Hacettepe Journal of Mathematics and Statistics
Volume 40 (4) (2011), 537 – 554

$\mathcal{X}$-GORENSTEIN PROJECTIVE AND $\mathcal{Y}$-GORENSTEIN INJECTIVE MODULES

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Received 21 : 06 : 2010 : Accepted 26 : 01 : 2011

Abstract

Let $\mathcal{X}$ be a class of right R-modules that contains all projective right R-modules. The notion of $\mathcal{X}$-Gorenstein projective modules was introduced by D. Bennis and K. Ouarghi ($\mathcal{X}$-*Gorenstein projective modules*, International Mathematical Forum **5** (10), 487–491, 2010). In this paper, we introduce $\mathcal{Y}$-Gorenstein injective right R-modules and $\mathcal{Y}$-Gorenstein flat left R-modules, where $\mathcal{Y}$ is a class of right R-modules that contains all injective right R-modules. We show that the principal results on Gorenstein modules remain true for $\mathcal{X}$-Gorenstein projective right R-modules, $\mathcal{Y}$-Gorenstein injective right R-modules and $\mathcal{Y}$-Gorenstein flat left R-modules.

Keywords: $\mathcal{X}$-Gorenstein projective modules, $\mathcal{Y}$-Gorenstein injective modules, $\mathcal{Y}$-Gorenstein flat modules.

2000 AMS Classification: 16 E 10, 16 E 30.

1. Introduction

In [6], Enochs and Jenda defined the Gorenstein injective modules over an arbitrary ring R. Recall that a right R-module M is called Gorenstein injective if there is an exact sequence

$$\mathcal{E} \equiv \ \cdots \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow E^{0} \longrightarrow E^{1} \longrightarrow \cdots$$

of injective right R-modules with $M = \ker(E^0 \to E^1)$, and which remains exact whenever $\mathrm{Hom}_R(E, -)$ is applied for any injective right R-module E.

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In [2], Bennis *et al.* introduced the notion of $\mathcal{X}$-Gorenstein projective modules. Let $\mathcal{X}$ be a class of right R-modules that contains all projective right R-modules. A right R-module M is called $\mathcal{X}$-Gorenstein projective if there exists an exact sequence

$$\mathcal{P} \equiv \cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

of projective right R-modules such that $M = \ker(P^0 \to P^1)$ and $\mathrm{Hom}_R(\mathcal{P}, F)$ is exact whenever $F \in \mathcal{X}$.

In this paper, we introduce $\mathcal{Y}$-Gorenstein injective right R-modules and $\mathcal{Y}$-Gorenstein flat left R-modules, where $\mathcal{Y}$ is a class of right R-modules that contains all injective right R-modules. A right R-module M is called $\mathcal{Y}$-Gorenstein injective if there exists an exact sequence

$$\mathcal{E} \equiv \cdots \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

of injective right R-modules such that $M = \ker(E^0 \to E^1)$, and which remains exact whenever $\mathrm{Hom}_R(H, -)$ is applied for any $H \in \mathcal{Y}$. A left R-module M is called $\mathcal{Y}$-Gorenstein flat if there exists an exact sequence

$$\mathcal{F} \equiv \cdots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

of flat left R-modules such that $M = \ker(F^0 \to F^1)$, and which remains exact whenever $G \otimes_R -$ is applied for any $G \in \mathcal{Y}$.

We mainly show that principal results on Gorenstein modules remain true for $\mathcal{X}$-Gorenstein projective right R-modules, $\mathcal{Y}$-Gorenstein injective right R-modules and $\mathcal{Y}$-Gorenstein flat left R-modules.

Section 2 introduces $\mathcal{Y}$-Gorenstein injective modules and studies their relations with Gorenstein injective modules. For a ring R with $\mathrm{r.Ggldim}(R) < \infty$, it is shown that the class of $\mathcal{Y}$-Gorenstein injective right R-modules coincides with the class of Gorenstein injective right R-modules if and only if every module in $\mathcal{Y}$ has finite injective dimension. We also define the $\mathcal{Y}$-Gorenstein injective dimension of a module and a ring. Using the functors $\mathrm{Ext}^i_R(-,-)$, we give some characterizations of a module with finite $\mathcal{Y}$-Gorenstein injective dimension. For a ring R with $r\mathcal{Y}\text{-GID}(R) < \infty$, we get that $\big({}^{\perp}(\mathcal{Y}\text{-}\mathcal{GI}(R)), \mathcal{Y}\text{-}\mathcal{GI}(R)\big)$ is a complete hereditary cotorsion theory.

