Research Article GU J Sci 32(4): 1254-1269 (2019) DOI: 10.35378/gujs.433174

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Lower Separation Axioms in Čech Fuzzy Soft Closure Spaces

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Highlights

- In this work, some properties of Cech fuzzy soft closure spaces are introduced and studied.
- Also, several types of lower separation axioms in Cech fuzzy soft closure space are introduced.
- In addition, the relationship between these types of separation axioms are discussed.
- Moreover, several examples are given to support our study.

Article Info

Received: 12/06/2018
Accepted: 11/04/2019

Keywords

Fuzzy soft set
Fuzzy soft point
Čech fuzzy soft operator
Distinct fuzzy soft points
T_i-Čech closure spaces

Abstract

By considering Čech fuzzy soft closure spaces (X,θ,K), we provide a basic structure of a fuzzy soft topological space (X,τ_θ,K) associated with Čech fuzzy soft closure space (X,θ,K). Separation axioms, namely, T_i $(i=0,1,2)$, semi- (respectively, pseudo and Uryshon) T_2 are studied in both Čech fuzzy soft closure spaces and its associative fuzzy soft topological spaces. It is shown that hereditary property is satisfied for T_i, $i=0,1$ with respect to Čech fuzzy soft closure space but for other mentioned types of separations axioms, hereditary property satisfies for closed subspaces of Čech fuzzy soft closure space. Several examples are given to illustrate each type of the separation axioms and to study the relationship between them.

1. INTRODUCTION

The notion of fuzzy sets was firstly proposed by Zadeh [1] in 1965 as one of the effective mathematical tools to deal with uncertainties, where each element in a fuzzy set has a grade of membership. In 1999, Molodsov [2] initiated a new theory called soft set theory, by using classical sets, that also deals with uncertainties and ambiguity. A combination of fuzzy sets and soft sets, namely fuzzy soft sets, is then formulated by Maji et al. [3]. By using fuzzy soft sets, Tanay and Kandemir [4] defined fuzzy soft topology.

The concept of Čech closure spaces $(X,\mathcal{C})$ was introduced by Čech [5] in 1966, where $\mathcal{C}: P(X) \to P(X)$ is a mapping satisfying $\mathcal{C}(\emptyset)=\emptyset, A \subseteq \mathcal{C}(A)$ and $\mathcal{C}(A \cup B)=\mathcal{C}(A) \cup \mathcal{C}(B)$. The mapping $\mathcal{C}$ is referred to as Čech closure operator on X. In 1985, Mashhour and Ghanim [6] introduced the concept of Čech fuzzy closure space by replacing ordinary sets with fuzzy sets in the definition of Čech closure space. On the other hand, in 2014, Gowri and Jegadeesan [7] and Krishnaveni and Sekar [8], have introduced and studied soft Čech closure spaces. Note that the soft closure operator in the sense of Gowri and Jegadeesan is defined from the power set $P(X_{F_A})$ of X_{F_A} to itself (where F_A is a soft set over the universe set X with the set of parameters K, and $A \subseteq K$) while, Krishnaveni and Sekar, defined soft closure operator from the set of all soft sets over X to itself. Very recently, Majeed [9] generalized soft Čech closure space, in the sence of Krishnaveni and Sekar, into Čech fuzzy soft closure spaces. Also, Majeed and Maibed [10] have further studied some structures of Čech fuzzy soft closure spaces. Majeed and Maibed show that every Čech fuzzy soft closure space gives a parameterized family of Čech fuzzy closure spaces.
In the present paper, we continue studying the properties of Čech fuzzy soft closure spaces and their separation axioms. Some of the provided properties of Čech fuzzy soft closure spaces are essential for

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studying the separation axioms. In Section 3, basic structure of fuzzy soft topological space (X, τ_θ, K) associated with $\check{C}$ech fuzzy soft closure space (X, θ, K) is studied. The fuzzy soft topological closure τ_θ-cl (respectively, interior τ_θ- int) is defined and its relationship with the $\check{C}$ech fuzzy soft closure operator θ (respectively, interior operator Int) is given (see Theorem 4). In Section 4, separation axioms T_0 and T_1 in $\check{C}$ech fuzzy soft closure spaces are defined and their basic properties are discussed. Finally, in Section 5, T_2 $\check{C}$ech fuzzy soft closure space and some other types of separation axioms, namely, semi- (respectively, pseudo and Uryshon) T_2 are defined. Some properties of each type are discussed and the relationship between aforementioned and T_0 (respectively, T_1) are given. Finally, several examples are given to support our study.

2. PRELIMINARIES

In this section we review some basic definitions and results related to fuzzy soft theory and $\check{C}$ech fuzzy soft closure spaces that will be needed in the sequel, and we foresee the reader be familiar with the usual notions and most basic ideas of fuzzy set theory. Throughout our paper, X will refer to the initial universe, $I = [0,1]$, $I_0 = (0,1]$, I^X be the family of all fuzzy sets of X, and K the set of parameters for X.

Definition 1. [11, 12] A fuzzy soft set (fss, for short) λ_A on X is a mapping from K to I^X, i.e., $\lambda_A: K \to I^X$, where $\lambda_A(h) \neq \bar{0}$ if $h \in A \subseteq K$ and $\lambda_A(h) = \bar{0}$ if $h \notin A \subseteq K$, where $\bar{0}$ is the empty fuzzy set on X. The family of all fuzzy soft sets over X denoted by $\mathcal{F}_{ss}(X, K)$.

In the next definition, the basic operations between fuzzy soft sets are given.

Definition 2. [12] Let $\lambda_A, \mu_B \in \mathcal{F}_{ss}(X, K)$, then

1. λ_A is called a fuzzy soft subset of μ_B, denoted by $\lambda_A \subseteq \mu_B$, if $\lambda_A(h) \leq \mu_B(h)$, for all $h \in K$.
2. λ_A and μ_B are said to be equal, denoted by $\lambda_A = \mu_B$ if $\lambda_A \subseteq \mu_B$ and $\mu_B \subseteq \lambda_A$.
3. The union of λ_A and μ_B, denoted by $\lambda_A \cup \mu_B$ is the fss $\sigma_{(A \cup B)}(h)$ defined by $\sigma_{(A \cup B)}(h) = \lambda_A(h) \vee \mu_B(h)$, for all $h \in K$.
4. The intersection of λ_A and μ_B, denoted by $\lambda_A \cap \mu_B$ is the fss $\sigma_{(A \cap B)}$ defined by $\sigma_{(A \cap B)}(h) = \lambda_A(h) \wedge \mu_B(h)$, for all $h \in K$.

Definition 3. [12] The null fss, denoted by $\bar{0}_K$, is a fss defined by $\bar{0}_K(h) = \bar{0}$, for all $h \in K$, where $\bar{0}$ is the empty fuzzy set of X.

Definition 4. [12] The universal fss, denoted by $\bar{1}_K$, is a fss defined by $\bar{1}_K(h) = \bar{1}$, for all $h \in K$, where $\bar{1}$ is the universal fuzzy set of X.

Definition 5. [12] The complement of a fss $\lambda_A \in \mathcal{F}_{ss}(X, K)$, denoted $\bar{1}_K - \lambda_A$, is the fss defined by $(\bar{1}_K - \lambda_A)(h) = \bar{1} - \lambda_A(h)$, for each $h \in K$, Its clear that $\bar{1}_K - (\bar{1}_K - \lambda_A) = \lambda_A$.

Definition 6. [13] Two fss's $\lambda_A, \mu_B \in \mathcal{F}_{ss}(X, K)$ are said to be disjoint, denoted by $\lambda_A \cap \mu_B = \bar{0}_K$, if $\lambda_A(h) \cap \mu_B(h) = \bar{0}$ for all $h \in K$.

Definition 7. [14] A fuzzy soft set $\lambda_A \in \mathcal{F}_{SS}(X, K)$ is called fuzzy soft point, denoted by ${x_t}^h$, if there exist $x \in X$ and $h \in K$ such that $\lambda_A(h)(x) = t$ ($0 < t \leq 1$) and 0 otherwise for all $y \in X - \{x\}$.

Definition 8. [14] The fuzzy soft point ${x_t}^h$ is said to be belongs to the fss λ_A, denoted by ${x_t}^h \tilde{\in} \lambda_A$ if for the element $h \in K, t \leq \lambda_A(h)(x)$.

Definition 9. [15] Two fuzzy soft points x_t^h and $y_s^{h'}$ are said to be distinct if $x \neq y$ or $h \neq h'$.

Definition 10. [5,12] A fuzzy soft topological space(fsts, for short) (X, τ, K) where X is a nonempty set with a fixed set of parameters and τ is a family of fuzzy soft sets over X satisfying the following properties:

1. $\overline{0}_K, \overline{1}_K \in \tau$,
2. If $\lambda_A, \mu_B \in \tau$, then $\lambda_A \cap \mu_B \in \tau$,
3. If $(\lambda_A)_i \in \tau$, then $\bigcup_{i \in J \in} (\lambda_A)_i \in \tau$.

τ is called a topology of fuzzy soft sets on X. Every member of τ is called open fuzzy soft set (open-fss, for short). The complement of open-fss is called a closed fuzzy soft set (closed-fss, for short).

Definition 11. [9] An operator $\theta: \mathcal{F}_{ss}(X, K) \to \mathcal{F}_{ss}(X, K)$ is called $\check{C}$ech fuzzy soft closure operator ($\check{C}$-fsco, for short) on X, if the following axioms are satisfied:

(C1) $\theta(\overline{0}_K) = \overline{0}_K$,
(C2) $\lambda_A \subseteq \theta(\lambda_A)$, for all $\lambda_A \in \mathcal{F}_{ss}(X, K)$,
(C3) $\theta(\lambda_A \cup \mu_B) = \theta(\lambda_A) \cup \theta(\mu_A)$, for all $\lambda_A, \mu_B \in \mathcal{F}_{ss}(X, K)$.

The triple (X, θ, K) is called a Čech fuzzy soft closure space ($\check{C}\mathcal{F}$-fscs, for short).

A fss λ_A is said to be closed-fss in (X, θ, K) if $\lambda_A = \theta(\lambda_A)$. And a fss λ_A is said to be an open-fss if $\overline{1}_K - \lambda_A$ is a closed-fss.

Proposition 1. [9] Let (X, θ, K) be a $\check{C}\mathcal{F}$-scs, and λ_A, $\mu_B \in \mathcal{F}_{ss}(X, K)$ such that $\lambda_A \subseteq \mu_B$, then $\theta(\lambda_A) \subseteq \theta(\mu_B)$.

Definition 12. [9] Let (X, θ, K) be a $\check{C}\mathcal{F}$-scs, and let $\lambda_A \in \mathcal{F}_{ss}(X, K)$. The interior of λ_A, denoted by $Int(\lambda_A)$ is defined as $Int(\lambda_A) = \overline{1}_K - (\theta(\overline{1}_K - \lambda_A))$.

