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ON VARIOUS TYPES OF COMPATIBLE MAPS AND COMMON FIXED POINT THEOREMS FOR NON-CONTINUOUS MAPS

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Abstract

Concepts of S- and A- compatible of type (E), S- and A- reciprocally continuous are introduced. In this paper, we give a brief discussion on various types of compatible maps and obtain some common fixed point theorems for non-continuous self maps on a metric space. Here the utility of the newly introduced maps viz. S- compatible of type (E) and S- reciprocally continuous (*one sided*) in lieu of continuity is shown. Our results improve, generalize or extend the results of H. Bouhadjera (*General common fixed point theorem for compatible mappings of type (C)*, Sarajevo J. Math. **1** (14), 261–270, 2005), X. P. Ding (*Some common fixed point theorems of commuting mappings* II, Math. Seminar Note **11**, 301–305, 1983), M. L. Diviccaro and S. Sessa (*Some remarks on common fixed points of four mappings*, Jnanabha **15**, 139–149, 1985), G. Jungck (*Compatible mappings and common fixed points* (2), Internat. J. Math. & Math. Sci. **11** (2), 285–288, 1988), S. M. Kang, Y. J. Cho and G. Jungck (*Common fixed points of compatible mappings*, Internat. J. Math. & Math. Sci. **13** (1), 61–66, 1990), V. Popa (*Some fixed point theorems for compatible mappings satisfying an implicit relation*, Demonstr. Math. **32**(1) (1999), 157–163, 1999), and others.

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1. Introduction and preliminaries

In 1976, Jungck[6] used the notion of commuting maps to prove the existence of a common fixed point theorem on a metric space (X, d). Self maps A and S of a metric space (X, d) are said to be commuting if $ASx = SAx$, for all x in X. Since then, he and many authors have investigated various concepts of commuting maps in the weaker sense (see [7], [9], [10], [13]–[17], [22]). For example, Sessa [22] introduced the notion of weakly commuting maps, which is a generalization of commuting maps.

This paper has three sections. In Section 2, we give a brief discussion of various types of compatible maps (compatible of type (A), compatible of type (B), compatible of type (C) and compatible of type (P)) with compatible of type (E). Further, S- and A-compatible of type (E) (S- and A- reciprocally continuous) are introduced by splitting the concept of compatible of type (E) (reciprocally continuous maps [17]). In Section 3, we demonstrate the utility of these new concepts by proving some common fixed point theorems on four non-continuous self maps on a complete metric space.

1.1. Definition. [22] Self maps A and S of a metric space (X, d) are said to be *weakly commuting* if $d(ASx, SAx) \le d(Ax, Sx)$, for all x in X.

Clearly, commuting mappings are weakly commuting but the converse is not true (see [8],[22], [23]).

1.2. Definition. [8] Self maps A and S of a metric space (X, d) are said to be *compatible* iff $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$.

1.3. Definition. [9] Self maps A and S of a metric space (X, d) are said to be *compatible of type (A)* if $\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$ and $\lim_{n\to\infty} d(ASx_n, SSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$.

1.4. Definition. [13] Self maps A and S of a metric space (X, d) are said to be *compatible of type (P)* if $\lim_{n\to\infty} d(SSx_n, AAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$.

The notions of compatible and compatible of type (A) (compatible of type (P)) are equivalent under the continuity of A and S (see [3], [9], [14]).

1.5. Definition. [15] Self maps A and S of a metric space (X, d) are said to be *compatible of type (B)* if

$$\lim_{n\to\infty} d(ASx_n, SSx_n) \le \frac{1}{2}(\lim_{n\to\infty} d(ASx_n, At) + \lim_{n\to\infty} d(At, AAx_n)),$$
$$\lim_{n\to\infty} d(SAx_n, AAx_n) \le \frac{1}{2}(\lim_{n\to\infty} d(SAx_n, St) + \lim_{n\to\infty} d(St, SSx_n)),$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Clearly, compatible of type (A) implies compatible of type (B), however the implication is not reversible (see [15, Ex. 2.4]).

