Research Article GU J Sci 32(4): 1228-1236 (2019) DOI: 10.35378/gujs.513478

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

A Voronovskaja-Type Theorem for a Kind of Durrmeyer-Bernstein-Stancu Operators

Ulku DINLEMEZ KANTAR [1,*], Gizem ERGELEN [2]

[1]Gazi University, Faculty of Science, Department of Mathematics, Teknikokullar, 06500, Ankara, Turkey

[2]Yukselen College, Mamak Campus, Cengizhan Mahallesi, Dogukent Cd., 06480 Mamak/Ankara, Turkey

Highlights

•A Durrmeyer variant of Bernstein-Stancu operators is studied.
•These operators are introduced with shifted knots.
•Asymptotic behavior of these operators is examined by using Voronovskaja-type theorem.

Article Info

Received: 16/01/2019
Accepted: 25/03/2019

Keywords

Durrmeyer type operators
Bernstein-Stancu type operators
Voronovskaja-type theorem

Abstract

In this paper, we study on a Durrmeyer variant of Bernstein-Stancu operators. We give a Voronovskaja-type theorem for these type operators.

1. INTRODUCTION

In [1], for $f \in C[0,1]$, the Bernstein polynomials $(B_n f)$ are defined by

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{nk}(x) \ ,n \in \mathbb{N} \tag{1}$$

where $b_{nk}(x) := \binom{n}{k} x^k (1-x)^{n-k}$, $k = 0,1,\ldots,n$.

In [2], Durrmeyer introduced modified Bernstein polynomials to approximate Lebesgue integrable function on $[0,1]$, then he motivated the following integral modification of Bernstein polynomials

$$D_n(f,x) := (n+1)\sum_{k=0}^{n} b_{nk}(x)\int_0^1 f(t) b_{nk}(t)\,dt.$$

In [3], the authors introduced the following new generalized Bernstein-Stancu type operators with shifted knots:

$$\tilde{B}_n^{\alpha,\beta}(f,x) := \left(\frac{n+\beta_2}{n}\right)^n \ \sum_{k=0}^{n} f\left(\frac{k+\alpha_1}{n+\beta_1}\right) T_{nk}(x), \tag{2}$$

where $\in \ A_n := \left[\frac{\alpha_2}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}\right]$, and $T_{nk}(x) := \binom{n}{k} \left(x - \frac{\alpha_2}{n+\beta_2}\right)^k \left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{n-k}$,

*Corresponding author, e-mail: ulku@gazi.edu.tr

$k=0,1,\dots n$ with $\alpha_k,\beta_k, k=1,2$ positive numbers satisfying $0\leq \alpha_1\leq\beta_1, 0\leq\alpha_2\leq\beta_2$. Apparently, substituting $\alpha_1=\alpha_2=\beta_2=\beta_1=0$ in (2) one finds the classical Bernstein operators in (1) , substituting $\alpha_2=\beta_2=0$ in (2), one obtains the Bernstein-Stancu operators introduced by Stancu in [4]

$$B_{n,\alpha,\beta}(f,x)\coloneqq\sum_{k=0}^{n} f\left(\frac{k+\alpha_1}{n+\beta_1}\right)T_{nk}(x).$$

In [5], for $f\in C[0,1]$, authors defined a Durrmeyer variant of Bernstein-Stancu operators

$$S_n^{\alpha,\beta}(f,x)\coloneqq\left(\frac{n+\beta_2}{n}\right)^{2n+1}\sum_{k=0}^{n} T_{nk}(x)(n+1)\int_{A_n} T_{nk}(t)f\left(\frac{nt+\alpha_1}{n+\beta_1}\right)dt, \tag{3}$$

$A_n\coloneqq\left[\frac{\alpha_2}{n+\beta_2},\frac{n+\alpha_2}{n+\beta_2}\right]$, and $T_{nk}(x)\coloneqq\binom{n}{k}\left(x-\frac{\alpha_2}{n+\beta_2}\right)^k\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-k}, k=0,1,\dots n$

with $\alpha_k,\beta_k, k=1,2$ positive numbers satisfying $0\leq\alpha_1\leq\beta_1, 0\leq\alpha_2\leq\beta_2$. They obtained the direct and converse results of approximation by these operators. In [6-11], many authors have studied some special cases of the operators $S_n^{\alpha,\beta}(f,x)$.

In this manuscript, we give a Voronovskaja-type theorem for the operators defined in (3).

2. MAIN RESULT

In this section, we give some lemmas in order to prove our main result. For the sake of shortness, throughout the paper we use the following notations

$$H_s(n)=\sum_{i=2}^{s}(n+i),\quad G_{\beta_i}(n)=(n+\beta_i)\text{ for } i=1,2.$$

Now, we give the following lemma which is proved in [5].