Proposition 2. [9] Let (X, θ, K) be a $\check{C}\mathcal{F}$-scs, and let λ_A, $\mu_B \in \mathcal{F}_{ss}(X, K)$. Then

1. $Int(\overline{0}_K) = \overline{0}_K$ *and* $Int(\overline{1}_K) = \overline{1}_K$,
2. $Int(\lambda_A) \subseteq \lambda_A$,
3. $Int(\lambda_A \cap \mu_B) = Int(\lambda_A) \cap Int(\mu_B)$,
4. *If* $\lambda_A \subseteq \mu_B$, *then* $Int(\lambda_A) \subseteq Int(\mu_B)$,
5. λ_A *is an open-fss* $\Leftrightarrow Int(\lambda_A) = \lambda_A$,
6. $Int(\lambda_A) \cup Int(\mu_B) \subseteq Int(\lambda_A \cup \mu_B)$.

Theorem 1. [9] Let (X, θ, K) be a $\check{C}\mathcal{F}$-scs and let $\tau_\theta \subseteq \mathcal{F}_{ss}(X, K)$, defined as follows

$$\tau_\theta = \{\overline{1}_K - \lambda_A : \theta(\lambda_A) = \lambda_A\}.$$

Then τ_θ is a fuzzy soft topology on X and (X, τ_θ, K) is called an associative fsts of (X, θ, K).

Definition 13. [9] Let V be a non-empty subset of X, then $\overline{V}_K$ denotes the fuzzy soft set V_K over X for which $V(h) = \overline{1}_V$ for all $h \in K$, (where $\overline{1}_V: X \to I$ such that $\overline{1}_V(x) = 1$ if $x \in V$ and $\overline{1}_V(x) = 0$ if $x \notin V$).

Theorem 2. [9] Let (X, θ, K) be a $\check{C}\mathcal{F}$-scs, $V \subseteq X$ and let $\theta_V: \mathcal{F}_{ss}(V, K) \to \mathcal{F}_{ss}(V, K)$ defined as $\theta_V(\lambda_A) = \overline{V}_K \cap \theta(\lambda_A)$. Then θ_V is a $\check{C}\mathcal{F}$-sco. The triple (V, θ_V, K) is said to be $\check{C}$ech fuzzy soft closure subspace ($\check{C}\mathcal{F}$-sc subspace, for short) of (X, θ, K).

Definition 14. [9] Let θ_1 and θ_2 be two $\check{C}$-fsco's on X. Then θ_1 is said to finer than θ_2, or equiventily θ_2 is coarser than θ_1, if for each $\lambda_A \in \mathcal{F}_{ss}(X, K)$, $\theta_2(\lambda_A) \subseteq \theta_1(\lambda_A)$.

3. SOME PROPERTIES OF ASSOCIATIVE FUZZY SOFT TOPOLOGICAL SPACES

In [10], Majeed and Maibed show that from each $\check{C}\mathcal{F}$-scs (X, θ, K) there exists an associated fsts (X, τ_θ, K) (see Theorem 1). In this section we study the associated fsts (X, τ_θ, K). Namely, we define the fuzzy soft topological closure τ_θ-cl (respectively, interior τ_θ- int) and study it is basic properties. In addition, we discuss the relation between the $\check{C}$ech fuzzy soft closure operator θ (respectively, interior operator Int) and the fuzzy soft topological closure τ_θ-cl (respectively, Interior τ_θ-int).

Definition 15. Let (X, τ_θ, K) be an associative fsts of (X, θ, K) and let $\lambda_A \in \mathcal{F}_{ss}(X, K)$. The fuzzy soft topological closure of λ_A with respect to θ, denoted by $\tau_\theta\text{-}cl(\lambda_A)$, is the intersection of all closed fuzzy soft super sets of λ_A . i.e.,

$$\tau_\theta\text{-}cl(\lambda_A) = \cap \{\rho_C : \lambda_A \subseteq \rho_C \ and \ \theta(\rho_C) = \rho_C \}. \quad (1)$$

From Theorem 1, it is clear that $\tau_\theta\text{-}cl(\lambda_A)$ is the smallest closed-fss over X which contains λ_A .

Proposition 3. Let (X, τ_θ, K) be an associative fsts of (X, θ, K) and let $\lambda_A, \mu_B \in \mathcal{F}_{ss}(X, K)$. Then

1. $\tau_\theta\text{-} cl\ (\bar{0}_K) = \bar{0}_K$ and $\tau_\theta\text{-}cl(\bar{1}_K) = \bar{1}_K$,
2. $\lambda_A \subseteq \tau_\theta\text{-}cl(\lambda_A)$,
3. If $\lambda_A \subseteq \mu_B$, then $\tau_\theta\text{-}cl(\lambda_A) \subseteq \tau_\theta\text{-}cl(\mu_B)$,
4. $\tau_\theta\text{-}cl(\lambda_A \cup \mu_B) = \tau_\theta\text{-}cl(\lambda_A) \cup \tau_\theta\text{-}cl(\mu_B)$,
5. $\tau_\theta\text{-}cl(\tau_\theta\text{-}cl(\lambda_A)) = \tau_\theta\text{-}cl(\lambda_A)$,
6. λ_A is a closed-fss if and only if $\lambda_{A=}\tau_\theta\text{-}cl(\lambda_A)$.

Proof. The proof of parts from 1 to 3 and 5 are follows directly from the Definition 15. To prove part 4, since $\lambda_A \subseteq \lambda_A \cup \mu_B$ and $\mu_B \subseteq \lambda_A \cup \mu_B$, then by 3, we have $\tau_\theta\text{-}cl(\lambda_A) \subseteq \tau_\theta\text{-}cl(\lambda_A \cup \mu_B)$ and $\tau_\theta\text{-}cl(\mu_B) \subseteq \tau_\theta\text{-}cl(\lambda_A \cup \mu_B)$.Thus,$\tau_\theta\text{-}cl(\lambda_A) \cup \tau_\theta\text{-}cl(\mu_B) \subseteq \tau_\theta\text{-}cl(\lambda_A \cup \mu_B)$. Conversely, by 2, $\lambda_A \cup \mu_B \subseteq \tau_\theta\text{-}cl(\lambda_A) \cup \tau_\theta\text{-}cl(\mu_B)$. Since $\tau_\theta\text{-}cl(\lambda_A)$ and $\tau_\theta\text{-}cl(\mu_B)$ are closed-fss. Then from Theorem 1, $\tau_\theta\text{-}cl(\lambda_A) \cup \tau_\theta\text{-}cl(\mu_B)$ is the smallest closed-fss contains $\lambda_A \cup \mu_B$. But $\tau_\theta\text{-}cl(\lambda_A \cup \mu_B)$ is the smallest closed-fss contains $\lambda_A \cup \mu_B$, this implies $\tau_\theta\text{-}cl(\lambda_A \cup \mu_B) \subseteq \tau_\theta\text{-}cl(\lambda_A) \cup \tau_\theta\text{-}cl(\mu_B)$. Hence, we obtain the equality. Finally, to prove 6, let λ_A be a closed-fss. By 2, we get $\lambda_A \subseteq \tau_\theta\text{-}cl(\lambda_A)$ and $\tau_\theta\text{-}cl(\lambda_A)$ is the smallest closed-fss which contains λ_A and since λ_A is a closed-fss, then $\tau_\theta\text{-}cl(\lambda_A) \subseteq \lambda_A$. Hence, $\lambda_A = \tau_\theta\text{-}cl(\lambda_A)$. Conversely, suppose that $\lambda_A = \tau_\theta\text{-}cl(\lambda_A)$. Since $\tau_\theta\text{-}cl(\lambda_A)$ is the closed-fss, then λ_A is a closed-fss. ■

Now, we introduce the definition of fuzzy soft interior of a fss in the associative fsts of (X, τ_θ, K) and give some properties of it.

Definition 16. Let (X, τ_θ, K) be an associative fsts of (X, θ, K) and let $\lambda_A \in \mathcal{F}_{ss}(X, K)$. The fuzzy soft topological interior of λ_A with respect to θ, denoted by $\tau_\theta\text{-}\ int(\lambda_A)$ is the union of all open fuzzy soft subset of λ_A. i.e.,

$$\tau_\theta\text{-}\ int(\lambda_A) = \cup\{\rho_C : \rho_C \subseteq \lambda_A \ and \ \theta(\bar{1}_K - \rho_C) = \bar{1}_K - \rho_C\}. \quad (2)$$

Clearly, $\tau_\theta\text{-}\ int(\lambda_A)$ is the largest open-fss contained in λ_A.

Proposition 4. Let (X, τ_θ, K) be an associative fuzzy soft topological space of (X, θ, K) and let $\lambda_A, \mu_B \in \mathcal{F}_{ss}(X, K)$. Then

1. $\tau_\theta\text{-}\ int(\bar{0}_K) = \bar{0}_K$ and $\tau_\theta\text{-}\ int(\bar{1}_K) = \bar{1}_K$,
2. $\tau_\theta\text{-}\ int(\lambda_A) \subseteq \lambda_A$,
3. If $\lambda_A \subseteq \mu_B$, then $\tau_\theta\text{-}\ int(\lambda_A) \subseteq \tau_\theta\text{-}\ int(\mu_B)$,
4. $\tau_\theta\text{-}int(\lambda_A \cap \mu_B) = \tau_\theta\text{-}int(\lambda_A) \cap \tau_\theta\text{-}int(\mu_B)$,
5. $\tau_\theta\text{-}int(\tau_\theta\text{-}int(\lambda_A)) = \tau_\theta\text{-}int(\lambda_A)$,
6. λ_A is an open-fss if and only if $\lambda_A = \tau_\theta\text{-}int(\lambda_A)$.

Proof. The prove is similar to the proof of Proposition 3. ■

Theorem 3. Let (X, τ_θ, K) be an associative fsts of (X, θ, K) and $\lambda_A \in \mathcal{F}_{ss}(X, K)$. Then

1. $\bar{1}_K - (\tau_\theta\text{-}int(\lambda_A)) = \tau_\theta\text{-}cl(\bar{1}_K - \lambda_A)$,
2. $\bar{1}_K - (\tau_\theta\text{-}cl(\lambda_A)) = \tau_\theta\text{-}int(\bar{1}_K - \lambda_A)$.

Proof. The proof is obtained by using Proposition 4 part 2, and Proposition 3 part 3. ■

In the next theorem, we discuss the relationship between the $\check{C}$ech fuzzy soft closure operator θ (respectively, interior operator Int) and the fuzzy soft topological closure $\tau_\theta\text{-}cl$ (respectively, Interior $\tau_\theta\text{-}int$) for any fss $\lambda_A \in \mathcal{F}_{ss}(X,K)$.

