:

(F3) $\mu(x \to z) \geq \min\{\mu(x \to (y \to z)), \mu(x \to y)\}$, for all $x, y, z \in L$.

3. Filteristic soft lattice implication algebras

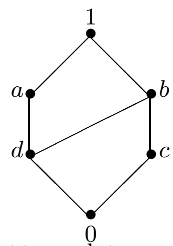
Molodtsov [9] defined a soft set in the following way: Let U be an initial universe set and E a set of parameters. Let $\mathcal{P}(U)$ denotes the power set of U and $A \subset E$.

A pair (F, A) is called a *soft set* over U, where F is a mapping given by $F : A \to \mathcal{P}(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F, A).

3.1. Definition. Let (F, A) be a soft set over L. Then (F, A) is called a *filteristic soft lattice implication algebra over* L if $F(x)$ is a filter of L for all $x \in A$, where for our convenience the empty set $\emptyset$ is regarded as a filter of L.

3.2. Example. Let $L = \{0, a, b, c, d, 1\}$ be a subset with an involution "$'$" and a binary operation " $\to$ " with Hasse diagram and Cayley tables as shown below:



x	x'
0	1
a	c
b	d
c	a
d	b
1	0

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Define the $\vee-$ and $\wedge-$ operations on L as follows

$$x \vee y = (x \to y) \to y, \; x \wedge y = ((x' \to y') \to y')'.$$

Then it can be verified that L is a lattice implication algebra. Let (F, A) be a soft set over L, where $A = (0, 1]$ and $F : A \to \mathcal{P}(L)$ is a set-valued function defined by

$$F(x) = \begin{cases} \{0, a, b, c, 1\} & \text{if } 0 < x \leq 0.3, \\ \{1, a\} & \text{if } 0.3 < x \leq 0.6, \\ \emptyset & \text{if } 0.6 < x \leq 1. \end{cases}$$

Thus, $F(x)$ is a filter of L for all $x \in A$, and so (F, A) is a filteristic soft lattice implication algebra over L.

Given a fuzzy set μ in any lattice implication algebra L and $A \subseteq [0, 1]$, consider two set-valued functions

$$F : A \to \mathcal{P}(L), \quad t \mapsto \{x \in L \mid x_t \in \mu\}$$

and

$$F_{\mathrm{q}} : A \to \mathcal{P}(L), \quad t \mapsto \{x \in L \mid x_t \, \mathrm{q} \, \mu\}.$$

Then (F, A) and (F_{q}, A) are called an $\in$-*soft set* and q-*soft set* over L, respectively.

3.3. Theorem. *Let μ be a fuzzy set of L and (F, A) an $\in$-soft set over L with $A = (0, 1]$. Then (F, A) is a filteristic soft lattice implication algebra over L if and only if μ is a fuzzy filter of L.*

Proof. Let μ be a fuzzy filter of L and $t \in A$. If $x \in F(t)$, then $x_t \in \mu$, and so $1_t \in \mu$, i.e., $1 \in F(t)$. Let $x, y \in L$ be such that $x, x \to y \in F(t)$. Then $x_t \in \mu$ and $(x \to y)_t \in \mu$, and so $y_{\min\{t,t\}} = y_t \in \mu$. Hence $y \in F(t)$. Hence (F, A) is a filteristic soft lattice implication algebra over L.

Conversely, assume that (F, A) is a filteristic soft lattice implication algebra over L. If there exists $a \in L$ such that $\mu(1) < \mu(a)$, then we can choose $t \in A$ such that $\mu(1) < t \leq \mu(a)$. Thus, $1_t \overline{\in} \mu$, i.e., $1 \overline{\in} F(t)$. This is a contradiction. Thus, $\mu(1) \geq \mu(x)$, for all $x \in L$. If there exist $a, b \in L$ such that $\mu(b) < s \leq \min\{\mu(a \to b), \mu(a)\}$, then $(a \to b)_s \mu$ and $a_s \in \mu$, but $b_s \overline{\in} \mu$, that is, $a \to b \in F(s)$ and $a \in F(s)$, but $b \overline{\in} F(s)$, contradiction, and so, $\mu(y) \geq \min\{\mu(x \to y), \mu(x)\}$, for all $x, y \in L$. Therefore, μ is a fuzzy filter of L. □