Lemma 1. [5] We have the following equality

$$\int_{\frac{\alpha_2}{n+\beta_2}}^{\frac{n+\alpha_2}{n+\beta_2}} T_{nk}(t)dt=\left(\frac{n}{n+\beta_2}\right)^{n+1}\frac{1}{(n+1)},\quad k=0,1,2\dots n. \tag{4}$$

Lemma 2. Let $e_j=t^j,\ j=0,1,2,3,4,$ we obtain

(i) $S_n^{\alpha,\beta}(e_0,x)=1$,

(ii) $S_n^{\alpha,\beta}(e_1,x)=\frac{n^2}{G_{\beta_1}(n)\,G_{\beta_2}(n)}x+\frac{n^2+2n\alpha_2}{G_{\beta_2}(n)H_2(n)}+\frac{\alpha_1}{G_{\beta_1}(n)}$,

(iii)
$$S_n^{\alpha,\beta}(e_2,x)=\frac{n^4-n^3}{[G_{\beta_1}(n)]^2H_3(n)}x^2+\left\{\frac{4n^4+8n^3\alpha_2}{[G_{\beta_1}(n)]^2\,G_{\beta_2}(n)\,H_3(n)}+\frac{2n^2\alpha_1}{[G_{\beta_1}(n)]^2\,H_2(n)}\right\}x$$
$$+\frac{2n^4-2\alpha_2n^4-2\alpha_2^2n^3+6n^3\alpha_2+6n^2\alpha_2^2}{[G_{\beta_1}(n)]^2[G_{\beta_2}(n)]^2H_3(n)}+\frac{2n^2\alpha_1+4n\alpha_1\alpha_2}{[G_{\beta_1}(n)]^2G_{\beta_2}(n)\,H_2(n)}+\frac{\alpha_1^2}{[G_{\beta_1}(n)]^2},$$

(iv)
$$S_n^{\alpha,\beta}(e_3,x)=\frac{n^6-3n^5-2n^4}{[G_{\beta_1}(n)]^3H_4(n)}x^3+\left\{\frac{9n^6+18n^5\alpha_2-9n^5-18n^4\alpha_2}{[G_{\beta_1}(n)]^3G_{\beta_2}(n)\,H_4(n)}+\frac{3n^4\alpha_1-3n^3\alpha_1}{[G_{\beta_1}(n)]^3H_3(n)}\right\}x^2$$
$$+\left\{\frac{18n^6-9n^6\alpha_2-6n^5\alpha_2^2+54n^5\alpha_2+66n^4\alpha_2^2}{[G_{\beta_1}(n)]^3[G_{\beta_2}(n)]^2H_4(n)}+\frac{12n^4\alpha_1+24n^3\alpha_1\alpha_2}{[G_{\beta_1}(n)]^3G_{\beta_2}(n)H_3(n)}+\frac{3n^2\alpha_1^2}{[G_{\beta_1}(n)]^3H_2(n)}\right\}x$$

$$+\frac{3n^6\alpha_2^2-12n^6\alpha_2+6n^6-24n^5\alpha_2^2+24n^5\alpha_2-24n^4\alpha_2^3+36n^4\alpha_2^2+24n^3\alpha_2^3}{\left[G_{\beta_1}(n)\right]^3\left[G_{\beta_2}(n)\right]^3H_4(n)}$$

$$+\frac{6n^4\alpha_1-6n^4\alpha_1\alpha_2-6n^3\alpha_1\alpha_2^2+18n^3\alpha_1\alpha_2+18n^2\alpha_1\alpha_2^2}{\left[G_{\beta_1}(n)\right]^3\left[G_{\beta_2}(n)\right]^2H_3(n)}+\frac{3n^2\alpha_1^2+6n\alpha_1^2\alpha_2}{\left[G_{\beta_1}(n)\right]^3G_{\beta_2}(n)\,H_2(n)}+\frac{\alpha_1^3}{\left[G_{\beta_1}(n)\right]^3},$$

(v) $$S_n^{\alpha,\beta}(e_4,x)=\frac{n^8-6n^7+11n^6-6n^5}{\left[G_{\beta_1}(n)\right]^4H_5(n)}x^4+\left\{\frac{24n^7\alpha_2-4n^8\alpha_2-44n^6\alpha_2+24n^5\alpha_2+16n^8-48n^7+32n^6}{\left[G_{\beta_1}(n)\right]^4G_{\beta_2}(n)H_5(n)}\right.$$

$$\left.+\frac{4n^7\alpha_2-12n^6\alpha_2+8n^5\alpha_2)(n+5)+(4n^4\alpha_1-12n^5\alpha_1+8n^4\alpha_1)G_{\beta_2}(n)(n+5)}{\left[G_{\beta_1}(n)\right]^4G_{\beta_2}(n)H_5(n)}\right\}x^3$$

$$+\left\{\frac{6n^8\alpha_2^2-36n^7\alpha_2^2+66n^6\alpha_2^2-36n^5\alpha_2^2-48n^8\alpha_2+144n^7\alpha_2-96n^6\alpha_2+72n^8-72n^7}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^2H_5(n)}\right.$$