Theorem 4. Let (X,θ,K) be $\check{\mathrm{C}}\mathcal{F}$-scs and $(\mathrm{X},\tau_\theta,K)$ be an associative fuzzy soft topological space of (X,θ,K). Then for any fss $\lambda_A \in \mathcal{F}_{SS}(X,K)$

$$\tau_\theta\text{-}int(\lambda_A) \subseteq Int(\lambda_A) \subseteq \lambda_A \subseteq \theta(\lambda_A) \subseteq \tau_\theta\text{-}cl(\lambda_A). \tag{3}$$

Proof. First, we prove $\tau_\theta\text{-}int(\lambda_A) \subseteq Int(\lambda_A) \subseteq \lambda_A$. Since $\tau_\theta\text{-}int(\lambda_A) \subseteq \lambda_A$, then by 4 of Proposition 2, $Int(\tau_\theta\text{-}int(\lambda_A)) \subseteq Int(\lambda_A)$. But $\tau_\theta\text{-}int(\lambda_A)$ is an open-fss, this implies $\tau_\theta\text{-}int(\lambda_A) \subseteq Int(\lambda_A)$, and from the definition of $\check{C}$ech fuzzy soft interior operator, we get $\tau_\theta\text{-}int(\lambda_A) \subseteq Int(\lambda_A) \subseteq \lambda_A$.

Now, to prove $\lambda_A \subseteq \theta(\lambda_A) \subseteq \tau_\theta\text{-}cl(\lambda_A)$. Since $\lambda_A \subseteq \tau_\theta\text{-}cl(\lambda_A)$ and $\tau_\theta\text{-}cl(\lambda_A)$ is a closed-fss, then $\theta(\lambda_A) \subseteq \theta(\tau_\theta\text{-}cl(\lambda_A)) = \tau_\theta\text{-}cl(\lambda_A)$ and by (C2) of Definition 11, we have $\lambda_A \subseteq \theta(\lambda_A) \subseteq \tau_\theta\text{-}cl(\lambda_A)$. Hence, we obtain the required result. ■

Remark 1. In the above theorem the equality hold for the part $\tau_\theta\text{-}int(\lambda_A) \subseteq int(\lambda_A) \subseteq \lambda_A$ (respectively, $\lambda_A \subseteq \theta(\lambda_A) \subseteq \tau_\theta\text{-}cl(\lambda_A)$) if λ_A is an open- (respectively, closed-) fss in (X,θ,K). But in general, the equality of the above theorem is not true, so we give an example to explain that.

Example 1. Let $X = \{a,b,c\}$, $K = \{h_1,h_2\}$. Define $\mu_B \in \mathcal{F}_{ss}(X,K)$ such that$\mu_B = \{(h_1, b_{0.5}),(h_2,b_{0.5})\}$. Define $\check{C}$-fsco $\theta: \mathcal{F}_{ss}(X,K) \to \mathcal{F}_{ss}(X,K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \overline{0}_K & if \;\; \lambda_A = \overline{0}_{K,} \\ \{(h_1,a_{0.5} \vee b_{0.5}),(h_2,a_{0.5} \vee b_{0.5})\} & if \;\; \lambda_A \subseteq \mu_B, \\ \overline{1}_K & otherwise. \end{cases}$$

It is clear that θ satisfy the axioms $(C1)$, $(C2)$ and $(C3)$ of Definition 11, in addition $\theta(\mu_B) \neq \theta(\theta(\mu_B))$. Thus, (X,θ,K) is a $\check{\mathrm{C}}\mathcal{F}$-scs. The associative fuzzy soft topology τ_θ of (X,θ,K) is $\{\overline{0}_K,\overline{1}_K\}$. Now, consider the fss $\lambda_A = \{(h_1,a_1 \vee b_{0.7} \vee c_1),(h_2,a_1 \vee b_{0.9} \vee c_1)\}$. Then, $Int(\lambda_A) = \{(h_1,a_{0.5} \vee b_{0.5} \vee c_1),(h_2,a_{0.5} \vee b_{0.5} \vee c_1)\}$ and $\tau_\theta\text{-}int(\lambda_A) = \overline{0}_K$. Hence, $Int(\lambda_A) \nsubseteq \tau_\theta\text{-}int(\lambda_A)$. On the other hand consider $\lambda_A = \{(h_1,b_{0.4}),(h_2,b_{0.2})\}$. Then $\tau_\theta\text{-}cl(\lambda_A)) = \overline{1}_K \nsubseteq \{(h_1,a_{0.5} \vee b_{0.5}),(h_2,a_{0.5} \vee b_{0.5})\} = \theta(\lambda_A)$. Thus, $\tau_\theta\text{-}cl(\lambda_A)) \nsubseteq \theta(\lambda_A)$.

4. $\mathrm{T_i}$-ČECH FUZZY SOFT CLOSURE SPACES, $i = 0,1$

This section is devoted to defining separation axioms $\mathrm{T_0}$ and $\mathrm{T_1}$ in $\check{\mathrm{C}}\mathcal{F}$-scs's and its associated fsts's. We discuss the relation between $\mathrm{T_0}$ and $\mathrm{T_1}$, and study the hereditary property on $\check{\mathrm{C}}\mathcal{F}$-scs's. Also, we give the relation between $\check{\mathrm{C}}\mathcal{F}$-scs (X,θ,K) and its associated fsts (X,τ_θ,K) when (X,τ_θ,K) is $\mathrm{T_i}$, $i = 0,1$.

Definition 17. A $\check{\mathrm{C}}\mathcal{F}$-scs (X,θ,K) is said to be $\mathrm{T_0}$-$\check{\mathrm{C}}\mathcal{F}$-scs, if for every pair of distinct fuzzy soft points x_t^h and $y_s^{h'}$, either $x_t^h \,\widetilde{\notin}\, \theta(y_s^{h'})$ or $y_s^{h'} \,\widetilde{\notin}\, \theta(x_t^h)$.

Now we give some examples to illustrate Definition 17.

Example 2. Let (X,θ,K) be the discrete $\check{\mathrm{C}}\mathcal{F}$-scs (i.e., $\theta(\lambda_A) = \lambda_A$ for all $\lambda_A \in \mathcal{F}_{\mathrm{ss}}(\mathrm{X},K)$), then (X,θ,K) is a $\mathrm{T_0}$-$\check{\mathrm{C}}\mathcal{F}$-scs.

Example 3. Let (X,θ,K) be the trivial $\check{\mathrm{C}}\mathcal{F}$-scs (i.e., $\theta(\lambda_A) = \overline{1}_K$ for all $\lambda_A \in \mathcal{F}_{\mathrm{ss}}(\mathrm{X},K)$), then (X,θ,K) is not $\mathrm{T_0}$-$\check{\mathrm{C}}\mathcal{F}$-scs, because for any distinct fuzzy soft points x_t^h and $y_s^{h'}$, we have $x_t^h \,\widetilde{\in}\, \overline{1}_K = \theta(y_s^{h'})$ and $y_s^{h'} \,\widetilde{\in}\, \overline{1}_K = \theta(x_t^h)$.

Example 4. Let X={a, b}, K={h_1, h_2} and let $\lambda_A^* \subseteq \mathcal{F}_{ss}(\mathrm{X}, K)$ such that λ_A^* ={(h_1,$a_{t_1} \vee b_{s_1}$),($h_2, a_{t_2} \vee b_{s_2}$); $t_1, t_2, s_1, s_2 \in I_0$}. Define $\theta: \mathcal{F}_{ss}(\mathrm{X}, K) \to \mathcal{F}_{ss}(\mathrm{X}, K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if \;\; \lambda_A = \bar{0}_K\,, \\ a_1^{h_1} & if \;\; \lambda_A \in \{a_{t_1}^{h_1};\;\; t_1 \in I_0\}, \\ a_1^{h_2} & if \;\; \lambda_A \in \{a_{t_2}^{h_2};\;\; t_2 \in I_0\}, \\ b_{s_1+0.2}^{h_1} & if \;\; \lambda_A \in \{b_{s_1}^{h_1};\; 0 < s_1 \; < \; 0.8\}, \\ b_1^{h_1} & if \;\; \lambda_A \in \{b_{s_1}^{h_1};\; 0.8 \le \; s_1 \le 1\}, \\ \left\{\left(h_1, \theta\left(a_{t_1}^{h_1}\right) \cup \theta\left(b_{s_1}^{h_1}\right)\right), \left(h_2, \theta\left(a_{t_2}^{h_2}\right) \cup \theta\left(b_{s_2}^{h_2}\right)\right)\right\} & if \;\; \lambda_A \in \; \lambda_A^*\,. \end{cases}$$

Then (X, θ, K) is a Č$\mathcal{F}$-scs. To show (X, θ, K) is T_0-Č$\mathcal{F}$-scs, we have three cases for distinct fuzzy soft points in X.

<u>Case(1).</u> If $a \neq b$ and $h_1 = \; h_2$, then we have $a_{t_1}^{h_1}$ and $b_{s_1}^{h_1}$ are distinct fuzzy soft points. It is clear that $b_{s_1}^{h_1}$ $\tilde{\notin}\; \theta\left(a_{t_1}^{h_1}\right)$ because $s_1 > (a_1^{h_1})(h_1)(b)$=$(a_1)(b) = 0$. Similarly, $a_{t_1}^{h_2}$ and $b_{s_1}^{h_2}$ are distinct fuzzy soft points and $b_{s_1}^{h_2} \;\tilde{\notin}\; \theta\left(a_{t_1}^{h_2}\right)$.

<u>Case(2).</u> If $a = b$ and $h_1 \neq \; h_2$, then $a_{t_1}^{h_1}$ and $a_{t_2}^{h_2}$ are distinct fuzzy soft points. It is clear that $a_{t_1}^{h_1} \;\tilde{\notin}\; \theta\left(a_{t_2}^{h_2}\right)$ because $t_1 > \theta(a_1^{h_2})(h_1)(a)$=$\bar{0}(a) = 0$. Similarly, $b_{s_1}^{h_1}$ and $b_{s_2}^{h_2}$ are distinct fuzzy soft points and $b_{s_2}^{h_2}$ $\tilde{\notin}\; \theta\left(b_{s_1}^{h_1}\right)$.

<u>Case(3).</u> If $a \neq b$ and $h_1 \neq \; h_2$, then we have $a_t^{h_1}$ and $b_s^{h_2}$ are distinct fuzzy soft points such that $b_s^{h_2}$ $\tilde{\notin}\; \theta\left(a_t^{h_1}\right)$. Similarly, $a_t^{h_2}$ and $b_s^{h_1}$ are distinct fuzzy soft points and $b_s^{h_1} \;\tilde{\notin}\; \theta\left(a_t^{h_2}\right)$. Hence, (X, θ, K) is T_0-Č$\mathcal{F}$-scs.