3.4. Theorem. *Let μ be a fuzzy set of L and (F_q, A) a q-soft set over L with $A = (0, 1]$. Then the following are equivalent:*

(i) *μ is a fuzzy filter of L;*
(ii) *$\forall t \in A, F_{\mathrm{q}}(t) \neq \emptyset \Rightarrow F_q(t)$ is a filter of L.*

Proof. Let μ be a fuzzy filter of L. Take any $t \in A$ with $F_{\mathrm{q}}(t) \neq \emptyset$. If $1 \overline{\in} F_{\mathrm{q}}(t)$, then $1_t \overline{q} \mu$, and so $\mu(1) + t < 1$. Then $\mu(x) + t \leq \mu(1) + t < 1$ for all $x \in L$, and so $F_{\mathrm{q}}(t) = \emptyset$, contradiction. Hence $1 \in F_{\mathrm{q}}(t)$. Let $x, y \in L$ be such that $x \to y \in F_{\mathrm{q}}(t)$ and $x \in F_{\mathrm{q}}(t)$. Then $(x \to y)_t q \mu$ and $x_t q \mu$, or equivalently, $\mu(x \to y) + t > 1$ and $\mu(x) + t > 1$. Thus,

$$\begin{aligned} \mu(y) + t &\geq \min\{\mu(x \to y), \mu(x)\} + t \\ &= \min\{\mu(x \to y) + t, \mu(x) + t\} \\ &> 1, \end{aligned}$$

and so $y_t q \mu$, i.e., $x \in F_{\mathrm{q}}(t)$. Hence $F_{\mathrm{q}}(t)$ is a filter of L.

Conversely, assume that the condition (ii) holds. If $\mu(1) < \mu(a)$ for some $a \in L$, then $\mu(1) + t \leq 1 < \mu(a) + t$ for some $t \in A$. Thus, $a_t q \mu$, and so $F_{\mathrm{q}}(t) \neq \emptyset$. Hence $1 \in F_{\mathrm{q}}(t)$, and so $1_t q \mu$, i.e., $\mu(1) + t > 1$, a contradiction. Hence $\mu(1) \geq \mu(x)$ for all $x \in L$.

If there exist $a, b \in L$ such that $\mu(b) < \min\{\mu(a \to b), \mu(a)\}$. Then $\mu(b) + s \leq 1 < \min\{\mu(a \to b), \mu(a)\} + s$ for some $s \in A$. Hence $(a \to b)_s q \mu$ and $a_s q \mu$, i.e., $a \to b \in F_{\mathrm{q}}(s)$ and $a \in F_{\mathrm{q}}(s)$. Since $F_{\mathrm{q}}(s)$ is a filter of L, we have $b \in F_{\mathrm{q}}(s)$, and so $b_s q \mu$, that is, $\mu(b) + s > 1$, a contradiction. Hence $\mu(y) \geq \min\{\mu(x \to y), \mu(x)\}$, for all $x, y \in L$. Therefore μ is a fuzzy filter of L. □

3.5. Definition. [6] A fuzzy set μ of L is an $(\in, \in \vee \mathrm{q})$-*fuzzy filter* of L if it satisfies:

(F4) $\mu(1) \geq \min\{\mu(x), 0.5\}$;
(F5) $\mu(y) \geq \min\{\mu(x \to y), \mu(x), 0.5\}$.

3.6. Theorem. *Let μ be a fuzzy set of L and (F, A) an $\in$-soft set over L with $A = (0, 0.5]$. Then the following are equivalent:*

(i) *μ is an $(\in, \in \vee \mathrm{q})$-fuzzy filter of L;*
(ii) *(F, A) is a filteristic soft lattice implication algebra over L.*

Proof. Let μ be an $(\in, \in \vee \mathrm{q})$-fuzzy filter of L. For any $t \in A$, we have $\mu(1) \geq \min\{\mu(x), 0.5\}$ for all $x \in F(t)$ by Definition 3.5 (F4). Hence $\mu(1) \geq \min\{\mu(x), 0.5\} \geq \min\{t, 0.5\} = t$, which implies, $1_t \in \mu$, and so $1 \in F(t)$. If $x \to y \in F(t)$ and $x \in F(t)$, then $(x \to y)_t \in \mu$ and $x_t \in \mu$, that is, $\mu(x \to y) \geq t$ and $\mu(x) \geq t$. By Definition 3.5 (F5), we have

$$\begin{aligned}\mu(y) &\geq \min\{\mu(x \to y), \mu(x), 0.5\}\\ &\geq \min\{t, 0.5\}\\ &= t,\end{aligned}$$

which implies, $y_t \in \mu$, and so $y \in F(t)$. Thus, (F, A) is a filteristic soft lattice implication algebra over L.