$$+\frac{\left(-12n^7\alpha_2^2+36n^6\alpha_2^2-24n^5\alpha_2^2+36n^7\alpha_2-36n^6\alpha_2\right)(n+5)+(6n^6\alpha_2^2-6n^5\alpha_2^2)(n+5)(n+4)}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^2H_5(n)}$$

$$\left.+\frac{\left(36n^6\alpha_1+72n^5\alpha_1\alpha_2-36n^5\alpha_1-74n^4\alpha_1\alpha_2\right)(n+\beta_2)(n+5)+(6n^4\alpha_1^2-6n^3\alpha_1^2)(n+\beta_2)^2(n+5)(n+4)}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^2H_5(n)}\right\}x^2$$

$$+\left\{\frac{-4n^8\alpha_2^3+24n^7\alpha_2^3-44n^6\alpha_2^3+24n^5\alpha_2^3+48n^8\alpha_2^2-144n^7\alpha_2^2+96n^6\alpha_2^2-144n^8\alpha_2+144n^7\alpha_2+96n^4}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^3H_5(n)}\right.$$

$$+\frac{\left(12n^7\alpha_2^3-36n^6\alpha_2^3+24n^5\alpha_2^3-72n^7\alpha_2^2+72n^6\alpha_2^2+72n^7\alpha_2\right)(n+5)}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^3H_5(n)}$$

$$+\frac{\left(-12n^6\alpha_2^3+12n^5\alpha_2^3+24n^6\alpha_2^2\right)(n+5)(n+4)+4n^4\alpha_2^3F_5(n)}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^3H_5(n)}$$

$$+\frac{-36n^6\alpha_1\alpha_2+72n^6\alpha_1+216n^5\alpha_1\alpha_2-24n^5\alpha_1\alpha_2^2+264n^4\alpha_1\alpha_2^2}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^2H_4(n)}+\frac{24n^4\alpha_1^2+48n^3\alpha_1^2\alpha_2}{\left[G_{\beta_1}(n)\right]^4G_{\beta_2}(n)H_3(n)}$$

$$\left.+\frac{4n^2\alpha_1^3}{\left[G_{\beta_1}(n)\right]^4H_3(n)}\right\}x$$

$$+\frac{n^8\alpha_2^4-6n^7\alpha_2^4+11n^6\alpha_2^4-6n^5\alpha_2^4-16n^8\alpha_2^3+48n^7\alpha_2^3-32n^6\alpha_2^3+72n^8\alpha_2^2-72n^7\alpha_2^2}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^4H_5(n)}$$

$$+\frac{24n^6-96n^8\alpha_2}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^4H_5(n)}+\frac{\left(-4n^7\alpha_2^4+12n^6\alpha_2^4-8n^5\alpha_2^4+36n^7\alpha_2^3-36n^6\alpha_2^3-72n^7\alpha_2^2+24n^7\alpha_2\right)}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^4H_5(n)}$$

$$+\frac{(6n^2\alpha_2^4-6n\alpha_2^4-24n^2\alpha_2^3+12n^2\alpha_2^2)(n+5)(n+4)}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^4H_5(n)}+\frac{\left(4\alpha_2^3-3\alpha_2^4\right)F_5(n)}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^4H_5(n)}$$

$$+\frac{12n^6\alpha_1\alpha_2^2-48n^6\alpha_1\alpha_2+24n^6\alpha_1-96n^5\alpha_1\alpha_2^2-96n^5\alpha_1\alpha_2-96n^4\alpha_1\alpha_2^3+144n^4\alpha_1\alpha_2^2+96n^3\alpha_1\alpha_2^3}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^3H_4(n)}$$

$$+\frac{12n^4\alpha_1^2-12n^4\alpha_1^2\alpha_2-12n^3\alpha_1^2\alpha_2^2+36n^2\alpha_1^2\alpha_2+36n^2\alpha_1^2\alpha_2^2}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^2H_3(n)}+\frac{4n^2\alpha_1^3+8n\alpha_1^3\alpha_2}{\left[G_{\beta_1}(n)\right]^4G_{\beta_2}(n)H_2(n)}+\frac{\alpha_1^4}{\left[G_{\beta_1}(n)\right]^4}.$$