1.6. Definition. [16] Self maps A and S of a metric space (X, d) are said to be *compatible of type (C)*, if

$$\lim_{n\to\infty} d(ASx_n, SSx_n) \le \frac{1}{3}(\lim_{n\to\infty} d(ASx_n, At) + \lim_{n\to\infty} d(At, AAx_n) + \lim_{n\to\infty} d(At, SSx_n))$$

and

$$\lim_{n\to\infty} d(SAx_n, AAx_n) \leq \frac{1}{3}(\lim_{n\to\infty} d(SAx_n, St) + \lim_{n\to\infty} d(St, SSx_n) + \lim_{n\to\infty} d(St, AAx_n))$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

Clearly, compatible of type (A) implies compatible of type (C), however compatible (compatible of type (A), compatible of type (B) and compatible of type (C)) are equivalent under the continuity of A and S (see [2, Prop. 2.1]). Further, in [2] it has been shown that there exists a pair of maps which are compatible of type (C) but neither compatible nor compatible of type (A) (compatible of type (B)).

1.7. Definition. [17] Self maps A and S of a metric space (X, d) are said to be *reciprocally continuous*, if $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

One can see that a continuous pair of self maps is reciprocally continuous, but the converse may not be true (see [17]) and it has to be noted that compatibility and reciprocally continuous are independent concepts (see [23]).

1.8. Definition. [10] Self maps A and S of a metric space (X, d) are said to be *weakly compatible*, if they commute at their coincidence points i.e. $At = St$ implies $ASt = SAt$, for some $t \in X$.

Notice that weakly compatible does not imply compatible (compatible of type (A), compatible of type (P)) (see [17, 19, 20]).

Through out this paper, $N_0 = N \cup \{0\}$, where N and (X, d) denote the set of natural numbers and a metric space respectively.

2. Compatible maps of type (E) and reciprocally continuous maps

We recall the concept compatible of type (E) (see [24]), and compare with compatible (compatible of type (A), compatible of type (B), compatible of type (C), compatible of type (P)). Further, S- and A- compatible of type (E) (S- and A-reciprocally continuous) are also obtained by splitting the concept of compatible of type (E) (reciprocally continuous).

2.1. Definition. Self maps A and S of a metric space (X, d) are said to be *compatible of type (E)*, if $\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} SAx_n = At$ and $\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} ASx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

2.2. Remark. If $At = St$, then compatible of type (E) implies compatible (compatible of type (A), compatible of type (B), compatible of type (C), compatible of type (P)); however, the converse may not be true. Further, if $At \neq St$ then compatible of type (E) is neither compatible nor compatible of type (A) (compatible of type (C), compatible of type (P)).

If $\{A, S\}$ is compatible of type (E), then it is compatible of type (B) but may not be compatible of type (C). For

$$\lim_{n\to\infty} d(ASx_n, SSx_n) \leq \frac{1}{2}(\lim_{n\to\infty} d(ASx_n, At) + \lim_{n\to\infty} d(At, SSx_n),$$

$$\text{i.e. } |At - St| \leq \frac{1}{2}(|St - At| + |At - St|) = |St - At|$$

and

$$\lim_{n\to\infty} d(SAx_n, AAx_n) \le \frac{1}{2}\left(\lim_{n\to\infty} d(SAx_n, St) + \lim_{n\to\infty} d(St, AAx_n)\right),$$
$$\text{i.e. } |At - St| \le \frac{1}{2}\left(|St - At| + |At - St|\right) = |St - At|$$

whenever $Ax_n, Sx_n \to t$ for some $t \in X$. Thus, the pair $\{A, S\}$ is compatible of type (B). Further,

$$\lim_{n\to\infty} d(ASx_n, SSx_n) \le \frac{1}{3}(\lim_{n\leftarrow\infty} d(ASx_n, At) + \lim_{n\to\infty} d(At, AAx_n) + \lim_{n\to\infty} d(At, SSx_n)),$$
$$\text{i.e. } |St - At| \not\le \frac{1}{3}\left(|St - At| + |At - St| + |At - At|\right) = \frac{2}{3}\left|St - At\right|,$$

for $At \ne St$, and

$$\lim_{n\to\infty} d(SAx_n, AAx_n) \le \frac{1}{3}(\lim_{n\leftarrow\infty} d(SAx_n, St) + \lim_{n\to\infty} d(St, SSx_n) + \lim_{n\to\infty} d(St, AAx_n))$$
$$\text{i.e. } |At - St| \not\le \frac{1}{3}\left(|At - St| + |St - At| + |St - St|\right) = \frac{2}{3}\left|St - At\right|,$$

for $At \ne St$ whenever $Ax_n, Sx_n \to t$ for some $t \in X$. Thus, the pair $\{A, S\}$ is not compatible of type (C).

It may be noted that compatible of type (E) implies compatible of type (B) as shown above; however, the converse is not true (see Example 2.4).