Theorem 5. Let (X, θ, K) be a T_0- Č$\mathcal{F}$-scs, then for any two distinct fuzzy soft points x_t^h and $y_s^{h'}$, $\theta\left(x_t^h\right) \neq \theta\left(y_s^{h'}\right)$.

Proof. Let (X, θ, K) be a T_0-Č$\mathcal{F}$-scs, and let x_t^h and $y_s^{h'}$ be any two distinct fuzzy soft points. Suppose that $\theta\left(x_t^h\right) = \theta\left(y_s^{h'}\right)$. Since $x_t^h \;\tilde{\in}\; \theta(x_t^h)$ and $y_s^{h'} \;\tilde{\in}\; \theta\left(y_s^{h'}\right)$. Then from hypothesis, $x_t^h \;\tilde{\in}\; \theta\left(y_s^{h'}\right)$ and $y_s^{h'} \;\tilde{\in}\; \theta(x_t^h)$. This implies (X, θ, K) is not T_0- Č$\mathcal{F}$-scs, which is a contradiction. Hence, $\theta\left(x_t^h\right) \neq \theta\left(y_s^{h'}\right)$.■

The converse of above theorem is not true, as the following example show.

Example 5. Let X={a, b}, $K = \{h\}$. Define $\theta: \mathcal{F}_{ss}(\mathrm{X}, K) \to \mathcal{F}_{ss}(\mathrm{X}, K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if \;\; \lambda_A = \bar{0}_K, \\ \{(h, a_{0.2} \vee b_{0.1})\} & if \;\; \lambda_A \subseteq a_{0.2}^h, \\ \{(h, a_{0.1} \vee b_{0.2})\} & if \;\; \lambda_A \subseteq b_{0.2}^h, \\ \lambda_A & if \;\; \lambda_A \in \{a_t^h;\; 0.2 < t \; \le 1\}, \\ \lambda_A & if \;\; \lambda_A \in \{b_s^h; 0.2 < s \; \le 1\}, \\ \theta\left(a_t^h\right) \cup \theta\left(b_s^h\right) & if \;\; \lambda_A \in \{(h, a_t \vee b_s)\;;\; t, s \in I_0\}. \end{cases}$$

Then (X, θ, K) is Č$\mathcal{F}$-scs. It is clear that from the definition of θ, $\theta\left(a_t^h\right) \neq \theta\left(b_s^h\right)$ for any two-distinct fuzzy soft points a_t^h and b_s^h. However, (X, θ, K) is not T_0-Č$\mathcal{F}$-scs, since there exist two distinct fuzzy soft point $a_{0.1}^h$ and $b_{0.1}^h$ such that $a_{0.1}^h \;\tilde{\in}\; \theta\left(b_{0.1}^h\right)$ and $b_{0.1}^h \;\tilde{\in}\; \theta\left(a_{0.1}^h\right)$.

Next, we show that T_0 is hereditary property on Č$\mathcal{F}$-scs's.

Theorem 6. A Č$\mathcal{F}$-sc subspace of a T_0-Č$\mathcal{F}$-scs, is a T_0-Č$\mathcal{F}$-sc subspace.

Proof. Let (X,θ,K) be a T_0-$\check{C}\mathcal{F}$-scs and (V,θ_V,K) be a $\check{C}\mathcal{F}$-sc subspace of (X,θ,K). Let x_t^h and $y_s^{h'}$ be any two-distinct fuzzy soft points in (V,θ_V,K). Since $\mathcal{F}_{ss}(V,K)\subseteq\mathcal{F}_{ss}(X,K)$, then x_t^h and $y_s^{h'}$ are distinct fuzzy soft points in $\mathcal{F}_{ss}(X,K)$. Since (X,θ,K) is T_0-$\check{C}$-fscs. Then $x_t^h\,\tilde{\notin}\,\theta(y_s^{h'})$ or $y_s^{h'}\,\tilde{\notin}\,\theta(x_t^h)$. This implies either $x_t^h\,\tilde{\notin}\,\theta(y_s^{h'})\cap\bar{V}_K$ or $y_s^{h'}\,\tilde{\notin}\,\theta(x_t^h)\cap\bar{V}_K$. Then $x_t^h\,\tilde{\notin}\,\theta_V(y_s^{h'})$ or $y_s^{h'}\,\tilde{\notin}\,\theta_V(x_t^h)$. Hence (V,θ_V,K) is T_0-$\check{C}\mathcal{F}$-sc subspace. ■

Definition 18. An associative fsts (X,τ_θ,K) of $\check{C}\mathcal{F}$-fscs (X,θ,K) is said to be T_0-fsts, if for every two distinct fuzzy soft points x_t^h and $y_s^{h'}$, either $x_t^h\,\tilde{\notin}\,\tau_\theta\text{-}cl(y_s^{h'})$ or $y_s^{h'}\,\tilde{\notin}\,\tau_\theta\text{-}cl(x_t^h)$.

The next theorem give the relationship between the associative fsts (X,τ_θ,K) which is T_0-fsts and $\check{C}\mathcal{F}$-fscs (X,θ,K).

Theorem 7. If (X,τ_θ,K) is a T_0-fsts, then (X,θ,K) is also T_0-$\check{C}\mathcal{F}$-scs.

Proof. Let x_t^h and $y_s^{h'}$ be any two distinct fuzzy soft point in (X,θ,K). Since (X,τ_θ,K) is a T_0-fsts, then $x_t^h\,\tilde{\notin}\,\tau_\theta\text{-}cl(y_s^{h'})$ or $y_s^{h'}\,\tilde{\notin}\,\tau_\theta\text{-}cl(x_t^h)$. By Theorem 4 we get, $x_t^h\,\tilde{\notin}\,\theta(y_s^{h'})$ or $y_s^{h'}\,\tilde{\notin}\,\theta(x_t^h)$. This implies (X,θ,K) is T_0-$\check{C}\mathcal{F}$-scs.■

The converse Theorem 7 is not true, as we shown in the following example.

Example 6. Let $X=\{a,b\}$, $K=\{h_1,h_2\}$. Define $\theta:\mathcal{F}_{ss}(X,K)\to\mathcal{F}_{ss}(X,K)$ as follows:

$$\theta(\lambda_A)=\begin{cases}\bar{0}_K & if \quad \lambda_A=\bar{0}_K,\\ \{(h_1,a_1),(h_2,b_1)\} & if \quad \lambda_A\subseteq a_1^{h_1},\\ \{(h_1,b_1),(h_2,a_1)\} & if \quad \lambda_A\subseteq b_1^{h_1},\\ \{(h_1,a_1),(h_2,a_1)\} & if \quad \lambda_A\subseteq a_1^{h_2},\\ \{(h_1,b_1),(h_2,b_1)\} & if \quad \lambda_A\subseteq b_1^{h_2},\\ \{(h_1,a_1),(h_2,a_1\vee b_1)\} & if \quad \lambda_A\subseteq\{(h_1,a_1),(h_2,a_1)\};\ \lambda_A\notin\{a_t^{h_i}, i=1,2, t\in I_0\},\\ \{(h_1,a_1\vee b_1),(h_2,b_1)\} & if \quad \lambda_A\subseteq\{(h_1,a_1),(h_2,b_1)\};\ \lambda_A\notin\{a_t^{h_1},b_s^{h_2};\ t,s\in I_0\},\\ \{(h_1,a_1\vee b_1),(h_2,a_1)\} & if \quad \lambda_A\subseteq\{(h_1,b_1),(h_2,a_1)\};\ \lambda_A\notin\{a_t^{h_2},b_s^{h_1};\ t,s\in I_0\},\\ \{(h_1,b_1),(h_2,a_1\vee b_1)\} & if \quad \lambda_A\subseteq\{(h_1,b_1),(h_2,b_1)\};\ \lambda_A\notin\{b_s^{h_i}; i=1,2, s\in I_0\},\\ \bar{1}_K & otherwise.\end{cases}$$

Then (X,θ,K) is T_0-$\check{C}\mathcal{F}$-scs. But (X,τ_θ,K) is not T_0-fsts, because $\tau_\theta=\{\bar{1}_K-\lambda_A:\theta(\lambda_A)=\lambda_A\}=\{\bar{0}_K,\bar{1}_K\}$.

Definition 19. A $\check{C}$-fscs (X,θ,K) is said to be T_1-$\check{C}\mathcal{F}$-scs, if for every two distinct fuzzy soft points x_t^h and $y_s^{h'}$ we have $x_t^h\,\tilde{\notin}\,\theta(y_s^{h'})$ and $y_s^{h'}\,\tilde{\notin}\,\theta(x_t^h)$.

Example 7. Let $X=\{a,b\}$, $K=\{h_1,h_2\}$, and let $\lambda_A^*\subseteq\mathcal{F}_{ss}(X,K)$ such that $\lambda_A^*=\{(h_1,a_{t_1}\vee b_{s_1}),(h_2,a_{t_2}\vee b_{s_2});\ t_1,t_2,s_1,s_2\in I_0\}$. Define $\theta:\mathcal{F}_{ss}(X,K)\to\mathcal{F}_{ss}(X,K)$ as follows:

$$\theta(\lambda_A)=\begin{cases}\bar{0}_K & if\ \lambda_A=\bar{0}_K,\\ a_1^{h_1} & if\ \lambda_A\subseteq a_1^{h_1},\\ a_1^{h_2} & if\ \lambda_A\subseteq a_1^{h_2},\\ b_1^{h_1} & if\ \lambda_A\subseteq b_1^{h_1},\\ b_{0.9}^{h_2} & if\ \lambda_A\in\{b_s^{h_2}, 0<s<0.9\},\\ b_1^{h_2} & if\ \lambda_A\in\{b_s^{h_2}, 0.9\le s\le 1\},\\ \left\{\left(h_1,\theta(a_{t_1}^{h_1})\cup\theta(b_{s_1}^{h_1})\right),\left(h_2,\theta(a_{t_2}^{h_2})\cup\theta(b_{s_2}^{h_2})\right)\right\} & if\ \lambda_A\in\lambda_A^*.\end{cases}$$

Then (X,θ,K) is T_1-$\check{C}\mathcal{F}$-scs. Since for any two distinct fuzzy soft points a_t^h and $b_s^{h'}$ we have $a_t^h \tilde{\notin} \theta(b_s^{h'})$ and $b_s^{h'} \tilde{\notin} \theta(a_t^h)$.

Proposition 5. Every T_1-$\check{C}\mathcal{F}$-scs is T_0-$\check{C}\mathcal{F}$-scs.

Proof. Follows directly from the definition of T_1- $\check{C}\mathcal{F}$-scs. ∎

The converse of Proposition 5 is not true, as seen in the following example.