Proof. Using equality (4) in the Durrmeyer variant of Bernstein-Stancu operators (3) for $j=0$, we get

$$S_n^{\alpha,\beta}(e_0,x)=\left(\frac{n+\beta_2}{n}\right)^{2n+1}\sum_{k=0}^{n} T_{nk}(x)(n+1)\int_{A_n} T_{nk}(t)\,dt$$

$$=\left(\frac{n+\beta_2}{n}\right)^{n}\sum_{k=0}^{n} T_{nk}(x)$$

$$=1.$$

So the proof of (i) is finished. Using the direct computation, we obtain (ii) as follows

$$S_n^{\alpha,\beta}(e_1,x)=\left(\frac{n+\beta_2}{n}\right)^{2n+1}\sum_{k=0}^{n} T_{nk}(x)(n+1)\int_{A_n} T_{nk}(t)\left(\frac{nt+\alpha_1}{n+\beta_1}\right)dt$$

$$=\left(\frac{n+\beta_2}{n}\right)^{2n+1}\frac{n}{n+\beta_1}\sum_{k=0}^{n} T_{nk}(x)(n+1)\binom{n}{k}\int_{A_n}\left(t-\frac{\alpha_2}{n+\beta_2}\right)^{k}\left(\frac{n+\alpha_2}{n+\beta_2}-t\right)^{n-k} t\,dt$$

$$+\left(\frac{n+\beta_2}{n}\right)^{n}\frac{\alpha_1}{n+\beta_1}\sum_{k=0}^{n} T_{nk}(x)\ .$$

If we take $t=u+\frac{\alpha_2}{n+\beta_2}$ in the last equality, then we get

$$S_n^{\alpha,\beta}(e_1,x)=\left(\frac{n+\beta_2}{n}\right)^{2n+1}\frac{n}{n+\beta_1}\sum_{k=0}^{n} T_{nk}(x)(n+1)$$

$$\text{x}\binom{n}{k}\left\{\int_0^{\frac{n}{n+\beta_2}} u^{k+1}\left(\frac{n}{n+\beta_2}-u\right)^{n-k}du+\frac{\alpha_2}{n+\beta_2}\int_0^{\frac{n}{n+\beta_2}} u^{k}\left(\frac{n}{n+\beta_2}-u\right)^{n-k}du\right\},$$

$$S_n^{\alpha,\beta}(e_1,x)=\left(\frac{n+\beta_2}{n}\right)^{2n+1}\frac{n}{n+\beta_1}\sum_{k=0}^{n}\binom{n}{k}\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{k}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-k}(n+1)$$

$$\text{x}\left\{\left(\frac{n}{n+\beta_2}\right)^{n+2}\frac{k+1}{(n+1)(n+2)}+\frac{\alpha_2}{n+\beta_2}\frac{1}{n+1}\left(\frac{n}{n+\beta_2}\right)^{n+1}\right\}$$

$$S_n^{\alpha,\beta}(e_1,x)=\frac{n^2}{G_{\beta_1}(n)\,G_{\beta_2}(n)}x+\frac{n^2+2n\alpha_2}{G_{\beta_2}(n)H_2(n)}+\frac{\alpha_1}{G_{\beta_1}(n)}\,.$$

Thus, we have the proof of (ii). Finally, we apply the same process in (ii), as a result, we get (iii), (iv), and (v) easily.

Now, we give the following lemma for using the Voronovskaja-type theorem.

Lemma 3. Let α_k,β_k, k=1,2 be positive numbers such that 0≤ $\alpha_1\le\beta_1$, 0≤ $\alpha_2\le\beta_2$.

We obtain the following limits

$$(i)\ \lim_{n\to\infty} nS_n^{\alpha,\beta}((t-x)^2;x) = -2x^2+2x, \quad (5)$$

$$(ii)\ \lim_{n\to\infty} n^2S_n^{\alpha,\beta}((t-x)^4;x) = 28x^4-24x^3+(12\alpha_2+12)x^2. \quad (6)$$

Proof. (*i*) From Lemma 2, we get

$$\lim_{n\to\infty} nS_n^{\alpha,\beta}((t-x)^2;x) = \lim_{n\to\infty}\Bigg[\left\{\frac{-2n^4+n^3(6+\beta_1{}^2+4\beta_1)+n^2(5\beta_1{}^2+12\beta_1)+6\beta_1{}^2n}{[G_{\beta_1}(n)]^2H_3(n)}\right\}x^2$$

$$+\frac{2n^5+n^4(4\alpha_2+4\alpha_1-2\alpha_1\beta_1-2\beta_1-6)+n^3(-12\alpha_1-12\alpha_2-10\alpha_1\beta_1-4\alpha_1\beta_2-4\alpha_2\beta_1-2\alpha_1\beta_1\beta_2-6\beta_1)}{[G_{\beta_1}(n)]^2[G_{\beta_2}(n)]^2H_3(n)}$$

$$+\frac{n^2(-12\alpha_1\beta_1-12\alpha_1\beta_2-12\alpha_2\beta_1-10\alpha_1\beta_1\beta_2)}{[G_{\beta_1}(n)]^2[G_{\beta_2}(n)]^2H_3(n)}+\left.\frac{-12n\alpha_1\beta_1\beta_2}{[G_{\beta_1}(n)]^2[G_{\beta_2}(n)]^2H_3(n)}\right\}x$$