Section 3 deals with $\mathcal{X}$-Gorenstein projective right R-modules, in a way much similar to how we treat the $\mathcal{Y}$-Gorenstein injective right R-modules in Section 2.

Section 4 introduces $\mathcal{Y}$-Gorenstein flat modules and studies their relations with $\mathcal{X}$-Gorenstein projective modules and $\mathcal{Y}$-Gorenstein injective modules. Let $\mathcal{X}$ be a class of right R-modules that contains all projective right R-modules and $\mathcal{Y}$ a class of left R-modules that contains all injective left R-modules. If $\mathcal{Y}^+ \subseteq \mathcal{X}$, then every $\mathcal{X}$-Gorenstein projective right R-module is $\mathcal{Y}$-Gorenstein flat. For a right coherent ring, we get that M is a $\mathcal{Y}$-Gorenstein flat left R-module if and only if $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a $\mathcal{Y}$-Gorenstein injective right R-module. We also define the $\mathcal{Y}$-Gorenstein flat dimension of a module and a ring. Using the functors $\mathrm{Tor}^R_i(-,-)$, we give some characterizations of a left R-module with finite $\mathcal{Y}$-Gorenstein flat dimension over a right coherent ring R. If R is a right coherent ring with $l\mathcal{Y}\text{-GFD}(R) < \infty$, then $\big(\mathcal{Y}\text{-}\mathcal{GF}(R), \mathcal{Y}\text{-}\mathcal{GF}(R)^{\perp}\big)$ is a perfect complete hereditary cotorsion theory.

Next we recall some notions and facts required in the paper. In [4], Ding *et al.* introduced the notion of strongly Gorenstein flat modules. A right R-module M is called strongly Gorenstein flat if there exists an exact sequence

$$\mathcal{P} \equiv \cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

of projective right R-modules such that $M = \ker(P^0 \to P^1)$ and $\mathrm{Hom}_R(\mathcal{P}, F)$ is exact whenever F is flat. Obviously, $\mathcal{X}$-Gorenstein projective modules generalize both Gorenstein projective modules and strongly Gorenstein flat modules.

In [13], Mao and Ding introduced Gorenstein FP-injective modules. A right R-module M is called Gorenstein FP-injective if there exists an exact sequence

$$\mathcal{E} \equiv \ \cdots \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

of injective right R-modules such that $M = \ker(E^0 \to E^1)$, and which remains exact whenever $\mathrm{Hom}_R(H, -)$ is applied for any FP-injective module H. Obviously, $\mathcal{Y}$-Gorenstein injective modules generalize both Gorenstein injective modules and Gorenstein FP-injective modules.

Let $\mathcal{C}$ be a class of R-modules and M an R-module. Following [5], we say that a homomorphism $\phi : M \to C$ is a $\mathcal{C}$-*preenvelope* if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}(\phi, C') : \mathrm{Hom}(C, C') \to \mathrm{Hom}(M, C')$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope $\phi : M \to C$ is said to be a $\mathcal{C}$-*envelope* if every endomorphism $g : C \to C$ such that $g\phi = \phi$ is an isomorphism. Dually we have the definitions of a $\mathcal{C}$-*precover* and a $\mathcal{C}$-*cover*. $\mathcal{C}$-envelopes ($\mathcal{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

A module M is said to have a special $\mathcal{C}$-precover [7, Definition 7.1.6] if there is an exact sequence $0 \longrightarrow F \longrightarrow C \longrightarrow M \longrightarrow 0$ with $C \in \mathcal{C}$ and $F \in \mathcal{C}^{\perp}$. M is said to have a special $\mathcal{C}$-preenvelope [7, Definition 7.1.6] if there is an exact sequence $0 \longrightarrow M \longrightarrow C \longrightarrow F \longrightarrow 0$ with $C \in \mathcal{C}$ and $F \in {}^{\perp}\mathcal{C}$.