Example 8. Let X={a,b}, K={h}. Define $\theta: \mathcal{F}_{ss}(X,K) \to \mathcal{F}_{ss}(X,K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ a_{t+0.1}^h & if\ \lambda_A \in \{a_t^h, 0 < t < 0.9\}, \\ a_1^h & if\ \lambda_A \in \{a_t^h, 0.9 \le t \le 1\}, \\ \bar{1}_K & otherwise. \end{cases}$$

Then (X,θ,K) is T_0-$\check{C}\mathcal{F}$-scs. Since for any two distinct fuzzy soft points a_t^h and b_s^h, t, s ∈(0,1]. We have, $b_s^h \tilde{\notin} \theta(a_t^h)$. However, it is not T_1- $\check{C}\mathcal{F}$-scs because $a_t^h \tilde{\in} \theta(b_s^h) = \bar{1}_K$.

Theorem 8. If every fuzzy soft point in a $\check{C}\mathcal{F}$-scs (X,θ,K) is closed-fss, then (X,θ,K) is T_1- $\check{C}\mathcal{F}$-scs.

Proof: Let x_t^h and $y_s^{h'}$ be any two distinct fuzzy soft points in (X,θ,K). From hypothesis, we have $\theta(x_t^h)=x_t^h$ and $\theta(y_s^{h'}) = y_s^{h'}$. This implies $x_t^h \tilde{\notin} \theta(y_s^{h'})$ and $y_s^{h'} \tilde{\notin} \theta(x_t^h)$. Thus, (X,θ,K) is a T_1- $\check{C}\mathcal{F}$-scs. ∎

The converse of above theorem is not true in general as we seen in the following example.

Example 9. In Example 7, (X,θ,K) is T_1- $\check{C}\mathcal{F}$-scs, yet there exists fuzzy soft point $b_{0.6}^{h_2}$ such that $\theta(b_{0.6}^{h_2}) = b_{0.9}^{h_2}$.

Theorem 9. A $\check{C}$-fsc subspace of T_1-$\check{C}\mathcal{F}$-scs is T_1- $\check{C}\mathcal{F}$-sc subspace.

Proof. Similar to the proof of Theorem 6. ∎

Definition 20. An associative fsts (X,τ_θ,K) of $\check{C}\mathcal{F}$-scs, (X,θ,K) is said to be T_1-fsts, if for every two distinct fuzzy soft points x_t^h and $y_s^{h'}$, we have $x_t^h \tilde{\notin} \tau_\theta\text{-}cl(y_s^{h'})$ and $y_s^{h'} \tilde{\notin} \tau_\theta\text{-}cl(x_t^h)$.

Theorem 10. If (X,τ_θ,K) is a T_1-fsts , then (X,θ,K) is also T_1- $\check{C}\mathcal{F}$-scs.

Proof. Similar to the proof of Theorem 7. ∎

Proposition 6. If (X,θ_1,K) is T_i-$\check{C}\mathcal{F}$-scs and θ_2 is a $\check{C}$-fsco on X such that θ_2 is coarser than θ_1, then (X,θ_2,K) is T_i- $\check{C}\mathcal{F}$-scs, $i = 0,1$.

Proof. We prove the proposition when $i = 1$, and the proof is similar for $i = 0$. Let x_t^h and $y_s^{h'}$ be any two distinct fuzzy soft points in (X,θ_2,K). From hypothesis (X,θ_1,K) is T_1- $\check{C}\mathcal{F}$-scs, this yield $x_t^h \tilde{\notin} \theta_1(y_s^{h'})$ and $y_s^{h'} \tilde{\notin} \theta_1(x_t^h)$. Since θ_2 is coarser than θ_1, that means $\theta_2(\lambda_A) \subseteq \theta_1(\lambda_A)$ for all $\lambda_A \in \mathcal{F}_{ss}(X,K)$. This implies, $x_t^h \tilde{\notin} \theta_2(y_s^{h'})$ and $y_s^{h'} \tilde{\notin} \theta_2(x_t^h)$. Hence, (X,θ_2,K) is T_1- $\check{C}\mathcal{F}$-scs. ∎

5. T_2-ČECH FUZZY SOFT CLOSURE SPACES

In this section we define T_2- Č$\mathcal{F}$-scs and other types, namely, semi- (respectively, pseudo and Uryshon) T_2- Č$\mathcal{F}$-scs, the properties of each type are discussed as in Section 4. In addition, the relationships between separation axioms that introduced in the current section and in the previous section are obtained.

Definition 21. A Č$\mathcal{F}$-scs, (X,θ,K) is said to be T_2- Č$\mathcal{F}$-scs, if for every two distinct fuzzy soft points x_t^h and $y_s^{h'}$, there exist disjoint open-fss's λ_A and μ_B such that $x_t^h \tilde{\in} \lambda_A$ and $y_s^{h'} \tilde{\in} \mu_B$.

Example 10. Let X={a,b,c}, K={h}. Define θ: $\mathcal{F}_{ss}(X,K) \to \mathcal{F}_{ss}(X,K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ \{(h, a_{t+0.1} \vee b_{t+0.1})\} & if\ \lambda_A \in \{a_t^h;\ 0 < t < 0.9\}, \\ \{(h, a_1 \vee c_1)\} & if\ \lambda_A \in \{a_t^h;\ 0.9 \le t \le 1\}, \\ \{(h, b_{s+0.1})\} & if\ \lambda_A \in \{b_s^h;\ 0 < s < 0.9\}, \\ \{(h, b_1)\} & if\ \lambda_A \in \{b_s^h;\ 0.9 \le s \le 1\}, \\ \{(h, c_{r+0.1})\} & if\ \lambda_A \in \{c_r^h;\ 0 < r < 0.9\}, \\ \{(h, c_1)\} & if\ \lambda_A \in \{c_r^h; 0.9 \le r \le 1\}, \\ \theta(a_t^h) \cup \theta(b_s^h) & if\ \lambda_A \in \{(h, a_t \vee b_s); t, s \in I_0\}, \\ \theta(a_t^h) \cup \theta(c_r^h) & if\ \lambda_A \in \{(h, a_t \vee c_r); t, r \in I_0\}, \\ \theta(b_s^h) \cup \theta(c_r^h) & if\ \lambda_A \in \{(h, b_s \vee c_r); s, r \in I_0\}, \\ \theta(a_t^h) \cup \theta(b_s^h) \cup \theta(c_r^h) & if\ \lambda_A \in \{(h, a_t \vee b_s \vee c_r); t, s, r \in I_0\}. \end{cases}$$

Then (X,θ,K) is Č$\mathcal{F}$-scs. and (X,θ,K) is T_2- Č$\mathcal{F}$-scs. To explain that we have three cases for distinct fuzzy soft points as follows:

Case (1). a_t^h, b_s^h are distinct fuzzy soft points, it follows there exist disjoint open-fss's λ_A={(h,a_1)}, μ_B={(h,b_1)} such that $a_t^h \tilde{\in} \lambda_A$ and $b_s^h \tilde{\in} \mu_B$.

Case (2). a_t^h, c_s^h are distinct fuzzy soft points, it follows there exist disjoint open-fss's λ_A={(h,a_1)}, μ_B={(h,c_1)} such that $a_t^h \tilde{\in} \lambda_A$ and $c_s^h \tilde{\in} \mu_B$.

Case (3). b_s^h, c_s^h are distinct fuzzy soft points, it follows there exist disjoint open-fss's λ_A={(h,b_1)}, μ_B={(h,c_1)} such that $a_t^h \tilde{\in} \lambda_A$ and $c_s^h \tilde{\in} \mu_B$. Therefore, (X,θ,K) is T_2- Č$\mathcal{F}$-scs.

Remark 2. If (X,θ,K) is T_2-Č$\mathcal{F}$-scs, then (X,θ,K) need not to be T_1- Č$\mathcal{F}$-scs. To see that, in Example 10, (X,θ,K) is T_2- Č$\mathcal{F}$-scs but it is not (X,θ,K) is T_1- Č$\mathcal{F}$-scs. Since there exist $a_{0.5}^h$ and $b_{0.5}^h$ are distinct fuzzy soft points, and $b_{0.5}^h \tilde{\in} \theta(a_{0.5}^h)$.

In order to study the hereditary property in Č$\mathcal{F}$-scs's, we need first to give the following lemmas.

Lemma 1. Let (X,θ,K) be a Č$\mathcal{F}$-scs and (V,θ_V,K) be a Č$\mathcal{F}$-sc subspace of (X,θ,K). Then for any $\lambda_A \in \mathcal{F}_{ss}(X,K)$, we have

$$(\bar{1}_K - \lambda_A) \cap \bar{V}_K = \bar{V}_K - (\lambda_A \cap \bar{V}_K). \tag{4}$$

Proof. Let $\lambda_A \in \mathcal{F}_{ss}(X,K)$. From the definition of $\bar{V}_K$, it is clear that for any $h \in K$ and $x \in V$, $\lambda_A \cap \bar{V}_K = \lambda_A$. Now, for any $h \in K$ and $x \in V$

$[\bar{V}_K - (\lambda_A \cap \bar{V}_K)](h) = (\bar{V}_K - \lambda_A)(h) \in I^X$. Now

$(\bar{V}_K - \lambda_A)(h)(x) = (\bar{1}_V - \lambda_A(h))(x)$

$= ((\bar{1}_X - \lambda_A(h)) \cap \bar{1}_V)(x)$

$= (\bar{1}_V - \lambda_A) \cap \bar{V}_K$. ■

Lemma 2. Let (X,θ,K) be a $\check{C}\mathcal{F}$-scs and let (V,θ_V,K) be a closed $\check{C}\mathcal{F}$-sc subspace of (X,θ,K). If λ_A is an open-fss of (X, θ,K) then $\lambda_A \cap \bar{V}_K$ is also open-fss in (V,θ_V,K).

Proof. Let λ_A be an open-fss in (X, θ,K). Then $\bar{1}_K - \lambda_A$ is a closed-fss in (X, θ,K). Since $\bar{V}_K$ is a closed fuzzy soft set in (X, θ,K), then $(\bar{1}_K - \lambda_A) \cap \bar{V}_K$ is a closed-fss in X. That means, $\theta((\bar{1}_K - \lambda_A) \cap \bar{V}_K)$=$(\bar{1}_K - \lambda_A) \cap \bar{V}_K$. From Lemma 1, we have $(\bar{1}_K - \lambda_A) \cap \bar{V}_K$= $\bar{V}_K - (\lambda_A \cap \bar{V}_K)$. To complete the prove we must show that $\theta_V(\bar{V}_K - (\lambda_A \cap \bar{V}_K))$= $\bar{V}_K - (\lambda_A \cap \bar{V}_K)$. Now,

$\theta_V\big(\bar{V}_K - (\lambda_A \cap \bar{V}_K)\big) = \bar{V}_K \cap \theta(\bar{V}_K - (\lambda_A \cap \bar{V}_K))$ (By Theorem 2)

$= \bar{V}_K \cap \theta\big((\bar{1}_K - \lambda_A) \cap \bar{V}_K\big)$ (By Lemma 1)

$= \bar{V}_K \cap ((\bar{1}_K - \lambda_A) \cap \bar{V}_K)$ (since $\theta\big((\bar{1}_K - \lambda_A) \cap \bar{V}_K\big) = (\bar{1}_K - \lambda_A) \cap \bar{V}_K$)

$= \bar{V}_K \cap (\bar{V}_K - (\lambda_A \cap \bar{V}_K))$ (By Lemma 1)

$= \bar{V}_K - (\lambda_A \cap \bar{V}_K)$.