Conversely, assume that the condition (ii) holds. If there exists $a \in L$ such that $\mu(1) < \min\{\mu(a), 0.5\}$, then $\mu(1) < t \leq \min\{\mu(a), 0.5\}$ for some $t \in A$. It follows that $1_t \overline{\in} \mu$, i.e., $1 \overline{\in} F(t)$, a contradiction. Hence $\mu(1) \geq \min\{\mu(x), 0.5\}$ for all $x \in L$. If there exist $a, b \in L$ such that $\mu(b) < \min\{\mu(a \to b), \mu(a), 0.5\}$ then taking $t = \frac{1}{2}(\mu(b) + \min\{\mu(a \to b), \mu(a), 0.5\})$, we have $t \in A$ and

$$\mu(b) < t < \min\{\mu(a \to b), \mu(a), 0.5\},$$

which implies, $a \to b \in F(t), a \in F(t)$. But $b \overline{\in} F(t)$, a contradiction. It follows from Definition 3.5 that μ is an $(\in, \in \vee \mathrm{q})$-fuzzy filter of L. □

3.7. Definition. [19] A fuzzy set μ of L is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-*fuzzy filter* of L if and only if it satisfies:

(F6) $\forall x \in L, \max\{\mu(1), 0.5\} \geq \mu(x)$;
(F7) $\forall x, y \in L, \max\{\mu(y), 0.5\} \geq \min\{\mu(x \to y), \mu(x)\}$.

3.8. Theorem. *Let μ be a fuzzy set of L and (F, A) an $\in$-soft set over L with $A = (0.5, 1]$. Then the following are equivalent:*

(i) *μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L;*
(ii) *(F, A) is a filteristic soft lattice implication algebra over L.*

Proof. Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L. For any $t \in A$, by Definition 3.7 (F6), we have $\mu(x) \leq \max\{\mu(1), 0.5\}$ for all $x \in F(t)$. Thus, $t \leq \mu(x) \leq \max\{\mu(1), 0.5\} = \mu(1)$, which implies $1_t \in \mu$, i.e., $1 \in F(t)$. Let $x, y \in L$ be such that $x \to y \in F(t)$ and $x \in F(t)$, then $(x \to y)_t \in \mu$ and $x_t \in \mu$, i.e., $\mu(x \to y) \geq t$ and $\mu(x) \geq t$. It follows from Definition 3.7 (F7) that

$$t \leq \min\{\mu(x \to y), \mu(x)\} \leq \max\{\mu(y), 0.5\} = \mu(y),$$

which implies, $y_t \in \mu$, i.e., $y \in F(t)$. Hence $F(t)$ is a filter of L for all $t \in A$, and so (F, A) is a filteristic soft lattice implication algebra over L.

Conversely, assume that (F, A) is a filteristic soft lattice implication algebra over L. If there exists $a \in L$ such that $\mu(a) \geq \max\{\mu(1), 0.5\}$, then $\mu(a) \geq t > \max\{\mu(1), 0.5\}$ for some $t \in A$ and so $\mu(1) < t$, hence $1 \overline{\in} F(a)$, a contradiction. Thus, $\mu(x) \leq \max\{\mu(1), 0.5\}$ for all $x \in L$. If there exist $a, b \in L$ such that $\min\{\mu(a \to b), \mu(a)\} \geq t > \max\{\mu(b), 0.5\}$ for some $t \in A$ then $(a \to b)_t \in \mu, a_t \in \mu$, but $b_t \overline{\in} \mu$, which implies $a \to b \in F(t), a \in F(t)$. But $b \overline{\in} F(t)$, a contradiction. It follows from Definition 3.7 that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L. □

Next, we give the following two important results by q-soft sets.

