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SUBORDINATION AND SUPERORDINATION FOR HIGHER-ORDER DERIVATIVES OF p-VALENT ANALYTIC FUNCTIONS

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Abstract

In this paper some subordination and superordination results for higher-order derivatives of certain p-valent analytic functions in the open unit disc are derived. Relevant connections of the results, which are obtained in this paper, with various known results are also considered.

Keywords: Analytic functions, Differential subordinations, Superordination, Subordinants, Dominants, Higher-order derivatives.

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1. Introduction

Let $A(p)$ denote the class of analytic functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \; (p \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1.1}$$

which are p-valent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $A(1) = A$. Upon differentiating both sides of (1.1) m-times with respect to z, we obtain (see [6])

$$\begin{gathered} f^{(m)}(z) = \delta(p, m) z^{p-m} + \sum_{k=1}^{\infty} \delta(k, m) a_{k+p} z^{k+p-m}, \\ (p \in \mathbb{N}; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\; p > m), \end{gathered} \tag{1.2}$$

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where

$$\delta(p,m)=\begin{cases}1 & \text{if } m=0,\\ p(p-1)\cdots(p-m+1) & \text{if } m\neq 0.\end{cases} \tag{1.3}$$

Several researchers have investigated higher-order derivatives of multivalent functions, see, for example, [1–3, 6–11, 15, 17, 20–22].

Let $H(U)$ be the class of analytic functions in U and $H[a,p]$ the subclass of $H(U)$ consisting of functions of the from:

$$f(z)=a+a_pz^p+a_{p+1}z^{p+1}+\cdots \quad (a\in\mathbb{C}).$$

For $f,g\in H(U)$, we say that the function f is *subordinate* to g, or the function g is *superordinate* to f, if there exists a Schwarz function w, i.e., $w\in H(U)$ with $w(0)=0$ and $|w(z)|<1,\ z\in U$, such that $f(z)=g(w(z))$ for all $z\in U$. This subordination is usually denoted by $f(z)\prec g(z)$. It is well-known that, if the function g is univalent in U, then $f(z)\prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$ (cf e.g. [12] see also [5]).

Supposing that p,h are two analytic functions in U, let

$$\varphi(r,s,t;z):\mathbb{C}^3\times U\to\mathbb{C}.$$

If p and $\varphi(p(z),zp'(z),z^2p''(z);z)$ are univalent functions in U and if p satisfies the second-order superordination

$$h(z)\prec\varphi(p(z),zp'(z),z^2p''(z);z), \tag{1.4}$$

then p is called a *solution of the differential superordination* (1.4). A function $q\in H(U)$ is called a *subordinant* of (1.4), if $q(z)\prec p(z)$ for all the functions $p(z)$ satisfying (1.4). A univalent subordinant $\widetilde{q}$ that satisfies $q(z)\prec\widetilde{q}(z)$ for all of the subordinants q of (1.4), is called the *best subordinant* (cf., e.g., [12], see also [5]).

Recently, Miller and Mocanu [13] obtained sufficient conditions on the functions h,q and φ for which the following implication holds:

$$h(z)\prec\varphi(p(z),zp'(z),\ z^2p''(z);z)\Rightarrow q(z)\prec p(z).$$

Now we introduce the class $B_p^m(\lambda,\alpha,\rho)$ defined by

$$B_p^m(\lambda,\alpha,\rho)=\Big\{f\in A(p):\mathrm{Re}\Big\{(1-\lambda)\left(\tfrac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha} + \lambda\left(\tfrac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha}\left[\tfrac{zf^{(m+1)}(z)}{(p-m)f^{(m)}(z)}\right]\Big\}>\rho\Big\}, \tag{1.5}$$

where $\lambda\geq 0,\ \alpha>0,\ \rho\geq 0,\ p\in\mathbb{N},\ m\in\mathbb{N}_0$ and $p>m$.

The main object of this paper is to apply a method of differential subordination in order to derive several subordination and superordination results involving higher-order derivatives. Further, we obtain some previous results as special cases of some of the results obtained here.

2. Preliminaries

In order to prove our subordination and superordination results, we make use of the following known definition and results.

2.1. Definition. [13] Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U}\backslash E(f)$, where

$$E(f)=\Big\{\zeta:\zeta\in\partial U \text{ and } \lim_{z\to\zeta}f(z)=\infty\Big\} \tag{2.1}$$

and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial U\backslash E(f)$.