$$+\frac{n^5(4\alpha_2^2-12\alpha_2+\alpha_1^2+2\alpha_1+2)+n^4(10\alpha_2^2+6\alpha_2+5\alpha_1^2+6\alpha_1+5\alpha_1^2\beta_2+2\alpha_1\alpha_2+2\alpha_1\beta_2}{[G_{\beta_1}(n)]^2[G_{\beta_2}(n)]^2H_3(n)}$$

$$+\frac{n^3(6\alpha_2^2+6\alpha_1^2+10\alpha_1^2\beta_2+\alpha_1^2\beta_2{}^2+12\alpha_1\alpha_2+6\alpha_1\beta_2+4\alpha_1\alpha_2\beta_2)+n^2(5\alpha_1^2\beta_2{}^2+12\alpha_1^2\beta_2+12\alpha_1\alpha_2\beta_2)+6\alpha_1^2\beta_2{}^2n}{[G_{\beta_1}(n)]^2[G_{\beta_2}(n)]^2H_3(n)}\Bigg]$$

$$= -2x^2+2x.$$

(ii) Using linearity of the operators $S_n^{\alpha,\beta}(f(t);x)$ and Lemma 2, we have

$$S_n^{\alpha,\beta}(f(t);x) = I_1^{\alpha,\beta}(n)x^4+I_2^{\alpha,\beta}(n)x^3+I_3^{\alpha,\beta}(n)x^2+I_4^{\alpha,\beta}(n)x+I_5^{\alpha,\beta}(n),$$

where

$$I_1^{\alpha,\beta}(n)= \frac{n^8-6n^7+11n^6-6n^5}{[G_{\beta_1}(n)]^4H_5(n)}-\frac{4n^6-12n^5-8n^4}{[G_{\beta_1}(n)]^3H_4(n)}+\frac{6n^4-6n^3}{[G_{\beta_1}(n)]^2H_3(n)}-\frac{4n^2}{G_{\beta_1}(n)\,H_2(n)}+1,$$

$$I_2^{\alpha,\beta}(n)=\frac{24n^7\alpha_2-4n^8\alpha_2-44n^6\alpha_2+24n^5\alpha_2+16n^8-48n^7+32n^6}{[G_{\beta_1}(n)]^4G_{\beta_2}(n)H_5(n)}+\frac{4n^7\alpha_2-12n^6\alpha_2+8n^5\alpha_2)}{[G_{\beta_1}(n)]^4G_{\beta_2}(n)H_4(n)}$$

$$+\frac{4n^4\alpha_1-12n^5\alpha_1+8n^4\alpha_1}{[G_{\beta_1}(n)]^4H_4(n)}-\frac{36n^6+72n^5\alpha_2-36n^7-72n^4\alpha_2}{[G_{\beta_1}(n)]^3G_{\beta_2}(n)H_4(n)}-\frac{12n^4\alpha_1-12n^3\alpha_1}{[G_{\beta_1}(n)]^3H_3(n)}$$

$$+\frac{24n^4+48n^3\alpha_2}{[G_{\beta_1}(n)]^2H_3(n)}+\frac{12n^2\alpha_1}{[G_{\beta_1}(n)]^2H_2(n)}-\frac{4n^2+8n\alpha_2}{G_{\beta_1}(n)G_{\beta_2}(n)H_2(n)}-\frac{4\alpha_1}{G_{\beta_1}(n)},$$

$$I_3^{\alpha,\beta}(n)=\frac{6n^8\alpha_2^2-36n^7\alpha_2^2+66n^6\alpha_2^2-36n^5\alpha_2^2-48n^8\alpha_2+144n^7\alpha_2-96n^6\alpha_2+72n^8-72n^7}{[G_{\beta_1}(n)]^4[G_{\beta_2}(n)]^2H_5(n)}$$

$$+\frac{-12n^7\alpha_2^2+36n^6\alpha_2^2-24n^5\alpha_2^2+36n^7\alpha_2-36n^6\alpha_2}{[G_{\beta_1}(n)]^4[G_{\beta_2}(n)]^2H_4(n)}+\frac{6n^6\alpha_2^2-6n^5\alpha_2^2}{[G_{\beta_1}(n)]^4[G_{\beta_2}(n)]^2H_3(n)}$$

$$+\frac{36n^6\alpha_1+72n^5\alpha_1\alpha_2-36n^5\alpha_1-74n^4\alpha_1\alpha_2}{[G_{\beta_1}(n)]^4G_{\beta_2}(n)H_4(n)}+\frac{6n^4\alpha_1^2-6n^3\alpha_1^2}{[G_{\beta_1}(n)]^4H_3(n)}$$

$$+\frac{-36n^6\alpha_2-72n^6+216n^5\alpha_2-24n^5\alpha_2^2+264n^4\alpha_2^2}{\left[G_{\beta_1}(n)\right]^3\left[G_{\beta_2}(n)\right]^2H_4(n)}-\frac{48n^4\alpha_1+96n^3\alpha_1\alpha_2}{\left[G_{\beta_1}(n)\right]^3G_{\beta_2}(n)H_3(n)}-\frac{12n^2\alpha_1^2}{\left[G_{\beta_1}(n)\right]^3H_2(n)}$$