2.3. Example. Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. We define self maps A and S as $Ax = 1$, $Sx = 0$ for $x \in [0, \frac{1}{2}] - \{\frac{1}{4}\}$, $Ax = 0$, $Sx = 1$ for $x = \frac{1}{4}$ and $Ax = \frac{1-x}{2}$, $Sx = \frac{x}{2}$ for $x \in (\frac{1}{2}, 1]$. Clearly, A and S are not continuous at $x = \frac{1}{2}, \frac{1}{4}$. Suppose that $x_n \to \frac{1}{2}$, $x_n > \frac{1}{2}$ for all n. Then, we have $Ax_n = \frac{1-x_n}{2} \to \frac{1}{4} = t$ and $Sx_n = \frac{x_n}{2} \to \frac{1}{4} = t$. Also, we have $AAx_n = A(\frac{(1-x_n)}{2}) = 1 \to 1$, $ASx_n = A(\frac{x_n}{2}) = 1 \to 1, S(t) = 1$ and $SSx_n = S(\frac{x_n}{2}) = 0 \to 0$, $SAx_n = S(\frac{(1-x_n)}{2}) = 0 \to 0$, $A(t) = 0$. Therefore, $\{A, S\}$ is compatible of type (E) but the pair $\{A, S\}$ is neither compatible nor compatible of type (A) (compatible of type (C), compatible of type (P)). Of course, $\{A, S\}$ is compatible of type (B) as compatible of type (E) implies compatible of type (B).

2.4. Example. Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. Define self maps A and S as $Ax = Sx = \frac{1}{2}$ for $x \in [0, \frac{1}{2})$, $Ax = Sx = \frac{2}{3}$ for $x = \frac{1}{2}$ and $Ax = 1 - x$, $Sx = x$ for $x \in (\frac{1}{2}, 1]$. Consider a sequence $\{x_n\}$ in X such that $x_n \to \frac{1}{2}$, $x_n > \frac{1}{2}$ for all n. Then, we have $Ax_n = (1 - x_n) \to \frac{1}{2} = t$, and $Sx_n = x_n \to \frac{1}{2} = t$. Since, $1 - x_n < \frac{1}{2}$ for all n we have $AAx_n = A(1 - x_n) = \frac{1}{2} \to \frac{1}{2}$, $ASx_n = A(x_n) = 1 - x_n \to \frac{1}{2}$ and $SSx_n = S(x_n) = x_n \to \frac{1}{2}$, $SAx_n = S(1 - x_n) = \frac{1}{2} \to \frac{1}{2}$. Also, we have $A(t) = \frac{2}{3} = S(t)$ but $AS(t) = AS(\frac{1}{2}) = A(\frac{2}{3}) = \frac{1}{3}$, $SA(t) = SA(\frac{1}{2}) = S(\frac{2}{3}) = \frac{2}{3}$. However, $\frac{1}{3} = AS(t) \ne SA(t) = \frac{2}{3}$, where $t = 1/2$. Therefore, $\{A, S\}$ is compatible (compatible of type (A), compatible of type (B), compatible of type (C), compatible of type (P)); but the maps are not compatible of type (E). Moreover, it has to be noted that the maps are not commuting at the coincidence point.

In [20], Popa remarked that every pair of compatible maps (compatible of type (A), compatible of type (B), compatible of type (P)) is weakly compatible (also see [19], [20, Remark 1]); however, in view of the above Example 2.3, we conclude that compatible maps (compatible of type (A), compatible of type (B), compatible of type (C)), compatible of type (P)) may not be weakly compatible.

In order to prove our main results, we introduce the following definitions by splitting the concepts of compatible of type (E) and reciprocally continuous[17].

2.5. Definition. Self maps A and S of a metric space (X, d) are said to be S-*compatible of type (E)*, if $\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} SAx_n = At$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

2.6. Definition. Self maps A and S of a metric space (X, d) are said to be A-*compatible of type (E)*, if $\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} ASx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

It is easy to see that compatible of type (E) implies both S- and A- compatible of type (E), however A- or S-compatible of type(E) do not imply compatible of type (E) (see Example 2.10).

2.7. Definition. Self maps A and S of a metric space (X, d) are said to be S-*reciprocally continuous*, if $\lim_{n\to\infty} SAx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

2.8. Definition. Self maps A and S of a metric space (X, d) are said to be A-*reciprocally continuous*, if $\lim_{n\to\infty} ASx_n = At$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

Clearly, the reciprocal continuity of $\{A, S\}$ implies both A- and S-reciprocal continuity, however A- or S- reciprocal continuity do not imply reciprocal continuity (see Example 2.11).