2.2. Lemma. [12] *Let the function $q(z)$ be univalent in the unit disc U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq^{'}(z)\varphi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

(i) *$Q(z)$ is starlike univalent in U,*

(ii) $\mathrm{Re}\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$ *for $z \in U$.*

If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.2}$$

then

$$p(z) \prec q(z)$$

and $q(z)$ is the best dominant. □

2.3. Lemma. [13] *Let q be a convex univalent function in U and let $\psi \in \mathbb{C}$, $\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ with*

$$\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\mathrm{Re}\left(\frac{\psi}{\gamma}\right)\right\}.$$

If $p(z)$ is analytic in U and

$$\psi p(z) + \gamma zp'(z) \prec \psi q(z) + \gamma zq'(z), \tag{2.3}$$

then

$$p(z) \prec q(z), \ (z \in U)$$

and q is the best dominant. □

2.4. Lemma. [12] *Let $q(z)$ be convex univalent in the unit disc U and let θ and φ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\mathrm{Re}\left\{\dfrac{\theta'(q(z))}{\varphi(q(z))}\right\} > 0$ *for $z \in U$;*

(ii) *$zq'(z)\varphi(q(z))$ is starlike univalent in U.*

If $p(z) \in H[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U, and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.4}$$

then

$$q(z) \prec p(z), \ (z \in U),$$

and $q(z)$ is the best subordinant. □

By taking $\theta(w) = w$ and $\varphi(w) = \gamma$ in Lemma 2.4, we get the following lemma.

2.5. Lemma. [13] *Let q be convex univalent in U and $\gamma \in \mathbb{C}$. Further assume that $\mathrm{Re}(\gamma) > 0$. If $p(z) \in H[q(0), 1] \cap Q$ and $p(z) + \gamma zp'(z)$ is univalent in U, then*

$$q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z), \tag{2.5}$$

implies

$$q(z) \prec p(z), \ (z \in U)$$

and q is the best subordinant. □

This last lemma gives us a necessary and sufficient condition for the univalence of a special function which will be used in some particular cases.

2.6. Lemma. [18] *The function $q(z) = (1 - z)^{-2ab}$, $(a, b \in \mathbb{C}^*)$ is univalent in U if and only if $|2ab - 1| \leq 1$ or $|2ab + 1| \leq 1$.* □

3. Subordination for analytic functions

Unless otherwise mentioned we shall assume throughout the paper that $\lambda > 0$, $\alpha > 0$, $p \in \mathbb{N}$, $m \in \mathbb{N}_0$, $p > m$ and the powers are understood as principle values.

By using Lemma 2.3, we first prove the following.

3.1. Theorem. *Let q be univalent in U. Suppose that q satisfies*

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} + \frac{\alpha(p-m)}{\lambda} > 0. \tag{3.1}$$

If a function $f \in A(p)$ satisfies

$$\Psi(f, \lambda, \alpha, p, m) \prec q(z) + \frac{\lambda z q'(z)}{\alpha(p-m)}, \tag{3.2}$$

where

$$\begin{aligned}&\Psi(f, \lambda, \alpha, p, m)\\ &\quad = (1-\lambda)\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha} + \lambda\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha}\left\{\frac{zf^{(m+1)}(z)}{(p-m)\, f^{(m)}(z)}\right\},\end{aligned} \tag{3.3}$$

then

$$\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha} \prec q(z), \tag{3.4}$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) = \left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha}, \ (z \in U). \tag{3.5}$$

Differentiating (3.5) logarithmically with respect to z, we have

$$\frac{zp'(z)}{p(z)} = \alpha\left\{\frac{zf^{(m+1)}}{f^{(m)}(z)} - (p-m)\right\},$$

which, in the light of hypothesis (3.1) of Theorem 3.1, yields the following subordination

$$p(z) + \frac{\lambda z p'(z)}{\alpha(p-m)} \prec q(z) + \frac{\lambda z q'(z)}{\alpha(p-m)}.$$

Now by application of Lemma 2.3, with $\gamma = \frac{\lambda}{\alpha(p-m)}$, $\psi = 1$, we obtain (3.4). □

3.2. Remark. (i) Putting $\alpha = \lambda = 1$ in Theorem 3.1 we obtain the result obtained by Ali *et al.* [1, Theorem 2.9];

(ii) Putting $p = 1$ and $m = 0$ in Theorem 3.1 we obtain the result obtained by Shanmugam *et al.* [19, Theorem 3.1, with correction of condition (3)] and Murugusundaramoorthy and Magesh [14, Corollary 3.3].

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3.1, we obtain the following corollary.