3.9. Theorem. *Let μ be a fuzzy set of L and (F_{q}, A) a q-soft set over L with $A = (0, 0.5]$. Then (F_{q}, A) is a filteristic soft lattice implication algebra over L if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L.*

Proof. Let (F_{q}, A) be a filteristic soft lattice implication algebra over L, then $F_{\mathrm{q}}(t)$ is a filter of L for all $t \in A$. If $\max\{\mu(1), 0.5\} < \mu(a)$ for some $a \in L$, then $\max\{\mu(1), 0.5\}+t \leq 1 < \mu(a)+t$ for some $t \in A$. Thus, $1_t\overline{q}\mu$, a contradiction. Hence $\max\{\mu(1), 0.5\} \geq \mu(x)$ for all $x \in L$.

If there exist $a, b \in L$ such that $\max\{\mu(b), 0.5\} < \min\{\mu(a \to b), \mu(a)\}$, then $\max\{\mu(b), 0.5\}+s \leq 1 < \min\{\mu(a \to b), \mu(a)\}+s$ for some $s \in A$. Hence $(a \to b)_s q\mu$ and $a_s q\mu$, i.e., $a \to b \in F_{\mathrm{q}}(s)$ and $a \in F_{\mathrm{q}}(s)$. Since $F_{\mathrm{q}}(s)$ is a filter of L, we have $b \in F_{\mathrm{q}}(s)$, and so $b_s q\mu$, that is, $\mu(b)+s > 1$, a contradiction. Hence $\max\{\mu(y), 0.5\} \geq \min\{\mu(x \to y), \mu(x)\}$, for all $x, y \in L$. Therefore μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L.

Conversely, let μ be an $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy filter of L. For any $t \in A$, by Definition 3.7 (F6), we have $\mu(x) \leq \max\{\mu(1), 0.5\}$ for all $x \in F_{\mathrm{q}}(t)$, and so $\max\{\mu(1), 0.5\}+t \geq \mu(x)+t > 1$. Hence $\mu(1)+t > 1$, that is, $1 \in F_{\mathrm{q}}(t)$. Let $x, y \in L$ be such that $x \to y \in F_{\mathrm{q}}(t)$ and $x \in F_{\mathrm{q}}(t)$. Then $(x \to y)_t q\mu$ and $x_t q\mu$, or equivalently, $\mu(x \to y)+t > 1$ and $\mu(x)+t > 1$. By (F7), we have

$$\begin{aligned}\max\{\mu(y), 0.5\}+t &\geq \min\{\mu(x \to y), \mu(x)\}+t\\ &= \min\{\mu(x \to y)+t, \mu(x)+t\}\\ &> 1,\end{aligned}$$

and so $y_t q\mu$, i.e., $y \in F_{\mathrm{q}}(t)$. Hence $F_{\mathrm{q}}(t)$ is a filter of L, and so (F_{q}, A) is a filteristic soft lattice implication algebra over L. □

3.10. Theorem. *Let μ be a fuzzy set of L and (F_{q}, A) a q-soft set over L with $A = (0.5, 1]$. Then (F_{q}, A) is a filteristic soft lattice implication algebra over L if and only if μ is an $(\in, \in\vee\mathrm{q})$-fuzzy filter of L.*

Proof. Let (F_{q}, A) be a filteristic soft lattice implication algebra over L. Then $F_{\mathrm{q}}(t)$ is a filter of L for all $t \in A$. If $\mu(1) < \min\{\mu(a), 0.5\}$ for some $a \in L$, then $\mu(1)+t \leq 1 < \min\{\mu(a), 0.5\}+t$ for some $t \in A$. Thus, $1_t\overline{q}\mu$, a contradiction. Hence $\mu(1) \geq \min\{\mu(x), 0.5\}$ for all $x \in L$.

If there exist $a, b \in L$ such that $\mu(b) < \min\{\mu(a \to b), \mu(a), 0.5\}$, then $\mu(b)+s \leq 1 < \min\{\mu(a \to b), \mu(a), 0.5\}+s$ for some $s \in A$. Hence $(a \to b)_s q\mu$ and $a_s q\mu$, i.e., $a \to b \in F_{\mathrm{q}}(s)$ and $a \in F_{\mathrm{q}}(s)$. Since $F_{\mathrm{q}}(s)$ is a filter of L, we have $b \in F_{\mathrm{q}}(s)$, and so $b_s q\mu$, that is, $\mu(b)+s > 1$, a contradiction. Hence $\mu(y) \geq \min\{\mu(x \to y), \mu(x), 0.5\}$, for all $x, y \in L$. Therefore μ is an $(\in, \in\vee\mathrm{q})$-fuzzy filter of L.