$$+\frac{-12n^4\alpha_2-12n^4-12n^3\alpha_2^2+36n^3\alpha_2+36n^2\alpha_2^2}{\left[G_{\beta_1}(n)\right]^2\left[G_{\beta_2}(n)\right]^2H_3(n)}+\frac{12n^2\alpha_1+24n\alpha_1\alpha_2}{\left[G_{\beta_1}(n)\right]^2G_{\beta_2}(n)H_2(n)}+\frac{6\alpha_1^2}{\left[G_{\beta_1}(n)\right]^2},$$

$$I_4^{\alpha,\beta}(n)=\frac{-4n^8\alpha_2^3+24n^7\alpha_2^3-44n^6\alpha_2^3+24n^5\alpha_2^3+48n^8\alpha_2^2-144n^7\alpha_2^2+96n^6-144n^8\alpha_2+144n^7\alpha_2+96n^4}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^3H_5(n)}$$

$$+\frac{12n^7\alpha_2^3-36n^6\alpha_2^3+24n^5\alpha_2^3-72n^7\alpha_2^2+72n^6\alpha_2^2+72n^7\alpha_2}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^3H_4(n)}+\frac{-12n^6\alpha_2^3+12n^5\alpha_2^3+24n^6\alpha_2^2}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^3H_3(n)}$$

$$+\frac{4n^4\alpha_2^3}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^3}+\frac{-36n^6\alpha_1\alpha_2+72n^6\alpha_1+216n^5\alpha_1\alpha_2-24n^5\alpha_1\alpha_2^2+264n^4\alpha_1\alpha_2^2}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^2H_4(n)}$$

$$-\frac{12n^6\alpha_2^2-48n^6\alpha_2+24n^6-96n^5\alpha_2^2-96n^5\alpha_2-96n^4\alpha_2^3+144n^4\alpha_2^2+96n^3\alpha_2^3}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^3H_4(n)}$$

$$-\frac{-24n^4\alpha_1^2\alpha_2+24n^4\alpha_1-24n^3\alpha_1\alpha_2^2-72n^3\alpha_1\alpha_2+72n^2\alpha_1\alpha_2^2}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^2H_3(n)}+\frac{24n^4\alpha_1^2+48n^3\alpha_1^2\alpha_2}{\left[G_{\beta_1}(n)\right]^4G_{\beta_2}(n)H_3(n)}$$

$$+\frac{3n^2\alpha_1^2+6n\alpha_1^2\alpha_2}{\left[G_{\beta_1}(n)\right]^3G_{\beta_2}(n)H_2(n)}+\frac{4n^2\alpha_1^3}{\left[G_{\beta_1}(n)\right]^4H_2(n)}+\frac{\alpha_1^3}{\left[G_{\beta_1}(n)\right]^3},$$

$$I_5^{\alpha,\beta}(n)=\frac{n^8\alpha_2^4-6n^7\alpha_2^4+11n^6\alpha_2^4-6n^5\alpha_2^4-16n^8\alpha_2^3+48n^7\alpha_2^3-32n^6\alpha_2^3+72n^8\alpha_2^2-72n^7\alpha_2^2+24n^6-96n^8\alpha_2}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^4H_5(n)}$$

$$+\frac{-4n^7\alpha_2^4+12n^6\alpha_2^4-8n^5\alpha_2^4+36n^7\alpha_2^3-36n^6\alpha_2^3-72n^7\alpha_2^2+24n^7\alpha_2}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^4H_4(n)}+\frac{6n^2\alpha_2^4-6n^5\alpha_2^4-24n^2\alpha_2^3+12n^6\alpha_2^2}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^4H_3(n)}$$

$$+\frac{12n^6\alpha_1\alpha_2^2-48n^6\alpha_1\alpha_2+24n^6\alpha_1-96n^5\alpha_1\alpha_2^2-96n^5\alpha_1\alpha_2-96n^4\alpha_1\alpha_2^3+144n^4\alpha_1\alpha_2^2+96n^3\alpha_1\alpha_2^3}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^3H_4(n)}$$

$$+\frac{\alpha_1^4}{\left[G_{\beta_1}(n)\right]^4}+\frac{-12n^4\alpha_1^2\alpha_2+12\alpha_1^2n^4-12n^3\alpha_1^2\alpha_2^2+36n^2\alpha_1^2\alpha_2+36n^2\alpha_1^2\alpha_2^2}{\left[G_{\beta_1}(n)\right]^2\left[G_{\beta_2}(n)\right]^2H_3(n)}+\frac{4n^2\alpha_1^3+8n\alpha_1^3\alpha_2}{\left[G_{\beta_1}(n)\right]^4G_{\beta_2}(n)H_2(n)}$$

$$+\frac{(4n^4\alpha_2^3-3n^4\alpha_2^4)}{\left[G_{\beta_1}(n)\right]^4\left[G_{\beta_2}(n)\right]^4}.$$