2.9. Proposition. *Let A and S be self maps of a metric space (X, d) into itself. Suppose that $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to t$, for some $t \in X$. If one of the following conditions is satisfied:*

(i) *$\{A, S\}$ is S-compatible of type (E) and S- reciprocally continuous;*
(ii) *$\{A, S\}$ is A-compatible of type (E) and A- reciprocally continuous;*

Then (a) *$At = St$, and* (b) *if there exists $u \in X$ such that $Au = Su = t$ then, $ASu = SAu$.*

Proof. Follows immediately. □

2.10. Example. Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. We define self maps A and S as $Ax = 1$, $Sx = \frac{1}{5}$, for $x \in [0, \frac{1}{2}] - \{\frac{1}{4}\}$, $Ax = 0$, $Sx = 1$, for $x = \frac{1}{4}$ and $Ax = \frac{1-x}{2}$, $Sx = \frac{x}{2}$, for $x \in (\frac{1}{2}, 1]$. Since A and S are not continuous at $x = \frac{1}{2}, \frac{1}{4}$, suppose that $x_n \to 1/2$, $x_n > 1/2$, for all n. Then, we have $Ax_n = \frac{1-x_n}{2} \to \frac{1}{4} = t$ and $Sx_n = \frac{x_n}{2} \to \frac{1}{4} = t$. Consequently, we have $AAx_n = A(\frac{1-x_n}{2}) = 1 \to 1, ASx_n = A(\frac{x_n}{2}) = 1 \to 1\, S(t) = 1$ and $SSx_n = S(\frac{x_n}{2}) = \frac{1}{5} \to \frac{1}{5}, SAx_n = S(\frac{1-x_n}{2}) = \frac{1}{5} \to \frac{1}{5}$, $A(t) = 0$. Therefore, $\{A, S\}$ is A-compatible of type (E) but not compatible of type (E).

2.11. Example. Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. We define self maps A and S as Ax =1, $Sx = 0$, for $x \in [0, \frac{1}{2})$ and $Ax = 1 - x$, $Sx = x$, for $x \in [\frac{1}{2}, 1]$. Let $\{x_n\}$ be a sequence in X such that $x_n \to \frac{1}{2}$, $x_n > \frac{1}{2}$, for all n. Then, we have $Ax_n = (1 - x_n) \to \frac{1}{2} = t$ and $Sx_n = x_n \to \frac{1}{2} = t$ since $(1 - x_n) < \frac{1}{2}$ for all n. Consequently, we have $ASx_n = A(x_n) = 1 - x_n \to \frac{1}{2}$, $A(t) = \frac{1}{2}$ and $SAx_n = S(1-x_n) = 0 \to 0$, $S(t) = \frac{1}{2}$. It follows that $\lim_{n\to\infty} ASx_n = \frac{1}{2} = A(t)$ and $\lim_{n\to\infty} SAx_n = 0 \neq S(t) = 1/2$. Therefore, the pair $\{A, S\}$ is A- reciprocally continuous. However the pair is neither S- reciprocally continuous nor reciprocally continuous.

2.12. Example. Let $X = [0, 1]$ with usual metric $d(x, y) = |x - y|$. We define self maps A and S as $Ax = \frac{1+x}{2}$, $Sx = \frac{1}{2} + x$, for $x \in [0, \frac{1}{2})$, $Ax = x - \frac{1}{4}$, for $x \in [\frac{1}{2}, 1]$, $Sx = \frac{1}{4}$, for $x = \frac{1}{2}$, and $Sx = 1 - x$, for $x \in (\frac{1}{2}, 1]$. Let $\{x_n\}$ be a sequence in X such that $x_n \to 0$ and $x_n > 0$, for all n. Then, $Ax_n = \frac{1+x_n}{2} \to \frac{1}{2} = t$ and $Sx_n = \frac{1}{2} + x_n \to \frac{1}{2} = t$. Consequently, $AAx_n = A(\frac{1+x_n}{2}) = \frac{1+x_n}{2} - \frac{1}{4} \to \frac{1}{4}$, $ASx_n = A(\frac{1}{2} + x_n) = \frac{1}{2} + x_n - \frac{1}{4} \to \frac{1}{4}$ and $S(t) = S(\frac{1}{2}) = \frac{1}{4}$. Also, we have $SSx_n = S(\frac{1}{2} + x_n) = 1 - (\frac{1}{2} + x_n) \to \frac{1}{2}$, $SAx_n = S(\frac{1+x_n}{2}) = 1 - \frac{1+x_n}{2} \to \frac{1}{2}$ and $A(t) = A(\frac{1}{2}) = \frac{1}{4}$. Therefore, $\{A, S\}$ is A-compatible of type (E) and A-reciprocally continuous; however $\{A, S\}$ is neither compatible of type (E) nor reciprocally continuous. Note that $\{A, S\}$ is neither compatible nor compatible of type (A) (compatible of type (P), compatible of type (B), compatible of type (C)).