Conversely, let μ be an $(\in, \in\vee\mathrm{q})$-fuzzy filter of L. For any $t \in A$, by Definition 3.6 (F4), we have $\mu(1) \geq \min\{\mu(x), 0.5\}$ for all $x \in F_{\mathrm{q}}(t)$, and so $\mu(1)+t \geq \min\{\mu(x), 0.5\}+t = \min\{\mu(x)+t, 0.5+t\} > 1$. Hence $\mu(1)+t > 1$, that is, $1 \in F_{\mathrm{q}}(t)$. Let $x, y \in L$ be such that $x \to y \in F_{\mathrm{q}}(t)$ and $x \in F_{\mathrm{q}}(t)$. Then $(x \to y)_t q\mu$ and $x_t q\mu$, or equivalently, $\mu(x \to y)+t > 1$ and $\mu(x)+t > 1$. By (F5), we have

$$\begin{aligned}\mu(y)+t &\geq \min\{\mu(x \to y), \mu(x), 0.5\}+t\\ &= \min\{\mu(x \to y)+t, \mu(x)+t, 0.5+t\}\\ &> 1,\end{aligned}$$

and so $y_t q\mu$, i.e., $y \in F_{\mathrm{q}}(t)$. Hence $F_{\mathrm{q}}(t)$ is a filter of L, and so (F_{q}, A) is a filteristic soft lattice implication algebra over L. □

3.11. Definition. [19] Given $\alpha, \beta \in (0, 1]$ and $\alpha < \beta$, we call a fuzzy set μ of L a *fuzzy filter with thresholds* $(\alpha, \beta]$ of L if for all $x, y \in L$, the following conditions are satisfied:

(F8) $\max\{\mu(1), \alpha\} \geq \min\{\mu(x), \beta\}$;
(F9) $\max\{\mu(y), \alpha\} \geq \min\{\mu(x \to y), \mu(x), \beta\}$.

3.12. Theorem. *Given $\alpha, \beta \in (0,1]$ and $\alpha < \beta$, let μ be a fuzzy set of L and (F, A) an $\in$-soft set over L with $A = (\alpha, \beta]$. Then*

(i) *μ is a fuzzy filter with thresholds $(\alpha, \beta]$ of L,*
(ii) *(F, A) is a filteristic soft lattice implication algebra over L.*

Proof. Let μ be a fuzzy filter with thresholds $(\alpha, \beta]$ of L. Take any $t \in A$. Then by Definition 3.11 (F8), we have $\max\{\mu(1), \alpha\} \geq \min\{\mu(x), \beta\}$ for all $x \in F(t)$. Thus, $\max\{\mu(1), \alpha\} \geq \min\{\mu(x), \beta\} \geq \min\{t, \beta\} = t > \alpha$, which implies, $\mu(1) \geq t$, i.e., $1_t \in \mu$. Hence $1 \in F(t)$. Let $x, y \in L$ be such that $x \to y \in F(t)$ and $x \in F(t)$. Thus, $(x \to y)_t \in \mu$ and $x_t \in \mu$, i.e., $\mu(x \to y) \geq t$ and $\mu(x) \geq t$. By Definition 3.11 (F9), we have $\max\{\mu(y), \alpha\} \geq \min\{\mu(x \to y), \mu(x), \beta\} \geq \min\{t, \beta\} = t > \alpha$, and so $\mu(y) \geq t$, i.e., $y_t \in \mu$, and so $y \in F(t)$. Therefore, (F, A) is a filteristic soft lattice implication algebra over L.