Thus, $\lambda_A \cap \bar{V}_K$ is an open-fss in (V,θ_V,K) . ■

Theorem 11. Let (X,θ,K) be a T_2- $\check{C}\mathcal{F}$-scs and let (V,θ_V,K) be a closed $\check{C}\mathcal{F}$-sc subspace of (X,θ,K). Then (V,θ_V,K) is a T_2-$\check{C}\mathcal{F}$-sc subspace of (X,θ,K).

Proof. Let x_t^h and $y_s^{h'}$ be any two distinct fuzzy soft points in (V,θ_V,K) . Then x_t^h and $y_s^{h'}$ are distinct fuzzy soft point in (X,θ,K). Since (X,θ,K) is a T_2-$\check{C}\mathcal{F}$-scs, there exist two disjoint open-fss's λ_A and μ_B such that $x_t^h \tilde{\in} \lambda_A$ and $y_s^{h'} \tilde{\in} \mu_B$. Consequently, $x_t^h \tilde{\in} \lambda_A \cap \bar{V}_K$, $y_s^{h'} \tilde{\in} \mu_B \cap \bar{V}_K$ and $(\lambda_A \cap \bar{V}_K) \cap (\mu_B \cap \bar{V}_K) = \bar{0}_K$. By Lemma 2, $\lambda_A \cap \bar{V}_K$ and $\mu_B \cap \bar{V}_K$ are open-fss's in (V,θ_V,K). Hence (V,θ_V,K) is a T_2-$\check{C}\mathcal{F}$-sc subspace of (X,θ,K). ■

Definition 22. An associative fsts (X,τ_θ,K) of (X,θ,K) is said to be T_2-fsts, if for every two distinct fuzzy soft points x_t^h and $y_s^{h'}$, there exist an open-fss's λ_A and μ_B in (X,τ_θ,K) such that $x_t^h \tilde{\in} \lambda_A$, $y_s^{h'} \tilde{\in} \mu_B$ and $\lambda_A \cap \mu_B$=$\bar{0}_K$.

Theorem 12. An associative fsts (X,τ_θ,K) is T_2-fsts of (X,θ,K) if and only if (X,θ,K) is T_2-$\check{C}\mathcal{F}$-scs.

Proof. Suppose (X,τ_θ,K) is T_2-fsts and let x_t^h and $y_s^{h'}$ be two distinct fuzzy soft points in X. Since (X,τ_θ,K) is T_2-fsts, there exist λ_A and μ_B open-fss's in (X,τ_θ,K) such that $x_t^h \tilde{\in} \lambda_A$, $y_s^{h'} \tilde{\in} \mu_B$ and $\lambda_A \cap \mu_B$=$\bar{0}_K$. Since λ_A and μ_B are open-fss's in (X,τ_θ,K), then τ_θ-*int*(λ_A) =λ_A and τ_θ-*int*(μ_B)= μ_B. From Theorem 4, we get Int(λ_A)= λ_A and Int(μ_B)= μ_B. Thus, λ_A and μ_B are open-fss's in (X,θ,K) such that $x_t^h \tilde{\in} \lambda_A$ and $y_s^{h'} \tilde{\in} \mu_B$ and $\lambda_A \cap \mu_B$=$\bar{0}_K$. Hence, (X,θ,K) is T_2-$\check{C}\mathcal{F}$-scs.

Conversely, similar to first direction. ■

Lemma 3. Let (X,θ_1,K), (X,θ_2,K) be $\check{C}\mathcal{F}$-scs's. For any $\lambda_A \in \mathcal{F}_{ss}(X,K)$, if $\theta_1(\lambda_A) \subseteq \theta_2(\lambda_A)$, then $Int_2(\lambda_A) \subseteq Int_1(\lambda_A)$.

Proof. Let $\lambda_A \in \mathcal{F}_{ss}(X,K)$. From hypothesis, $\theta_1(\bar{1}_K - \lambda_A) \subseteq \theta_2(\bar{1}_K - \lambda_A)$, implies $\bar{1}_K - \theta_2(\bar{1}_K - \lambda_A) \subseteq \bar{1}_K - \theta_1(\bar{1}_K - \lambda_A))$. Therefore, $\mathrm{Int}_2(\lambda_A) \subseteq \mathrm{Int}_1(\lambda_A)$. ■

Proposition 7. If (X,θ_1,K) is T_2-$\check{C}\mathcal{F}$-scs and θ_2 is coarser that θ_1. Then (X,θ_2,K) is T_2-$\check{C}\mathcal{F}$-scs.

Proof. Let x_t^h and $y_s^{h'}$ be any two distinct fuzzy soft points in X. Since (X,θ_1,K) is a T_2-$\check{C}\mathcal{F}$-scs, then there exist two disjoint open-fss's λ_A and μ_B such that $x_t^h \tilde{\in} \lambda_A$ and $y_s^{h'} \tilde{\in} \mu_B$. That is mean $\mathrm{Int}_1(\lambda_A) = \lambda_A$ and $\mathrm{Int}_1(\mu_B) = \mu_B$. Since θ_2 is coarser than θ_1, this yields by Lemma 3, $\lambda_A \subseteq \mathrm{Int}_1(\lambda_A) \subseteq \mathrm{Int}_2(\lambda_A) \subseteq \lambda_A$ and $\mu_B \subseteq \mathrm{Int}_1(\mu_B) \subseteq \mathrm{Int}_2(\mu_B) \subseteq \mu_B$. Therefore, there exist two disjoint open-fss's λ_A and μ_B in (X,θ_2,K) such that $x_t^h \tilde{\in} \lambda_A$ and $y_s^{h'} \tilde{\in} \mu_B$. Hence, (X,θ_2,K) is T_2-$\check{C}\mathcal{F}$-scs. ■

Definition 23. A $\check{C}\mathcal{F}$-scs (X,θ,K) be a is said to be semi T_2-$\check{C}\mathcal{F}$-scs, if for every two distinct fuzzy soft points x_t^h and $y_s^{h'}$, either there exists an open-fss λ_A such that $x_t^h \tilde{\in} \lambda_A$ and $y_s^{h'} \tilde{\notin} \theta(\lambda_A)$ or there exists an open-fss μ_B such that $y_s^{h'} \tilde{\in} \mu_B$ and $x_t^h \tilde{\notin} \theta(\mu_B)$.

Example 11. Let X={a,b}, K={h}. Define $\theta: \mathcal{F}_{ss}(\mathrm{X},K) \to \mathcal{F}_{ss}(\mathrm{X},K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K\,, \\ b_1^h & if\ \lambda_A \subseteq b_1^h, \\ a_{t+0.1}^h & if\ \lambda_A \in \{a_t^h, 0 < t < 0.9\}, \\ a_1^h & if\ \lambda_A \in \{a_t^h\,, 0 \le t \le 1\}, \\ \theta(a_t^h) \cup \theta(b_s^h) & if\ \lambda_A \in \{(h, a_t \vee b_s)\ ;\ t,s \in I_0\}. \end{cases}$$

(X,θ,K) is semi T_2-$\check{C}\mathcal{F}$-scs. Since a_t^h, b_s^h are distinct fuzzy soft points, there exists an open fuzzy soft set $\lambda_A = a_1^h$ such that $a_t^h \tilde{\in} \lambda_A$ and $b_t^h \tilde{\notin} \theta(\lambda_A)= a_t^h$.

Proposition 8. Every semi T_2-$\check{C}\mathcal{F}$-scs is T_0-$\check{C}\mathcal{F}$-scs.

Proof. Let x_t^h and $y_s^{h'}$ be any two distinct fuzzy soft points in (X,θ,K). From (X,θ,K) is semi T_2-$\check{C}\mathcal{F}$-scs, there exists an open-fss λ_A such that $x_t^h \tilde{\in} \lambda_A$ and $y_s^{h'} \tilde{\notin} \theta(\lambda_A)$ or there exists an open-fss μ_B such that $y_s^{h'} \tilde{\in} \mu_B$ and $x_t^h \tilde{\notin} \theta(\mu_B)$. This implies $x_t^h \tilde{\in} \theta(x_t^h) \subseteq \theta(\lambda_A)$ and $y_s^{h'} \tilde{\notin} \theta(x_t^h)$ or $y_s^{h'} \tilde{\in} \theta(y_s^{h'}) \subseteq \theta(\mu_B)$ and $x_t^h \tilde{\notin} \theta(y_s^{h'})$. Hence (X,θ,K) is a T_0-$\check{C}\mathcal{F}$-scs. ■

The converse of above proposition is not true.

Example 12. Let X={a,b}, K={$h_1 h_2,$}and let $\lambda_A^* \subseteq \mathcal{F}_{ss}(\mathrm{X},K)$ such that λ_A^*={(h_1,$a_{t_1} \vee b_{s_1}$),($h_2, a_{t_2} \vee b_{s_2}$); $t_1,t_2,s_1,s_2 \in I_0$}. Define $\theta: \mathcal{F}_{ss}(\mathrm{X},K) \to \mathcal{F}_{ss}(\mathrm{X},K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K\,, \\ a_1^{h_1} & if\ \lambda_A \in \{a_{t_1}^{h_1};\ t_1 \in I_0\}, \\ a_1^{h_2} & if\ \lambda_A \in \{a_{t_2}^{h_2};\ t_2 \in I_0\}, \\ b_{s_1+0.2}^{h_1} & if\ \lambda_A \in \{b_{s_1}^{h_1};\ 0 < s_1 < 0.8\}, \\ b_1^{h_1} & if\ \lambda_A \in \{b_{s_1}^{h_1};\ 0.8 \le s_1 \le 1\}, \\ \left\{\left(h_1, \theta(a_{t_1}^{h_1}) \cup \theta(b_{s_1}^{h_1})\right), \left(h_2, \theta(a_{t_2}^{h_2}) \cup \theta(b_{s_2}^{h_2})\right)\right\} & if\ \lambda_A \in \lambda_A^*\,. \end{cases}$$

Then (X,θ,K) is a T_0- $\check{C}\mathcal{F}$-scse (see the details in Example 4.4). But (X,θ,K) is not semi T_2-$\check{C}\mathcal{F}$-scs, since there exist $a_{0.5}^{h_1}$, $b_{0.5}^{h_2}$ distinct fuzzy soft points such that for any open-fss λ_A, we have $a_{0.5}^{h_1} \tilde{\in} \lambda_A$ and $b_{0.5}^{h_2} \tilde{\in} \theta(\lambda_A)$ and for any open-fss μ_B, we have $b_{0.5}^{h_2} \tilde{\in} \mu_B$, $a_{0.5}^{h_1} \tilde{\in} \theta(\mu_B)$.