3. Main results

Let $\phi : [0, \infty)^5 \to [0, \infty)$ satisfy the following conditions:

(ϕ_1) ϕ is non-decreasing and upper semi continuous in each coordinate variables,
(ϕ_2) For each $t > 0$,

$$\psi(t) = \max\{\phi(0, 0, t, t, t), \phi(0, t, 0, 0, t), \phi(t, t, t, 2t, 0), \phi(t, t, t, 0, 2t)\} < t.$$

3.1. Proposition. [12] *Suppose that* $\psi : [0, \infty) \to [0, \infty)$ *is non-decreasing and upper semi-continuous from the right. If* $\psi(t) < t$ *for every* $t > 0$*, then* $\lim_{n\to\infty} \psi^n(t) = 0$*, where* $\psi^n(t)$ *denotes the composition of* $\psi(t)$ *with itself* n *times.* □

3.2. Theorem. *Let* A, B, S *and* T *be self maps on a complete metric space* (X, d) *into itself satisfying:*

(i) $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X)$*;*
(ii) $d(Ax, By) \leq \phi(d(Ax, Sx), d(By, Ty), d(Sx, Ty), d(Ax, Ty), d(By, Sx))$*;*
for all $x, y \in X$*, where* ϕ *satisfies* (ϕ_1) *and* (ϕ_2)*.*

Then A, B, S *and* T *have a unique common fixed point if they satisfy either of the following one sided conditions:*

(a) $\{A, S\}$ *is* A*-compatible of type (E) and* A*-reciprocally continuous,* $\{B, T\}$ *is* B*-compatible of type (E) and* B*-reciprocally continuous.*
(b) $\{A, S\}$ *is* S*-compatible of type (E) and* S*-reciprocally continuous,* $\{B, T\}$ *is* T*-compatible of type (E) and* T*-reciprocally continuous.*

Proof. Suppose x_0 be an arbitrary point in X. By virtue of (i), it is guaranteed that we can choose a sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$; $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, where $n \in N_0$.

As in [11, Theorem 2.2], and using Proposition 3.1, one can show that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$$

and that the sequence $\{y_n\}$ is a Cauchy sequence. By the completeness of X, the sequence $\{y_n\}$ converges to a point $z \in X$. Consequently the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ converge to $z \in X$. Thus, $y_{2n} = Ax_{2n} = Tx_{2n+1} \to z$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z$. Suppose that $\{A, S\}$ is A-compatible of type (E) and

A-reciprocally continuous , then AAx_{2n}, $ASx_{2x} \to Sz$ and $ASx_{2n} \to Az$. Therefore, $Az = Sz$. Now, suppose that $z \neq Az$, then by (ii), we have

$$d(Az, Bx_{2n+1}) \leq \phi(d(Az, Sz), d(Bx_{2n+1}, Tx_{2n+1}), d(Sz, Tx_{2n+1}), d(Az, Tx_{2n+1}), d(Bx_{2n+1}, Sz)).$$

Letting $n \to \infty$, we obtain

$$d(Az, z) \leq \phi\,(0,\, 0,\, d(Sz, z), d(Az, z), d(z, Sz)) < \psi(d(Az, z)) < d(Az, z)$$

which is a contradiction. Therefore, $z = Az = Sz$. Since, $A(X) \subseteq T(X)$ and $Az = z$, z is in the range of T, so there exists $u \in X$ such that $Az = Tu = z$. If $Tu \neq Bu$, then by (ii), we obtain

$$\begin{aligned} d(Tu,\, Bu) &= d(Az,\, Bu) \\ &\leq \phi(d(Az,\, Sz),\, d(Bu,\, Tu),\, d(Sz,\, Tu),\, d(Az,\, Tu),\, d(Bu,\, Sz)) \\ &= \phi(0,\, d(Bu,\, Tu),\, 0,\, 0,\, d(Bu,\, Tu)) \\ &< \psi(d(Bu,\, Tu)) \\ &< d(Bu,\, Tu), \end{aligned}$$

which is a contradiction. Therefore, $Bu = Tu = z$. Since, $\{B, T\}$ is B- compatible of type (E) and B-reciprocally continuous, by Proposition 2.9, we obtain $Bz = BTu = TBu = Tz$. Moreover by (ii), we obtain