Conversely, assume that (F, A) is a filteristic soft lattice implication algebra over L. If there exists $a \in L$ such that $\max\{\mu(1), \alpha\} < \min\{\mu(a), \beta\}$, then $\max\{\mu(1), \alpha\} < t \leq \min\{\mu(a), \beta\}$ for some $t \in (\alpha, \beta]$. It follows that $1 \overline{\in} F(t)$, a contradiction. If there exist $a, b \in L$ such that $\max\{\mu(b), \alpha\} < t \leq \min\{\mu(a \to b), \mu(a), \beta\}$, then $(a \to b)_t \in \mu, a_t \in \mu$. But $b_t \overline{\in} \mu$, and so $a \to b \in F(t), a \in F(t)$, but $b \overline{\in} F(t)$, a contradiction. Therefore, μ is a fuzzy filter with thresholds $(\alpha, \beta]$ of L. □

4. Implicative filteristic soft lattice implication algebras

In this Section, we describe implicative filteristic soft lattice implication algebras.

4.1. Definition. Let (F, A) be a soft set over L. Then (F, A) is called an *implicative filteristic soft lattice implication algebra over* L if $F(x)$ is an implicative filter of L for all $x \in A$, where for our convenience the empty set $\emptyset$ is regarded as an implicative filter of L.

4.2. Example. Consider the lattice implication algebra L as in Example 3.2. Let (F, A) be a soft set over L, where $A = (0, 1]$ and $F : A \to \mathcal{P}(L)$ is a set-valued function defined by

$$F(x) = \begin{cases} \{0, a, b, c, d, 1\} & \text{if } 0 < x \leq 0.4, \\ \{1, b, c\} & \text{if } 0.4 < x \leq 0.6, \\ \emptyset & \text{if } 0.6 < x \leq 1. \end{cases}$$

Thus, $F(x)$ is an implicative filter of L for all $x \in A$, and so (F, A) is an implicative filteristic soft lattice implication algebra over L.

From the above definitions, we can get the following:

4.3. Proposition. *Every implicative filteristic lattice implication algebra is a filteristic lattice implication algebra, but the converse may not be true.* □

4.4. Theorem. *Let μ be a fuzzy set of L and (F, A) an $\in$-soft set over L with $A = (0, 1]$. Then (F, A) is an implicative filteristic soft lattice implication algebra over L if and only if μ is a fuzzy implicative filter of L.*

Proof. Let μ be a fuzzy implicative filter of L and $t \in A$. Then μ is also a fuzzy filter of L. It follows from Theorem 3.3 that (F, A) is a filteristic soft lattice implication algebra over L. Let $x, y, z \in L$ be such that $x \to (y \to z), x \to y \in F(t)$. Then $(x \to (y \to z))_t \in \mu$ and $(x \to y)_t \in \mu$. Hence $\mu(x \to z) \geq \min\{\mu(x \to (y \to z)), \mu(x \to y)\} \geq t$, and so $x \to z \in F(t)$. Hence (F, A) is an implicative filteristic soft lattice implication algebra over L.

Conversely, assume that (F, A) is an implicative filteristic soft lattice implication algebra over L. Then, (F, A) is a filteristic soft lattice implication algebra over L, and so μ is a fuzzy filter of L by Theorem 3.3. If there exist $a, b, c \in L$ such that $\mu(a \to c) < s \le \min\{\mu(a \to (b \to c)), \mu(a \to b)\}$ for some $s \in A$, then $(a \to (b \to c))_s\mu$ and $(a \to b)_s \in \mu$, but $(a \to c)_s \overline{\in} \mu$, that is, $a \to (b \to c) \in F(s)$ and $a \to b \in F(s)$, but $a \to c \overline{\in} F(s)$, a contradiction. Therefore, μ is a fuzzy implicative filter of L. □

4.5. Theorem. *Let μ be a fuzzy set of L and (F_{q}, A) a q-soft set over L with $A = (0, 1]$. Then the following are equivalent:*

(i) *μ is a fuzzy implicative filter of L;*
(ii) *$\forall t \in A, F_{\mathrm{q}}(t) \neq \emptyset \Rightarrow F_{\mathrm{q}}(t)$ is an implicative filter of L.*