A pair $(\mathcal{F}, \mathcal{C})$ of classes of right R-modules is called a cotorsion theory (cotorsion pair) [7, Definition 7.1.2] if $\mathcal{F}^{\perp} = \mathcal{C}$ and ${}^{\perp}\mathcal{C} = \mathcal{F}$. A pair of classes $(\mathcal{F}, \mathcal{F}^{\perp})$ is said to be cogenerated by a set $\mathcal{D}$ [9, Definition 1.1.7] if $\mathcal{F}^{\perp} = \mathcal{D}^{\perp}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called complete [11, Lemma 2.2.6] if every R-module has a special $\mathcal{C}$-preenvelope (and a special $\mathcal{F}$-precover). A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to be hereditary [8] if whenever

$$0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0$$

is exact with $L,\ L'' \in \mathcal{F}$ then L' is also in $\mathcal{F}$, or equivalently, if whenever

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

is exact with $C,\ C' \in \mathcal{C}$ then C'' is also in $\mathcal{C}$.

Throughout this paper, R is an associative ring with identity and all modules are unitary, $\mathrm{r.gldim}(R)$ (resp. $\mathrm{wdim}(R)$) stands for the right (resp. the weak) global dimension of R. For an R-module M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+, $\mathrm{fd}(M)$, $\mathrm{id}(M)$ and $\mathrm{pd}(M)$ stand for the flat, injective and projective dimensions of M respectively, $\mathrm{Gfd}(M)$, $\mathrm{Gid}(M)$ and $\mathrm{Gpd}(M)$ denote the Gorenstein flat, injective and projective dimensions of M respectively. $\mathrm{r.Ggldim}(R)$ (resp. $\mathrm{l.Ggldim}(R)$) denotes the right (resp. the left) Gorenstein global dimension of R.

2. $\mathcal{Y}$-Gorenstein injective modules

2.1. Definition. Let $\mathcal{Y}$ be a class of right R-modules that contains all injective right R-modules. A right R-module M is called $\mathcal{Y}$-*Gorenstein injective* if there exists an exact sequence

$$\mathcal{E} \equiv \ \cdots \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

of injective right R-modules such that $M = \ker(E^0 \to E^1)$, and which remains exact whenever $\mathrm{Hom}_R(H, -)$ is applied for any $H \in \mathcal{Y}$.

The sequence $\mathcal{E}$ is called a $\mathcal{Y}$-*complete injective resolution*.

2.2. Remark. (1) Obviously, we have the following implications:

injective modules $\Longrightarrow$ $\mathcal{Y}$-Gorenstein injective modules $\Longrightarrow$ Gorenstein injective modules.

(2) Let $\mathcal{Y}$ be the class of injective right R-modules, then $\mathcal{Y}$-Gorenstein injective right R-modules coincide with Gorenstein injective right R-modules.

(3) Let $\mathcal{Y}$ be the class of FP-injective right R-modules, then $\mathcal{Y}$-Gorenstein injective right R-modules coincide with Gorenstein FP-injective right R-modules [13].

(4) If $\mathcal{Y}$ is the class of Gorenstein injective right R-modules, then every $\mathcal{Y}$-Gorenstein injective right R-module is injective. Indeed, for any $\mathcal{Y}$-Gorenstein injective right R-module M, we have an exact sequence of right R-modules

$$0 \longrightarrow M \longrightarrow I \longrightarrow K \longrightarrow 0$$

with I injective and K $\mathcal{Y}$-Gorenstein injective, which remains exact whenever $\mathrm{Hom}_R(H,-)$ is applied for any module $H \in \mathcal{Y}$. Since every $\mathcal{Y}$-Gorenstein injective right R-module is Gorenstein injective, we let $H = K \in \mathcal{Y}$, then we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(K,M) \longrightarrow \mathrm{Hom}_R(K,I) \longrightarrow \mathrm{Hom}_R(K,K) \longrightarrow 0.$$

Thus M is a direct summand of I, hence M is injective.

2.3. Proposition. *A right R-module M is injective if and only if M belongs to $\mathcal{Y}$ and M is $\mathcal{Y}$-Gorenstein injective.*

Proof. If M is $\mathcal{Y}$-Gorenstein injective, then by the definition of $\mathcal{Y}$-Gorenstein injective modules, we have an exact sequence of right R-modules

$$0 \longrightarrow G \longrightarrow I \longrightarrow M \longrightarrow 0$$

with I injective and G $\mathcal{Y}$-Gorenstein injective, which remains exact whenever $\mathrm{Hom}_R(H,-)$ is applied for any $H \in \mathcal{Y}$. Since M belongs to $\mathcal{Y}$, we apply $\mathrm{Hom}_R(M,-)$ to the above exact sequence, then we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M,G) \longrightarrow \mathrm{Hom}_R(M,I) \longrightarrow \mathrm{Hom}_R(M,M) \longrightarrow 0.$$

Thus M is a direct summand of I, hence M is injective.