Theorem 13. Let (X,θ,K) be a semi T_2-$\check{C}\mathcal{F}$-scs and let (V,θ_V,K) be a closed $\check{C}$-fsc subspace of (X,θ,K). Then (V,θ_V,K) is a semi T_2-$\check{C}\mathcal{F}$-sc subspace of (X,θ,K).

Proof. Let x_t^h and $y_s^{h'}$ be any two distinct fuzzy soft points in (V,θ_V,K). Then x_t^h and $y_s^{h'}$are distinct fuzzy soft points in (X,θ,K). Since (X,θ,K) is a semi T_2-$\check{C}\mathcal{F}$-scs, then either there exists an open-fss λ_A such that $x_t^h \tilde{\in} \lambda_A$ and $y_s^{h'} \tilde{\notin} \theta(\lambda_A)$ or there exists an open-fss μ_B such that $y_s^{h'} \tilde{\in} \mu_B$ and $x_t^h \tilde{\notin} \theta(\mu_B)$. Now, if $x_t^h \tilde{\in} \lambda_A$ and $y_s^{h'} \tilde{\notin} \theta(\lambda_A)$, then by Lemma 2, $x_t^h \tilde{\in} \lambda_A \cap \bar{V}_K$ which is open-fss in (V,θ_V,K) . That is mean we find an open-fss $\lambda_A \cap \bar{V}_K$ in (V,θ_V,K) contains x_t^h. To complete the proof, we must show $y_s^{h'} \tilde{\notin} \theta_V(\lambda_A \cap \bar{V}_K)$. It is clear that from the definition of θ_V we have, $\theta_V(\lambda_A \cap \bar{V}_K)$= $\bar{V}_K \cap \theta(\lambda_A \cap \bar{V}_K) \subseteq \bar{V}_K \cap \theta(\lambda_A) \cap \theta(\bar{V}_K) = \bar{V}_K \cap \theta(\lambda_A)$. And since $y_s^{h'} \tilde{\notin} \theta(\lambda_A)$, then we have $y_s^{h'} \tilde{\notin} \theta_V(\lambda_A \cap \bar{V}_K)$. Similarly, if there exists an open-fss μ_B such that $y_s^{h'} \widetilde{\in} \mu_B$ and $x_t^h \widetilde{\notin} \theta(\mu_B)$. Hence, (V, θ_V, K) is a semi T_2-Č$\mathcal{F}$-sc subspace of (X, θ, K). ■

Definition 24. An associative fsts (X, τ_θ, K) of (X, θ, K) is said to be ***semi*** T_2*-fsts*, if for every distinct fuzzy soft points x_t^h and $y_s^{h'}$, either there exists a τ_θ-open-fss λ_A such that $x_t^h \widetilde{\in} \lambda_A$ and $y_s^{h'} \widetilde{\notin} \tau_\theta$-$cl(\lambda_A)$, or there exists a τ_θ-open-fss μ_B such that $y_s^{h'} \widetilde{\in} \mu_B$ and $x_t^h \widetilde{\notin} \tau_\theta$-$cl(\mu_B)$.

Theorem 14. If (X, τ_θ, K) is a semi T_2-fsts, then (X, θ, K) is also semi T_2-Č$\mathcal{F}$-scs.

Proof. Let x_t^h and $y_s^{h'}$ be any two distinct fuzzy soft points in (X, θ, K). From hypothesis, either there exists a τ_θ-open-fss λ_A such that $x_t^h \widetilde{\in} \lambda_A$ and $y_s^{h'} \widetilde{\notin} \tau_\theta$-$cl(\lambda_A)$, or there exists a τ_θ-open-fss μ_B such that $y_s^{h'} \widetilde{\in} \mu_B$ and $x_t^h \widetilde{\notin} \tau_\theta$-$cl(\mu_B)$. By Theorem 4, we have the following, either there exists an open-fss λ_A such that $x_t^h \widetilde{\in} \lambda_A$ and $y_s^{h'} \widetilde{\notin} \theta(\lambda_A)$ or there exists an open-fss μ_B such that $y_s^{h'} \widetilde{\in} \mu_B$ and $x_t^h \widetilde{\notin} \theta(\mu_B)$. Thus, (X, θ, K) is a semi T_2- Č$\mathcal{F}$-scs. ■

Proposition 9. If (X, θ_1, K) is a semi T_2-Č$\mathcal{F}$-scs, $\theta_2$coarser than θ_1. Then (X, θ_2, K) is semi T_2-Č$\mathcal{F}$-scs.

Proof. Let x_t^h and $y_s^{h'}$be any two distinct fuzzy soft points in X. Since (X, θ_1, K) is semi T_2- Č$\mathcal{F}$-scs, then either there exist an open-fss λ_A such that $x_t^h \widetilde{\in} \lambda_A$ and $y_s^{h'} \widetilde{\notin} \theta_1(\lambda_A)$, or there exist an open-fss μ_B such that $y_s^{h'} \widetilde{\in} \mu_B$ and $x_t^h \widetilde{\notin} \theta_1(\mu_B)$. Suppose, if there exists an open-fss λ_A in (X, θ_1, H) that is mean $Int_1(\lambda_A) = \lambda_A$. Since $\theta_2$coarser than θ_1, then by Lemma 3, we have $Int_1(\lambda_A) \subseteq Int_2(\lambda_A)$ and that is mean there exists an open-fss λ_A in(X, θ_2, K) such that $x_t^h \widetilde{\in} \lambda_A$. On the other hand since $y_s^{h'} \widetilde{\notin} \theta_1(\lambda_A)$ then $y_s^{h'} \widetilde{\notin} \theta_2(\lambda_A)$. This implies (X, θ_2, K) is a semi T_2- Č$\mathcal{F}$-scs. Similarly, if there exists an open-fss μ_B such that $y_s^{h'} \widetilde{\in} \mu_B$ and $x_t^h \widetilde{\notin} \theta_1(\mu_B)$.■

Definition 25. A Č$\mathcal{F}$-scs (X, θ, K) be a is said to be pseudo T_2- Č$\mathcal{F}$-scs, if for every two distinct fuzzy soft points x_t^h and $y_s^{h'}$, there exist open-fss's λ_A and μ_B such that $x_t^h \widetilde{\in} \lambda_A$, $y_s^{h'} \widetilde{\notin} \theta(\lambda_A)$ and $y_s^{h'} \widetilde{\in} \mu_B$, $x_t^h \widetilde{\notin} \theta(\mu_B)$.

Example 13. Let X=$\{a, b\}$, K=$\{h\}$. Define θ: $\mathcal{F}_{ss}(X, K) \to \mathcal{F}_{ss}(X, K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ \lambda_A & if\ \lambda_A \in \{(h, a_{1-t} \vee b_1); 0 \le t < 1\}, \\ \lambda_A & if\ \lambda_A \in \{(h, a_1 \vee b_{1-s});\ 0 \le s < 1\}, \\ \lambda_A & if\ \lambda_A \in \{(h, a_t); 0 < t \le 1\}, \\ \lambda_A & if\ \lambda_A \in \{(h, b_s); 0 < s \le 1\}, \\ \theta(a_t^h) \cup \theta(b_s^h) & if\ \lambda_A \in \{(h, a_t \vee b_s)\ ;\ t, s \in I_0\}. \end{cases}$$

Then (X, θ, K) is the discrete Č$\mathcal{F}$-scs. It is clear Č$\mathcal{F}$-scs (X, θ, K) is *pseudo* T_2- Č$\mathcal{F}$-scs, Since for any a_t^h, b_s^h are distinct fuzzy soft points there exist an open-fss's $\lambda_A = \{(h, a_t)\}$ and $\mu_B = \{(h, b_s)\}$ such that $a_t^h \widetilde{\in} \lambda_A = \{(h, a_t)\}$, $b_s^h \widetilde{\notin} \theta(\lambda_A) = \lambda_A$ and $b_s^h \widetilde{\in} \mu_B = \{(h, b_s)\}$, $a_t^h \widetilde{\notin} \theta(\mu_B) = \mu_B$.

Proposition 10. Every pseudo T_2- Č$\mathcal{F}$-scs is semi T_2-Č$\mathcal{F}$-scs.

Proof. Follows directly from the definition of pseudo T_2- Č$\mathcal{F}$-scs. ■

Proposition 11. Every pseudo T_2- Č$\mathcal{F}$-scs is T_1-Č$\mathcal{F}$-scs.

Proof. Let x_t^h and $y_s^{h'}$be any two distinct fuzzy soft points in X. From hypothesis (X, θ, K) is pseudo T_2-Č$\mathcal{F}$-scs, then there exist open-fss's λ_A and μ_B such that $x_t^h \widetilde{\in} \lambda_A$, $y_s^{h'} \widetilde{\notin} \theta(\lambda_A)$ and $y_s^{h'} \widetilde{\in} \mu_B$, $x_t^h \widetilde{\notin} \theta(\mu_B)$.

Since $x_t^h \tilde{\in} \lambda_A$ and $y_s^{h'} \tilde{\in} \mu_B$, then it follows $x_t^h \tilde{\in} \theta(x_t^h) \subseteq \theta(\lambda_A)$ and $y_s^{h'} \tilde{\in} \theta(y_s^{h'}) \subseteq \theta(\mu_B)$. And since $y_s^{h'} \tilde{\notin} \theta(\lambda_A)$ and $x_t^h \tilde{\notin} \theta(\mu_B)$, then $y_s^{h'} \tilde{\notin} \theta(x_t^h)$ and $x_t^h \tilde{\notin} \theta(y_s^{h'})$. Hence (X, θ, K) is a T_1-Č$\mathcal{F}$-scs. ■

The next example shows that converse of above proposition is not true.