Proof. Let μ be a fuzzy implicative filter of L. Then μ is also a fuzzy filter of L. By Theorem 3.4, $F_{\mathrm{q}}(t)$ is a filter of L. Let $x, y, z \in L$ be such that $x \to (y \to z) \in F_{\mathrm{q}}(t)$ and $x \to y \in F_{\mathrm{q}}(t)$. Then $(x \to (y \to z))_t q\mu$ and $(x \to y)_t q\mu$, or equivalently, $\mu(x \to (y \to z)) + t > 1$ and $\mu(x \to y) + t > 1$. Since μ is a fuzzy implicative of L, we have

$$\begin{aligned}\mu(x \to z) + t &\ge \min\{\mu(x \to (y \to z)), \mu(x \to y)\} + t\\ &= \min\{\mu(x \to (y \to z)) + t, \mu(x \to y) + t\}\\ &> 1,\end{aligned}$$

and so $(x \to z)_t q$, i.e., $x \to z \in F_{\mathrm{q}}(t)$. Hence $F_{\mathrm{q}}(t)$ is an implicative filter of L.

Conversely, assume that the condition (ii) holds. Then μ is a fuzzy filter of L by Theorem 3.4. If there exist $a, b, c \in L$ such that $\mu(a \to c) < \min\{\mu(a \to (b \to c)), \mu(a \to b)\}$, then $\mu(a \to c) + s \le 1 < \min\{\mu(a \to (b \to c)), \mu(a \to b)\} + s$ for some $s \in A$. Hence $(a \to (b \to c))_s q\mu$ and $(a \to b)_s q\mu$, but $(a \to c)_t \overline{q}\mu$, i.e., $a \to (b \to c) \in F_{\mathrm{q}}(s)$ and $a \to b \in F_{\mathrm{q}}(s)$, but $a \to c \overline{\in} F_{\mathrm{q}}(t)$, a contradiction. Therefore μ is a fuzzy implicative filter of L. □

4.6. Definition. [5] An $(\in, \in \vee q)$-fuzzy filter μ of L is called an $(\in, \in \vee q)$-*fuzzy implicative filter* of L if it satisfies:

(F10) $\mu(x \to z) \ge \min\{\mu(x \to (y \to z)), \mu(x \to y), 0.5\}$, for all $x, y, z \in L$.

4.7. Theorem. *Let μ be a fuzzy set of L and (F, A) an $\in$-soft set over L with $A = (0, 0.5]$. Then the following are equivalent:*

(i) *μ is an $(\in, \in \vee \mathrm{q})$-fuzzy implicative filter of L;*
(ii) *(F, A) is an implicative filteristic soft lattice implication algebra over L.*

Proof. Let μ be an $(\in, \in \vee \mathrm{q})$-fuzzy implicative filter of L. Then μ is also an $(\in, \in \vee \mathrm{q})$-fuzzy filter of L. By Theorem 3.6, (F, A) is a filteristic soft lattice implication algebra. For any $t \in A$, let $x, y, z \in L$ be such that $x \to (y \to z) \in F(t)$ and $x \to y \in F(t)$, that is, $\mu(x \to (y \to z)) \ge t$ and $\mu(x \to y) \ge t$. Thus,

$$\begin{aligned}\mu(x \to z) &\ge \min\{\mu(x \to (y \to z)), \mu(x \to y), 0.5\}\\ &\ge \min\{t, 0.5\}\\ &= t,\end{aligned}$$

which implies $(x \to z)_t \in \mu$, and so $x \to z \in F(t)$. Thus, (F, A) is an implicative filteristic soft lattice implication algebra over L.

Conversely, assume that the condition (ii) holds. Then (F, A) is also a filteristic soft lattice implication algebra over L. If there exist $a, b, c \in L$ such that $\mu(a \to c) <$

$\min\{\mu(a \to (b \to c)), \mu(a \to b), 0.5\}$. Taking $t = \frac{1}{2}(\mu(a \to c) + \min\{\mu(a \to (b \to c)), \mu(a \to b), 0.5\})$, we have $t \in A$ and

$$\mu(a \to c) < t < \min\{\mu(a \to (b \to c)), \mu(a \to b), 0.5\},$$

which implies, $a \to (b \to c) \in F(t), a \to b \in F(t)$. But $a \to c \overline{\in} F(t)$, a contradiction. It follows from Definition 4.6 that μ is an $(\in, \in \vee \mathrm{q})$-fuzzy implicative filter of L. □

4.8. Definition. [17] An $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-*fuzzy implicative filter* of L if it satisfies:

(F11) $\max\{\mu(x \to z), 0.5\} \geq \min\{\mu(x \to (y \to z)), \mu(x \to y)\}$, for all $x, y, z \in L$.