It is obvious that

$$\lim_{n\to\infty}n^2\left\{I_4^{\alpha,\beta}(n)+I_5^{\alpha,\beta}(n)\right\}=0. \quad (7)$$

Therefore, we have

$$\lim_{n\to\infty}n^2\left\{I_1^{\alpha,\beta}(n)+I_2^{\alpha,\beta}(n)+I_3^{\alpha,\beta}(n)\right\}=28x^4-24x^3+(12\alpha_2+12)x^2. \quad (8)$$

Adding the limits (7) and (8), we have the limits (6).

In [6], Gadzhiev proved the weighted Korovkin-type theorem. We give the Gadzhiev's results in weighted spaces. Let $\mu(x)=1+x^2$. $B_\mu[0,\infty)$ denotes the set of all functions from $[0,\infty)$ to $\mathbb{R}$, satisfying the growth condition $|f(x)|\leq N_f\mu(x)$. In this inequality, N_f is a constant depending only on f. $B_\mu[0,\infty)$ is a normed space with a norm $\|f\|_{\mu} = sup\left\{\frac{|f(x)|}{\mu(x)}\ ; x \in \mathbb{R}\right\}$. $C_{\mu}^{*}[0,\infty)$ denotes the subspace of continuos functions in $B_{\mu}[0,\infty)$ for which $\lim_{|x|\to\infty} \frac{|f(x)|}{\mu(x)}$ exists finitely.

And now we give a Voronovskaja-type theorem for $S_n^{\alpha,\beta}(f(t),x)$ operators.

Theorem 1. For any $f \in C_{\mu}^{*}[0,\infty)$ such that $f', f'' \in C_{\mu}^{*}[0,\infty)$. We get the following limit

$$\lim_{n\to\infty} n\left(S_n^{\alpha,\beta}(f(t);x) - f(x)\right) = (-x^2 + x)f''(x) + \left((-2-\beta_1)x + 1 + \alpha_2\right)f'(x).$$

Proof. Using Taylor's expansion of f, we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \rho(t,x)(t-x)^2,$$

where $\rho(t,x) \to 0$ as $t \to x$. From linearity of the operators $S_n^{\alpha,\beta}(f(t),x)$, we get

$$S_n^{\alpha,\beta}(f(t);x) = f(x) + f'(x)S_n^{\alpha,\beta}(t-x) + \frac{1}{2}f''(x)S_n^{\alpha,\beta}((t-x)^2;x)$$
$$+S_n^{\alpha,\beta}(\rho(t,x)(t-x)^2).$$

Thanks to Lemma 2, we obtain the following operators by making the necessary process,

$$\begin{aligned} S_n^{\alpha,\beta}(f(t);x) = {} & f(x) + f'(x)\left\{\frac{-n(2+\beta_1)-2\beta_2}{G_{\beta_1}(n)H_4(n)}x + \frac{n^2+2n\alpha_2}{G_{\beta_1}(n)G_{\beta_2}(n)H_2(n)} + \frac{\alpha_2}{G_{\beta_1}(n)}\right\} \\ & + \frac{1}{2}f''(x)\Bigg[\left\{\frac{-2n^3 + n^2(6+{\beta_1}^2+4\beta_1)+n(5{\beta_1}^2+12\beta_1)+6{\beta_1}^2}{\left[G_{\beta_1}(n)\right]^2 H_3(n)}\right\}x^2 \\ & + \Bigg\{\frac{2n^4 + n^3(4\alpha_2+4\alpha_1-2\alpha_1\beta_1-2\beta_1-6)}{\left[G_{\beta_1}(n)\right]^2\left[G_{\beta_2}(n)\right]^2 H_3(n)} \\ & - \frac{n^2(12\alpha_1+12\alpha_2+10\alpha_1\beta_1+4\alpha_1\beta_2+4\alpha_2\beta_1+2\alpha_1\beta_1\beta_2+6\beta_1)}{\left[G_{\beta_1}(n)\right]^2\left[G_{\beta_2}(n)\right]^2 H_3(n)} \\ & + \frac{12\alpha_1\beta_1+12\alpha_1\beta_2+12\alpha_2\beta_1+10\alpha_1\beta_1\beta_2-12\alpha_1\beta_1\beta_2}{\left[G_{\beta_1}(n)\right]^2\left[G_{\beta_2}(n)\right]^2 H_3(n)}\Bigg\}x \\ & + \frac{n^4(4\alpha_2^2-2\alpha_2+\alpha_1^2+2\alpha_1+2)+n^3(10\alpha_2^2+6\alpha_2+5\alpha_1^2+6\alpha_1+5\alpha_1^2\beta_2+2\alpha_1\alpha_2+2\alpha_1\beta_2)}{\left[G_{\beta_1}(n)\right]^2\left[G_{\beta_2}(n)\right]^2 H_3(n)} \\ & + \frac{2(6\alpha_2^2+6\alpha_1^2+10\alpha_1^2\beta_2+\alpha_1^2{\beta_2}^2+12\alpha_1\alpha_2+6\alpha_1\beta_2+4\alpha_1\alpha_2\beta_2)}{\left[G_{\beta_1}(n)\right]^2\left[G_{\beta_2}(n)\right]^2 H_3(n)} \\ & + \frac{n(5\alpha_1^2{\beta_2}^2+12\alpha_1^2\beta_2+12\alpha_1\alpha_2\beta_2)+6\alpha_1^2{\beta_2}^2 n}{\left[G_{\beta_1}(n)\right]^2\left[G_{\beta_2}(n)\right]^2 H_3(n)}\Bigg] + S_n^{\alpha,\beta}(\rho(t,x)(t-x)^2;x). \end{aligned} \tag{9}$$