$$\begin{aligned} d(z, Bz) &= d(Az, Bz) \\ &\leq \phi(d(Az, Sz), d(Bz, Tz),\, d(Sz, Tz), d(Az, Tz), d(Az, Tz), d(Bz, Sz)) \\ &= \phi(0,\, 0,\, d(z, Bz),\, d(z, Bz),\, d(z, Bz)) \\ &< \psi(d(z, Bz)) \\ &< d(z, Bz) \end{aligned}$$

if $z \neq Bz$, a contradiction.Therefore, z is a common fixed point of A, B, S and T. Similarly, we can prove z is a common fixed point of A, B, S and T if $\{A, S\}$ and $\{B, T\}$ satisfy condition (b). The uniqueness of z is immediately obtained from (ii). □

The following example illustrates the validity of the above Theorem 3.2.

3.3. Example. Let $X = [0,\, 1]$ with the usual metric $d(x, y) = |x - y|$. We define self maps A and S as $Ax = Bx = \frac{1+x}{2}$, $Sx = Tx = \frac{1}{2} + x$, for $x \in [0,\, \frac{1}{2})$, $Ax = Bx = \frac{1}{2}$, for $x \in [\frac{1}{2},\, 1]$, $Sx = \frac{1}{2}$, for $x = \frac{1}{2}$, and $Sx = \frac{4}{5}$, for $x \in (\frac{1}{2},\, 1]$. Clearly, $A(X) = [\frac{1}{2},\, \frac{3}{4}) \subseteq S(X) = [\frac{1}{2}, 1)$, and the maps are not continuous at $x = \frac{1}{2}$. Let $\{x_n\}$ be a sequence in X such that $x_n \to 0, x_n > 0$, for all n. Then $Ax_n, Sx_n \to 1/2 = t$, consequently $AAx_n = A\left(\frac{1+x_n}{2}\right) \to 1/2$, $ASx_n = A(\frac{1}{2} + x_n) \to 1/2$, $SSx_n = S(\frac{1}{2} + x_n) \to \frac{4}{5}$ and $SAx_n = S(\frac{1+x_n}{2}) \to \frac{4}{5}$. Also, we have $S(t) = S(\frac{1}{2}) = \frac{1}{2}$, $A(t) = A(\frac{1}{2}) = \frac{1}{2}$. Thus, AAx_n, $ASx_n \to \frac{1}{2} = S(t) = S(\frac{1}{2})$ and $ASx_n \to \frac{1}{2} = A(t) = A(\frac{1}{2})$. Therefore, the pair $\{A, S\}$ is A-compatible of type (E) and A-reciprocally continuous. In particular, if we take $A = B$, $S = T$, $\psi(t) = kt$ and $k = \frac{1}{2}$, then we have

$$d(Ax, By) \leq k \max(d(Ax, Sx), d(By, Ty), d(Sx, Ty), d(Ax, Ty), d(By, Sx)),$$

for all $x, y \in X$. Clearly, all the conditions of the theorem are satisfied and therefore, $1/2$ is the unique common fixed point of A, B, S and T.

The following Corollary is obtained from [11, Corollary 2.3] by employing (*one sided*) compatible of type (E) and reciprocally continuous in lieu of compatibility.

3.4. Corollary. *Let A, B, S and T be self maps on a complete metric space (X, d) into itself, such that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. If the inequality (ii) of Theorem 3.2 holds for all $x, y \in X$, where, ϕ satisfies (ϕ_1) and*

$$\psi(t) = \max\{\phi(t,\, t,\, t,\, t,\, t), \phi(t,\, t,\, t,\, 2t,\, 0), \phi(t,\, t,\, t,\, 0,\, 2t)\} < t,$$

for each $t > 0$, then A, B, S and T have a unique common fixed point if they satisfy either of the conditions (a) *and* (b) *of Theorem 3.2.*

Proof. Follows from Theorem 3.2. □

3.5. Remark. Theorem 3.2 and Corollary 3.4 improve the results of [4] and [5]. In particular, Theorem 3.2 and Corollary 3.4 improve Theorem 2.2 and Corollary 2.3 of [11], respectively. Of course our results do not require continuity conditions on the maps involved.

The following corollary improves [8, Theorem 3.1] by employing (*one sided*) compatible of type (E) and (*one sided*) reciprocal continuity in lieu of compatibility.