3.3. Corollary. *Let $-1 \le B < A \le 1$ and*

$$\operatorname{Re}\left\{\frac{1-Bz}{1+Bz}\right\} > \max\left\{0; -\frac{\alpha(p-m)}{\lambda}\right\}, \ z \in \mathrm{U}.$$

If $f \in A(p)$, and

$$\Psi(f, \lambda, \alpha, p, m) \prec \frac{1+Az}{1+Bz} + \frac{\lambda}{\alpha(p-m)}\frac{(A-B)z}{(1+Bz)^2},$$

where $\Psi(f,\lambda,\alpha,p,m)$ *given by* (3.3)*, then*

$$\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.* □

Taking $q(z)=\frac{1+z}{1-z}$ in Theorem 3.1, we obtain the following corollary.

3.4. Corollary. *If* $f\in A(p)$*, and*

$$\Psi(f,\lambda,\alpha,p,m) \prec \frac{1+z}{1-z} + \frac{2\lambda z}{\alpha(p-m)(1-z)^2},$$

where $\Psi(f,\lambda,\alpha,p,m)$ *is given by* (3.3)*, then*

$$\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha} \prec \frac{1+z}{1-z},$$

and $\dfrac{1+z}{1-z}$ *is the best dominant.* □

3.5. Theorem. *Let* q *be univalent in* U *such that* $q(0)=1$ *for all* $z\in U$ *and* $\gamma\in\mathbb{C}^*$*. Suppose that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* U*. Let* $f\in A(p)$*. If*

$$1+\gamma\alpha\left\{\frac{zf^{(m+1)}(z)}{f^{(m)}(z)}-(p-m)\right\} \prec 1+\gamma\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha} \prec q(z),$$

and q *is the best dominant.*

Proof. Define the function $p(z)$ by

$$p(z)=\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha},\ (z\in U). \tag{3.6}$$

Differentiating (3.6) logarithmically with respect to z, we have

$$\alpha\left\{\frac{zf^{(m+1)}(z)}{f^{(m)}(z)}-(p-m)\right\}=\frac{zp'(z)}{p(z)}.$$

By setting $\theta(w)=1$ and $\Phi(w)=\frac{\gamma}{w}$, it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\Phi(w)$ is analytic in $\mathbb{C}^*$, and that

$$\Phi(w)\neq 0,\ (w\in\mathbb{C}^*).$$

Also, we let

$$Q(z)=zq'(z)\Phi(q(z))=\gamma\frac{zq'(z)}{q(z)}$$

and

$$h(z)=\theta\{q(z)\}+Q(z)=1+\gamma\frac{zq'(z)}{q(z)}.$$

We find that $Q(z)$ is starlike univalent in U and that

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right)=\operatorname{Re}\left\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0.$$

Thus, by applying Lemma 2.2 our proof of Theorem 3.5 is completed. □

Taking $\alpha = 1$, $\gamma = \dfrac{e^{i\theta}}{ab\cos\theta}$, $(a, b \in \mathbb{C}^*,\ |\theta| < \frac{\pi}{2})$ and $q(z) = (1-z)^{-2ab\cos\theta e^{-i\theta}}$ in Theorem 3.5, and using Lemma 2.6, we obtain the following result.

3.6. Corollary. *Let $a, b \in \mathbb{C}^*$ and $|\theta| < \frac{\pi}{2}$, such that $\left|2ab\cos\theta e^{-i\theta} - 1\right| \leq 1$ or $\left|2ab\cos\theta e^{-i\theta} + 1\right| \leq 1$. If $f(z) \in A(p)$, and*

$$1 + \frac{e^{i\theta}}{ab\cos\theta}\left(\frac{zf^{(m+1)}(z)}{f^{(m)}(z)} - (p-m)\right) \prec \frac{1+z}{1-z}$$

then

$$\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right) \prec (1-z)^{-2ab\cos\theta e^{-i\theta}}$$

and $(1-z)^{-2ab\cos\theta e^{-i\theta}}$ is the best dominant. □

3.7. Remark. Taking $m = 0$ and $p = 1$ in Corollary 3.6, we obtain the result obtained by Aouf *et al.* [4, Theorem 1].

Taking, $q(z) = (1-z)^{-2b}$, $(b \in \mathbb{C}^*)$, $\gamma = \frac{1}{b}$ and $\alpha = 1$ in Theorem 3.5, we obtain the following result.