4.9. Theorem. *Let μ be a fuzzy set of L and (F, A) an $\in$-soft set over L with $A = (0.5, 1]$. Then the following are equivalent:*

(i) *μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy implicative filter of L;*
(ii) *(F, A) is an implicative filteristic soft lattice implication algebra over L.*

Proof. Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy implicative filter of L. Then μ is also an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L. By Theorem 3.8, (F, A) is a filteristic soft lattice implication algebra. For any $t \in A$, let $x, y, z \in L$ be such that $x \to (y \to z) \in F(t)$ and $x \to y \in F(t)$, then $(x \to (y \to z))_t \in \mu$ and $(x \to y)_t \in \mu$, i.e., $\mu(x \to (y \to z)) \geq t$ and $\mu(x \to y) \geq t$. Thus,

$$t \leq \min\{\mu(x \to (y \to z)), \mu(x \to y)\} \leq \max\{\mu(x \to z), 0.5\} = \mu(x \to z),$$

which implies, $(x \to z)_t \in \mu$, i.e., $x \to z \in F(t)$. Hence $F(t)$ is an implicative filter of L for all $t \in A$, and so (F, A) is an implicative filteristic soft lattice implication algebra over L.

Conversely, assume that (F, A) is an implicative filteristic soft lattice implication algebra over L. If there exist $a, b, c \in L$ such that $\min\{\mu(a \to (b \to c)), \mu(a \to b)\} \geq t > \max\{\mu(a \to c), 0.5\}$ for some $t \in A$, then $(a \to (b \to c))_t \in \mu, (a \to b)_t \in \mu$. But $(a \to c)_t \overline{\in} \mu$, which implies, $a \to (b \to c) \in F(t), a \to b \in F(t)$, whreras $a \to c \overline{\in} F(t)$, a contradiction. Therefore, μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy implicative filter of L. □

Next, we give the following two important results by q-soft sets.

4.10. Theorem. *Let μ be a fuzzy set of L and (F_{q}, A) a q-soft set over L with $A = (0, 0.5]$. Then (F_{q}, A) is an implicative filteristic soft lattice implication algebra over L if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy implicative filter of L.*

Proof. Similar to the proof of Theorem 3.9. □

4.11. Theorem. *Let μ be a fuzzy set of L and (F_{q}, A) a q-soft set over L with $A = (0.5, 1]$. Then (F_{q}, A) is an implicative filteristic soft lattice implication algebra over L if and only if μ is an $(\in, \in \vee \mathrm{q})$-fuzzy implicative filter of L.*

Proof. Similar to the prof of Theorem 3.10. □

4.12. Definition. [17] Given $\alpha, \beta \in (0, 1]$ and $\alpha < \beta$, a fuzzy set μ of L is called a *fuzzy implicative filter with thresholds* $(\alpha, \beta]$ of L if it satisfies (F8),(F9) and

(F12) $\max\{\mu(x \to z), \alpha\} \geq \min\{\mu(x \to (y \to z)), \mu(x \to y), \beta\}$, for all $x, y, z \in L$.

The following is a consequence of Theorems 3.12, 4.7 and 4.9.

4.13. Theorem. *Given $\alpha, \beta \in (0, 1]$ and $\alpha < \beta$, let μ be a fuzzy set of L and (F, A) an $\in$-soft set over L with $A = (\alpha, \beta]$. Then*

(i) *μ is a fuzzy implicative filter with thresholds $(\alpha, \beta]$ of L,*
(ii) *(F, A) is an implicative filteristic soft lattice implication algebra over L.* □

5. Conclusion

As a continuation of [5, 6, 17, 19], we apply fuzzy and soft set theory to lattice implication algebras. We hope that the research along this direction can be continued, and in fact, some results in this paper have already constituted a platform for further discussion concerning the future development of soft lattice implication algebras and other algebraic structure.

In our future study of lattice implication algebras, maybe the following topics should be considered:

(1) To describe soft lattice implication algebras based on rough sets;
(2) To discuss the relations between soft lattice implication algebras based on fuzzy sets and on rough sets;
(3) To consider the soft implication-based fuzzy implicative filters in lattice implication algebras.

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