Example 14. Let Let X={a, b}, K={h_1, h_2} and let $\lambda_A^* \subseteq \mathcal{F}_{SS}(X, K)$ such that λ_A^*={$(h_1, a_{t_1} \vee b_{s_1}), (h_2, a_{t_2} \vee b_{s_2}); t_1, t_2, s_1, s_2 \in I_0$}. Define $\theta: \mathcal{F}_{SS}(X, K) \to \mathcal{F}_{SS}(X, K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ a_1^{h_1} & if\ \lambda_A \subseteq a_1^{h_1}, \\ a_1^{h_2} & if\ \lambda_A \subseteq a_1^{h_2}, \\ b_1^{h_1} & if\ \lambda_A \subseteq b_1^{h_1}, \\ b_{0.9}^{h_2} & if\ \lambda_A \in \{b_s^{h_2};\ 0 < s < 0.9\}, \\ b_1^{h_2} & if\ \lambda_A \in \{b_s^{h_2}; 0.9 \le s \le 1\}, \\ \left\{\left(h_1, \theta(a_{t_1}^{h_1}) \cup \theta(b_{s_1}^{h_1})\right), \left(h_2, \theta(a_{t_2}^{h_2}) \cup \theta(b_{s_2}^{h_2})\right)\right\} & if\ \lambda_A \in \lambda_A^*. \end{cases}$$

Then (X, θ, K) is T_1-Č$\mathcal{F}$-scs. But (X, θ, K) is not pseudo T_2-Č$\mathcal{F}$-scs. To show that consider $a_{0.5}^{h_1}$ and $b_{0,7}^{h_2}$ are distinct fuzzy soft points. The open-fss's λ_A such that $a_{0.5}^{h_1} \tilde{\in} \lambda_A$ are:

1. $\lambda_A = \bar{1}_H$, implies $b_{0.7}^{h_2} \tilde{\in} \theta(\bar{1}_H)$.
2. $\lambda_A = \{(h_1, a_1 \vee b_1), (h_1, b_1)\}$, implies $b_{0.7}^{h_2} \tilde{\in} \theta(\lambda_A)$.
3. $\lambda_A = \{(h_1, a_1), (h_2, a_1 \vee b_1)\}$, implies $b_{0.7}^{h_2} \tilde{\in} \theta(\lambda_A)$.
4. $\lambda_A = \{(h_1, a_1 \vee b_1), (h_2, a_1)\}$, implies $b_{0.7}^{h_2} \tilde{\in} \theta(\lambda_A)$.

Hence, for all open-fss λ_A such that $a_{0.5}^{h_1} \tilde{\in} \lambda_A$, we have $b_{0,7}^{h_2} \tilde{\in} \theta(\lambda_A)$. Thus, (X, θ, K) is not pseudo T_2-Č$\mathcal{F}$-scs.

Theorem 15. Let (X, θ, K) be a pseudo T_2-Č$\mathcal{F}$-scs and let (V, θ_V, K) be a closed $\check{C}$-fsc subspace of (X, θ, K). Then (V, θ_V, K) is a pseudo T_2-Č$\mathcal{F}$-sc subspace of (X, θ, K).

Proof: Similar of Theorem 13. ■

Definition 26. An associative fsts (X, τ_θ, K) of (X, θ, K) is said to be a pseudo T_2-fsts, if for every distinct fuzzy soft points x_t^h and $y_s^{h'}$ there exist τ_θ-open-fss's λ_A and μ_B such that $x_t^h \tilde{\in} \lambda_A$, $y_s^{h'} \tilde{\notin} \tau_\theta$-$cl(\lambda_A)$ and $y_s^{h'} \tilde{\in} \mu_B$ and $x_t^h \tilde{\notin} \tau_\theta$-$cl(\mu_B)$.

Theorem 16. If (X, τ_θ, K) is a pseudo T_2-fsts, then (X, θ, K) is also pseudo T_2-Č$\mathcal{F}$-scs.

Proof. Similar of Theorem 14. ■

Proposition 12. If (X, θ_1, K) be a pseudo T_2- Č$\mathcal{F}$-scs, $\theta_2$coarser than θ_1, then (X, θ_2, K) is pseudo T_2- Č$\mathcal{F}$-scs.

Proof. Similar of proof Proposition 9. ■

Definition 27. A $\check{C}$ $\mathcal{F}$-scs (X, θ, K) be a is said to be Uryshon T_2- Č$\mathcal{F}$-scs, if for every two distinct fuzzy soft points x_t^h and $y_s^{h'}$, there exist open-fss's λ_A and μ_B such that $x_t^h \tilde{\in} \lambda_A$, $y_s^{h'} \tilde{\in} \mu_B$ and $\theta(\lambda_A) \cap \theta(\mu_B) = \bar{0}_K$.

Example 15. Let X={a, b}, K={h}. Define $\theta: \mathcal{F}_{SS}(X, K) \to \mathcal{F}_{SS}(X, K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ a_1^h & if\ \lambda_A \subseteq a_1^h, \\ b_1^h & if\ \lambda_A \subseteq b_1^h, \\ \bar{1}_K & other\ wise. \end{cases}$$

Then (X, θ, K) is a Uryshon T_2-Č$\mathcal{F}$-scs. Since Consider a_t^h and b_s^h are distinct fuzzy soft points, there exist open-fss's λ_A=a_1^h and μ_B=b_1^h such that $a_t^h \tilde{\in} \lambda_A$, $b_s^h \tilde{\in} \lambda_A$ and $\theta(\lambda_A) \cap \theta(\mu_B)$= $a_1^h \cap b_1^h$=$\bar{0}_K$.

Proposition 13. Every Uryshon T_2-Č$\mathcal{F}$-scs is pesudo T_2-Č$\mathcal{F}$-scs.

Proof. Let x_t^h and $y_s^{h'}$ be any two-distinct fuzzy soft points in (X, θ, H). Since (X, θ, H) is Uryshon T_2-Č$\mathcal{F}$-scs, there exist λ_A and μ_B open fuzzy soft sets such that $x_t^h \tilde{\in} \lambda_A$, $y_s^{h'} \tilde{\in} \mu_B$ and $\theta(\lambda_A) \cap \theta(\mu_B)$=$\bar{0}_K$. This implies $x_t^h \tilde{\notin} \theta(\mu_B)$ and $y_s^{h'} \tilde{\notin} \theta(\lambda_A)$. Therefore, (X, θ, K) is pesudo T_2- Č$\mathcal{F}$-scs. ■

Proposition 14. Every Uryshon T_2- Č$\mathcal{F}$-scs is T_2- Č$\mathcal{F}$-scs.

Proof. The proof follows immediately from the definition of Uryshon T_2- Č$\mathcal{F}$-scs and the property $\lambda_A \subseteq \theta(\lambda_A)$ for any fuzzy soft set λ_A. ■

Theorem 17. Let (X, θ, K) be an Uryshon T_2-Č$\mathcal{F}$-scs and let (V, θ_V, K) be a closed Č$\mathcal{F}$-sc subspace of (X, θ, K). Then (V, θ_V, K) is an Uryshon T_2- Č$\mathcal{F}$-sc subspace of (X, θ, K).

Proof. Let x_t^h and $y_s^{h'}$ be any two-distinct fuzzy soft points in (V, θ_V, K). Then x_t^h and $y_s^{h'}$ are distinct fuzzy soft points in (X, θ, K). Since (X, θ, K) is a Uryshon T_2-Č$\mathcal{F}$-scs, it follows there exist open-fss's λ_A and μ_B such that $x_t^h \tilde{\in} \lambda_A$, $y_s^{h'} \tilde{\in} \mu_B$ and $\theta(\lambda_A) \cap \theta(\mu_B)$=$\bar{0}_K$. By Lemma 2, $\lambda_A \cap \bar{V}_K$ and $\mu_B \cap \bar{V}_K$ are open fuzzy soft sets in $\bar{V}_K$ such that $x_t^h \tilde{\in} \lambda_A \cap \bar{V}_K$, $y_s^{h'} \tilde{\in} \mu_B \cap \bar{V}_K$. Next, we must show that $\theta_V(\lambda_A \cap \bar{V}_K) \cap \theta_V(\mu_B \cap \bar{V}_K)$ =$\bar{0}_K$. Now, from the definition of θ_V we get,

$$\begin{aligned} \theta_V(\lambda_A \cap \bar{V}_K) \cap \theta_V(\mu_B \cap \bar{V}_K) &= [\bar{V}_K \cap \theta(\lambda_A \cap \bar{V}_K)] \cap [\bar{V}_K \cap \theta(\mu_B \cap \bar{V}_K)] \\ &= [\theta(\lambda_A) \cap \theta(\mu_B)] \cap \bar{V}_K \\ &= \bar{0}_K \cap \bar{V}_K \\ &= \bar{0}_K. \end{aligned}$$

Therefore, (V, θ_V, K) is an Uryshon T_2-Č$\mathcal{F}$-sc subspace of (X, θ, K). ■

Definition 28. An associative fsts (X, τ_θ, K) of (X, θ, K) is said to be a Uryshon T_2-fst, if for every distinct fuzzy soft points x_t^h and $y_s^{h'}$, there exist τ_θ-open fuzzy soft sets λ_A and μ_B such that $x_t^h \tilde{\in} \lambda_A$, $y_s^{h'} \tilde{\in} \mu_B$ and τ_θ-*cl*$(\lambda_A) \cap \tau_\theta$-*cl*$(\mu_B)$=$\bar{0}_K$.

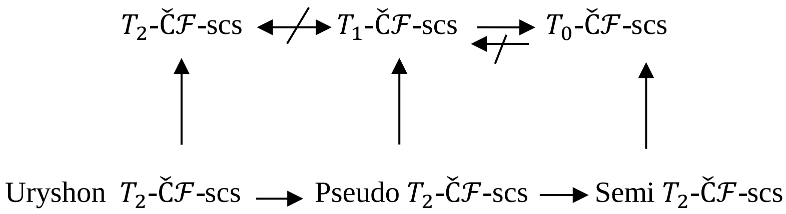
Theorem 18. If (X, τ_θ, K) is an Uryshon T_2-fsts, then (X, θ, K) is also Uryshon T_2-Č$\mathcal{F}$-scs.

Proof: Let x_t^h and $y_s^{h'}$ be any two-distinct fuzzy soft points in X. Since (X, τ_θ, K) is an Uryshon T_2-fsts, then there exist τ_θ-open-fss's λ_A and μ_B such that $x_t^h \tilde{\in} \lambda_A$, $y_s^{h'} \tilde{\in} \mu_B$ and τ_θ-*cl*$(\lambda_A) \cap \tau_\theta$-*cl*$(\mu_B)$=$\bar{0}_K$. By Theorem 4, we obtain λ_A and μ_B are open fuzzy soft sets in (X, θ, K) such that $x_t^h \tilde{\in} \lambda_A$ and $y_s^{h'} \tilde{\in} \mu_B$ and $\theta(\lambda_A) \cap \theta(\mu_B)$=$\bar{0}_K$. Hence, (X, θ, K) is an Uryshon T_2-Č$\mathcal{F}$-scs. ■

Proposition 15. If (X, θ_1, K) be a Uryshon T_2-Č$\mathcal{F}$-scs, $\theta_2$coarser than θ_1, then (X, θ_2, K) is Uryshon T_2-Č$\mathcal{F}$-scs.

Proof. Similar of proof Proposition 9. ■

Remark 3. The relationships between the above types of separation axioms in Č$\mathcal{F}$-scs's as shown in the following diagram.



CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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