3.8. Corollary. *Let $b \in \mathbb{C}^*$. If $f \in A(p)$, and*

$$1 + \frac{1}{b}\left\{\frac{zf^{(m+1)}(z)}{f^{(m)}(z)} - (p-m)\right\} \prec \frac{1+z}{1-z},\ (b \in \mathbb{C}^*).$$

Then

$$\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}} \prec (1-z)^{-2b},$$

and $(1-z)^{-2b}$ is the best dominant. □

3.9. Remark. Taking $m = 0$ and $p = 1$ in Corollary 3.8, we obtain the result obtained by Srivastava and Lashin [21] and Murugusundaramoorthy and Magesh [14, Corollary 3.6].

Taking $q(z) = \dfrac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, A \neq B, \gamma = 1$ in Theorem 3.5, we obtain the following result.

3.10. Corollary. *Let $-1 \leq B < A \leq 1$, $A \neq B$. If $f \in A(p)$, and*

$$\alpha\left\{\frac{zf^{(m+1)}(z)}{f^{(m)}(z)} - (p-m)\right\} \prec \frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha} \prec \frac{1+Az}{1+Bz}$$

and $\dfrac{1+Az}{1+Bz}$ is the best dominant. □

Taking $q(z) = (1+Bz)^{\mu\left(\frac{A-B}{B}\right)}$, $B \neq 0$, $\alpha = \mu$, $\gamma = \frac{1}{\alpha}$, $(\alpha \neq 0)$ in Theorem 3.5, we obtain the following result.

3.11. Corollary. *Let $-1 \leq B < A \leq 1, B \neq 0$. Also let μ, A, B, satisfy either*

$$\left|\frac{\mu(A-B)}{B} - 1\right| \leq 1 \ \text{or}\ \left|\frac{\mu(A-B)}{B} + 1\right| \leq 1.$$

If $f \in A(p)$, *and*

$$1 + \left\{ \frac{zf^{(m+1)}(z)}{f^{(m)}(z)} - (p-m) \right\} \prec \frac{1+Az}{1+Bz},$$

then

$$\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}} \right)^{\mu} \prec (1+Bz)^{\mu\left(\frac{A-B}{B}\right)}$$

and $(1+Bz)^{\mu\left(\frac{A-B}{B}\right)}$ *is the best dominant.* □

Taking $q(z) = e^{\mu Az} (|\mu A| < \pi)$, $\alpha = \mu$, $\gamma = \frac{1}{\alpha}$, $(\alpha \neq 0)$ in Theorem3.5, we obtain the following corollary.

3.12. Corollary. *If* $f \in A(p)$ *and*

$$1 + \left\{ \frac{zf^{(m+1)}(z)}{f^{(m)}(z)} - (p-m) \right\} \prec 1 + Az, \ (z \in U^*);$$

then

$$\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}} \right)^{\mu} \prec e^{\mu Az},$$

and $e^{\mu Az}$ *is the best dominant.* □

3.13. Remark. (i) Taking $m = 0$ and $p = 1$ in Corollary 3.11, we obtain the result obtained by Obradovic and Owa [16];

(ii) Taking $m = 0$ and $p = 1$ in Corollary 3.12, we obtain the result obtained by Obradovic and Owa [16].

4. Superordination for analytic functions

Next, applying Lemma 2.5, we obtain the following two theorems.

4.1. Theorem. *Let* q *be convex in* U *and* $\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}} \right)^{\alpha} \in H[q(0), 1] \cap Q$. *Let* $\Psi(f, \lambda, \alpha, p, m)$ *be univalent in* U, *where* $\Psi(f, \lambda, \alpha, p, m)$ *is given by* (3.3). *If* $f \in A(p)$ *satisfies the following superordination*

$$q(z) + \frac{\lambda z q'(z)}{\alpha(p-m)} \prec \Psi(f, \lambda, \alpha, p, m), \tag{4.1}$$

then

$$q(z) \prec \left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}} \right)^{\alpha}$$

and q *is the best subordinant.*

Proof. Define the function $p(z)$ by

$$p(z) = \left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}} \right)^{\alpha}, \ (z \in U). \tag{4.2}$$

Differentiating (4.2) logarithmically with respect to z, we have

$$p(z) + \frac{\lambda z p'(z)}{\alpha(p-m)} = \Psi(f, \lambda, \alpha, p, m). \tag{4.3}$$

Theorem 4.1 now follows by applying Lemma 2.5. □

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 4.1, we obtain the following corollary:

4.2. Corollary. *Let* $-1 \leq B < A \leq 1$*, and suppose* $\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha} \in H[q(0),1] \cap Q$*, and that* $\Psi(f,\lambda,\alpha,p,m)$ *is univalent in* U*, where* $\Psi(f,\lambda,\alpha,p,m)$ *is given by* (3.3)*. If* $f \in A(p)$ *satisfies the following superordination*

$$\frac{1+Az}{1+Bz} + \frac{\lambda}{\alpha(p-m)} \frac{(A-B)z}{(1+Bz)^2} \prec \Psi(f,\lambda,\alpha,p,m)$$

then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha},$$

and $\dfrac{1+Az}{1+Bz}$ *is the best subordinant.* □

Taking $q(z) = \frac{1+z}{1-z}$ in Theorem 4.1, we obtain the following corollary.