Applying Cauchy-Schwarz inequality to the last term of (9) we obtain

$$n\,S_n^{\alpha,\beta}(\rho(t,x)(t-x)^2;x) \leq S_n^{\alpha,\beta}(\rho(t,x)^2;x)^{\frac{1}{2}}\left(n^2\,S_n^{\alpha,\beta}((t-x)^4;x)\right)^{\frac{1}{2}},$$

and

$$\lim_{n\to\infty} n\, S_n^{\alpha,\beta}(\rho(t,x)(t-x)^2;x)$$

$$\leq \left(\lim_{n\to\infty} S_n^{\alpha,\beta}(\rho(t,x)^2;x)\right)^{\frac{1}{2}} \left(\lim_{n\to\infty} n^2\, S_n^{\alpha,\beta}((t-x)^4;x)\right)^{\frac{1}{2}}.$$

Inasmuch as $\lim_{n\to\infty} S_n^{\alpha,\beta}(\rho(t,x)^2;x) = 0$ and from (*ii*) of Lemma 3, $\lim_{n\to\infty} n^2\, S_n^{\alpha,\beta}((t-x)^4;x)$ is finite.

Therefore, we get

$$\lim_{n\to\infty} n\, S_n^{\alpha,\beta}(\rho(t,x)(t-x)^2;x)=0,$$

$$\lim_{n\to\infty} n\left(S_n^{\alpha,\beta}(f(t);x) - f(x)\right) = \left((-2-\beta_1)x + 1 + \alpha_2\right)f'(x) + (-x^2+x)\, f''(x).$$

Hence, the theorem is proved.

ACKNOWLEDGEMENTS

The authors would like to thank the referees for the careful reading of this paper and for their valuable suggestions to improve the paper.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Bernstein, S. N., “Demonstration du theoreme de Weierstrass fondee sur le calcul de probabilities”, Commun. Soc. Math. Kharkow, 13(2): 1-2, (1912).

[2] Durrmeyer, J. L., “Une Formule D’inversion de la transformée de Laplace: Aplications a la théorie des moments, Thése de 3e cycle”, Faculté des Sciences de I’Université de Paris, (1967).

[3] Gadhziev, A. D. and Chorbanalizadeh, A. M., “Approximation properties of a new type Bernstein-Stancu polynomials of one and two variables”, Appl. Math. Comput., 216: 890-901, (2010).

[4] Stancu, D. D., “Approximation of functions by a new class of linear polynomial operators”, Rev. Roum. Math. Pures Appl., 13: 1173-1194, (1968).

[5] Dong, L. X. and Yu, D. S., “Pointwise Approximation by a Durrmeyer Variant of Bernstein-Stancu Operators”, J. Inequal Appl., 2017, 28, (2017).

[6] Gadhziev, A. D., “Theorems of the type of P.P. Korovkin type theorems”, Math. Zametki, 20(5): 781-786, (1976). (English Translation, Math. Notes, 20(5/6): 996-998, (1976).

[7] Gupta, V. and Duman, O., “Bernstein–Durrmeyer type operators preserving linear function”, Matematicki Vesnik, 62(4): 259-264, (2010).

[8] Tasdelen, F., Bascanbaz-Tunca, G. and Erencin, A., “On a new type Bernstein-Stancu operators”, Fasc. Math., 48: 119-128, (2012).

[9] Acar, T., Aral, A. and Gupta, V., “On approximation properties of a new type Bernstein-Durrmeyer operators”, Math. Slovaca., 65: 1107-1122, (2015).

[10] Icoz, G., “A Kantorovich variant of a new type Bernstein Stancu polynomials”, Appl. Math. Comput., 218: 8552-8560, (2012).

[11] Wang, M. L., Yu, D. S. and Zhou, P., “On the approximation by operators of Bernstein-Stancu types”, Appl. Math. Comput., 246: 79-87, (2014).