3.6. Corollary. *Let A, B, S and T be self maps on a complete metric (X, d) into itself satisfying:*

(i) $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X)$;
(ii) $d(Ax, By) \le r \max(d(Ax, Sx), d(By, Ty), d(Sx, Ty), (d(Ax, Ty) + d(By, Sy))/2)$ *for all $x, y \in X$, where $0 < r < 1$,*

then there is a unique point $z \in X$ such that $z = Az = Bz = Sz = Tz$, provided the pairs $\{A, S\}$ and $\{B, T\}$ satisfy either of the conditions (a) *and* (b) *of Theorem 3.2.*

Proof. Follows from Theorem 3.2. □

Next, using an implicit relation as in Popa[18], let Φ be the set of all real continuous functions $g(t_1, t_2, t_3, t_4, t_5, t_6) : R_+^6 \to R$ satisfying the following conditions:

(g_1) g is non-increasing in the variables t_5 and t_6.
(g_2) There exists $h \in (0, 1)$ such that for every $u, v \ge 0$ with
  (g_a) $g(u,\, v,\, v,\, u,\, u + v,\, 0) \le 0$, or
  (g_a) $g(u,\, v,\, u,\, v,\, 0,\, u + v) \le 0$,
  then we have $u \le hv$.
(g_3) $g(u,\, u,\, 0,\, 0,\, u,\, u) > 0$ for all $u > 0$.

Recently, Aliouche and Djoudi [1], and Popa and Mocanu [21] proved common fixed point theorems satisfying an implicit relation in a metric space without assuming (g_1). Illustrative examples of such a type of implicit relation are given in [1], [18] and [21].

3.7. Theorem. *Let A, B, S and T be self maps on a complete metric space (X, d) into itself satisfying:*

(i) $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X)$;
(ii) $g\,(d(Ax, By), d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty,\, Ax)) \le 0$ *for all $x, y \in X$, where $g \in \Phi$.*

Then A, S, B and T have a unique common fixed point if they satisfy either of the conditions (a) *and* (b) *of Theorem 3.2.*

Proof. Suppose x_0 is an arbitrary point in X. By virtue of (i), it is guaranteed that we can choose sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$; $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, where $n \in N_0$. Using (ii), we obtain

$$g(d(Ax_{2n}, Bx_{2n+1}), d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, Ax_{2n}), d(Tx_{2n+1}, Bx_{2n+1}),$$
$$d(Sx_{2n}, Bx_{2n+1}), d(Tx_{2n+1}, Ax_{2n})) \le 0,$$

so

$$g(d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ d(y_{2n-1}, y_{2n+1}), d(y_{2n}, y_{2n})) \leq 0,$$

that is

$$g(d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ d(y_{2n-1}, y_{2n+1}), 0) \leq 0.$$

By (g_1), we have

$$g(d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}), 0) \leq 0.$$

By (g_a), we obtain

$$d(y_{2n}, y_{2n+1}) \leq \alpha d(y_{2n-1}, y_{2n}),$$

where $0 < \alpha < 1$. Similarly, we obtain

$$d(y_{2n-1}, y_{2n}) \leq \alpha d(y_{2n-2}, y_{2n-1}).$$

Therefore, by induction, we obtain

$$d(y_{2n}, y_{2n+1}) \leq \alpha^{2n} d(y_0, y_1).$$

Letting $n \to \infty$, we obtain, $d(y_{2n}, y_{2n+1}) \to 0$, where $0 < \alpha < 1$. It is easy to show that $\{y_n\}$ is Cauchy sequence. Since X is a complete metric space, there exists a point z in X such that $y_n \to z$. Therefore, we obtain $y_{2n} = Ax_{2n} = Tx_{2n+1} \to z$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z$.

Suppose that $\{A, S\}$ is A-compatible of type (E) and A-reciprocally continuous. Then, $ASx_{2n} \to Az$ and $AAx_{2n}, ASx_{2n} \to Sz$, therefore, $Az = Sz$. We claim that $Az = z$. By (ii), we have

$$g(d(Az, Bx_{2n+1}), d(Sz, Tx_{2n+1}), d(Sz, Az), d(Tx_{2n+1}, Bx_{2n+1}), \\ d(Sz, Bx_{2n+1}), d(Tx_{2n+1}, Az)) \leq 0.$$