4.3. Corollary. *Suppose* $\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha} \in H[q(0),1] \cap Q$ *and that* $\Psi(f,\lambda,\alpha,p,m)$ *is univalent in* U*, where* $\Psi(f,\lambda,\alpha,p,m)$ *is given by* (3.3)*. If* $f \in A(p)$ *satisfies the following superordination*

$$\frac{1+z}{1-z} + \frac{2\lambda z}{\alpha(p-m)(1-z)^2} \prec \Psi(f,\lambda,\alpha,p,m),$$

then

$$\frac{1+z}{1-z} \prec \left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha}$$

and $\dfrac{1+z}{1-z}$ *is the best subordinant.* □

4.4. Theorem. *Let* q *be univalent in* U *with* $\frac{zq'(z)}{q(z)}$ *starlike univalent in* U*, let* $\gamma \in \mathbb{C}$*,* $\mathrm{Re}\{\gamma\} > 0$*, and* $\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha} \in H[q(0),1] \cap Q$*.*

Let $\left\{1 + \gamma\alpha\left\{\frac{zf^{(m+1)}(z)}{f^{m}(z)} - (p-m)\right\}\right\}$ *be univalent in* U*. If* $f \in A(p)$ *satisfies the following superordination*

$$1 + \frac{\gamma z q'(z)}{q(z)} \prec 1 + \gamma\alpha\left\{\frac{zf^{(m+1)}(z)}{(f^{(m)}(z)} - (p-m)\right\},$$

then

$$q(z) \prec \left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha},$$

and q *is the best subordinant.* □

5. Sandwich results

Combining the results of differential subordination and supordination, we state the following "sandwich results".

5.1. Theorem. *Let* q_1 *and* q_2 *be convex univalent functions in* U *with* $q_1(0) = q_2(0) = 1$*, and suppose that* q_2 *satisfies* (3.1)*. If* $\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha} \in H[q(0),1] \cap Q$*, and* $\Psi(f,\lambda,\alpha,p,m)$ *is univalent in* U*, where* $\Psi(f,\lambda,\alpha,p,m)$ *is given by* (3.3)*, and if* $f \in A(p)$ *satisfies*

$$q_1(z) + \frac{\lambda}{\alpha(p-m)} z q_1'(z) \prec \Psi(f,\lambda,\alpha,p,m) \prec q_2(z) + \frac{\lambda}{\alpha(p-m)} z q_2'(z), \tag{5.1}$$

then

$$q_1(z) \prec \left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha} \prec q_2(z) \tag{5.2}$$

and q_1 and q_2 are, respectively, the best subordinant and best dominant. □

5.2. Theorem. *Let q_1 and q_2 be convex univalent functions in U with $q_1(0) = q_2(0) = 1$, let $\gamma \in \mathbb{C}$ and $\mathrm{Re}\{\gamma\} > 0$.*

If $\left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha} \in H[q(0), 1] \cap Q$, and $\left\{1 + \gamma\alpha\left\{\frac{zf^{(m+1)}(z)}{f^{(m)}(z)} - (p-m)\right\}\right\}$ is univalent in U, and if $f \in A(p)$ satisfies

$$1 + \gamma\frac{zq_1'(z)}{q_1(z)} \prec 1 + \gamma\alpha\left\{\frac{zf^{(m+1)}(z)}{f^{(m)}(z)} - (p-m)\right\} \prec 1 + \gamma\frac{zq_2'(z)}{q_2(z)}, \tag{5.3}$$

then

$$q_1(z) \prec \left(\frac{f^{(m)}(z)}{\delta(p,m)z^{p-m}}\right)^{\alpha} \prec q_2(z) \tag{5.4}$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant. □

5.3. Remark. (i) Putting $p = 1$ and $m = 0$ in Theorem 5.2 we correct the result obtained by Shanmugam *et al.* [19, Theorem 5.2];

(ii) Putting $p = 1$ and $m = 0$ in the above results we obtain the corresponding results obtained by Shanmugam *et al.* [19].

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