Letting $n \to \infty$ in the above inequality and by continuity of g, we obtain

$$g\,(d(Az\ , z), d(Az, z), 0\ , 0\ ,\ d(Az, z), d(Az, z\ )) \leq 0,$$

which is a contradiction to (g_3) if $Az \neq z$. Therefore, $z = Az = Sz$. Since $A(X) \subseteq T(X)$ and $Az = z$, z is in the range of T, so there exists $u \in X$ such that $Az = Tu = z$. If $Tu \neq Bu$ then, by (ii) we obtain

$$g\,(d(Az, Bu), d(Sz, Tu), d(Sz, Az), d(Tu, Bu), d(Sz, Bu), d(Tu,\ Az))\ \leq 0,$$

so $g\,(d(Tu, Bu), 0, 0, d(Tu, Bu), d(Tu, Bu), 0))\ \leq 0$, which is a contradiction to (g_a) and therefore, $Tu = Bu$. Since $Tu = Bu$, then $BBu = BTu = Bz$ and $TBu = TTu = Tz$. Also since, $\{B, T\}$ is B-compatible of type (E) and B-reciprocally continuous, by Proposition 2.9, we obtain $Bz = BTu = TBu = Tz$.

Again by (ii), we obtain

$$g\,(d(Az, Bz), d(Sz, Tz), d(Sz, Az), d(Tz, Bz), d(Sz, Bz), d(Tz,\ Az))\ \leq 0,$$

so $g\,(d(z, Bz), d(z, Bz), 0, 0, d(z, Bz), d(z,\ Bz))\ \leq 0$, which is a contradiction to (g_3) if $Bz \neq z$, and therefore $z = Bz = Tz$. Thus, $z = Az = Bz = Sz = Tz$. The same result holds if $\{A, S\}$ and $\{B, T\}$ satisfy condition (b) of Theorem 3.2.

For the uniqueness, suppose that A, B, S and T have another common fixed point $z' \neq z$. Then by (ii), we obtain

$$g\left(d(Az, Bz'), d(Sz, Tz'), d(Sz, Az), d(Tz', Bz'), d(Sz, Bz'), d(Tz', Az)\right) \leq 0,$$

so $g\left(d(z, z'), d(z, z'), 0, 0, d(z, z'), d(z', z)\right) \leq 0$, a contradiction to (g_3). Thus, z is the unique common fixed point of A, B, S and T. This completes the proof □

3.8. Theorem. *Let $\{A_i\}_{i \in N_0}$ and $\{B_i\}_{i \in N_0}$ be sequences of self maps of a complete metric space (X, d) into itself satisfying condition* (ii) *of Theorem 3.7. For any $n \in N_0$ satisfying:*

(i) *$A_{2n}(X) \subseteq B_{2n+1}(X)$ and $A_{2n+1}(X) \subseteq B_{2n}(X)$;*
(ii) *$\{A_{2n}, B_{2n}\}$ is A_{2n}-compatible of type (E) and A_{2n}-reciprocally continuous;*
(iii) *$\{A_{2n+1}, B_{2n+1}\}$ is A_{2n+1}- compatible of type (E) and A_{2n+1}- reciprocally continuous.*

Then, $\{A_i\}_{i \in N_0}$ and $\{B_i\}_{i \in N_0}$ have a unique common fixed point.

Proof. As in the proof of Theorem 3.7, for fixed $k \in N_0$, we can prove that A_{2k}, A_{2k+1}, B_{2k} and B_{2k+1} have a common fixed point. For the uniqueness, suppose that $A_{2n}z = B_{2n}z = A_{2n+1}z = B_{2n+1}z = z$ and $A_{2m}z' = B_{2m}z' = A_{2m+1}z' = B_{2m+1}z' = z'$, for any $n, m \in N_0$. Using (ii) of Theorem 3.7, we obtain

$$g(d(A_{2n}z, A_{2m+1}z'),\ d(B_{2n}z, B_{2m+1}z'),\ d(B_{2n}z, A_{2n}z),\ d(B_{2m+1}z'.A_{2m+1}z'),$$
$$d(B_{2n}z, A_{2m+1}z'),\ d(B_{2m+1}z', A_{2n}z)) \leq 0,$$

so $g(d(z, z'), d(z, z'), 0, 0, d(z, z'), d(z', z)) \leq 0$, which contradicts (g_3) if $z \neq z'$. Therefore, $z = z'$and hence z is a unique common fixed point of the sequences of maps $\{A_i\}_{i \in N_0}$ and $\{B_i\}_{i \in N_0}$. □

3.9. Remark. Theorem 3.7 improves [2, Theorem 4.1] by employing (*one sided*) compatibility of type (E) and reciprocal continuity in lieu of compatibility of type (C). Moreover, one can verify that Theorems 3.7 and 3.8 also hold true in the case of an implicit relation as used in Aliouche and Djoudi [1].

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