The converse is trivial. □

2.4. Corollary. *The following statements are equivalent for a ring R:*

(1) *$\mathcal{Y}$ is the class of injective right R-modules.*
(2) *Every $H \in \mathcal{Y}$ is $\mathcal{Y}$-Gorenstein injective.*

Proof. (1) $\Longrightarrow$ (2) is trivial by Remark 2.2 (1).

(2) $\Longrightarrow$ (1) By Proposition 2.3, we know that $\mathcal{Y}$ is the class of injective right R-modules. □

Let $\mathcal{Y}$ be the class of FP-injective right R-modules in Corollary 2.4. Then we have the following result which is a generalization of [13, Proposition 2.7].

2.5. Corollary. *The following statements are equivalent for a ring R:*

(1) *R is right noetherian.*
(2) *Every FP-injective right R-module is Gorenstein FP-injective.*

Proof. We only note that R is right noetherian if and only if every FP-injective right R-module is injective [14, Theorem 3]. □

2.6. Theorem. *Let R be a ring with* r.Ggldim$(R) < \infty$*, then the following statements are equivalent:*

(1) *The class of $\mathcal{Y}$-Gorenstein injective right R-modules coincides with the class of Gorenstein injective right R-modules.*
(2) *Every module in $\mathcal{Y}$ has finite injective dimension.*

Proof. (1) $\Longrightarrow$ (2) Let M be any Gorenstein injective right R-module, by hypothesis we know that M is also $\mathcal{Y}$-Gorenstein injective. Thus there is an exact sequence of right R-modules

$$0 \longrightarrow M \longrightarrow I \longrightarrow G \longrightarrow 0$$

with I injective and G $\mathcal{Y}$-Gorenstein injective, which remains exact whenever $\mathrm{Hom}_R(H, -)$ is applied for any $H \in \mathcal{Y}$. For any $H \in \mathcal{Y}$, we have a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(H, M) \longrightarrow \mathrm{Hom}_R(H, I) \longrightarrow \mathrm{Hom}_R(H, G) \longrightarrow \mathrm{Ext}^1_R(H, M) \longrightarrow 0.$$

Thus $\mathrm{Ext}^1_R(H, M) = 0$ for any Gorenstein injective right R-module M. Hence by [15, Proposition 2.5], H has finite injective dimension.

(2) $\Longrightarrow$ (1) Let M be any Gorenstein injective right R-module, then there is an exact sequence

$$\mathcal{E} \equiv \ \cdots \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

of injective right R-modules with $M = \ker(E^0 \to E^1)$, which remains exact whenever $\mathrm{Hom}_R(E, -)$ is applied for any injective right R-module E.

Let H be any module in $\mathcal{Y}$. By (2), we may assume $\mathrm{id}(H) = n < \infty$. We proceed by induction on n. If $n = 0$, by the definition of Gorenstein injective modules, $\mathrm{Hom}_R(H, \mathcal{E})$ is exact. For $n \geq 1$, we have an exact sequence

$$0 \longrightarrow H \longrightarrow E \longrightarrow N \longrightarrow 0,$$

where E is an injective right R-module and $\mathrm{id}(N) = n - 1$.

Then we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(N, \mathcal{E}) \longrightarrow \mathrm{Hom}_R(E, \mathcal{E}) \longrightarrow \mathrm{Hom}_R(H, \mathcal{E}) \longrightarrow 0$$

with the first two complexes exact by induction. Hence $\mathrm{Hom}_R(H, \mathcal{E})$ is exact. Thus M is $\mathcal{Y}$-Gorenstein injective. Since every $\mathcal{Y}$-Gorenstein injective right R-module is Gorenstein injective, the class of $\mathcal{Y}$-Gorenstein injective right R-modules coincides with the class of Gorenstein injective right R-modules. □

2.7. Corollary. *Let R be a ring with r.Ggldim$(R) < \infty$ and wdim$(R) < \infty$. Then the class of $\mathcal{Y}$-Gorenstein injective right R-modules coincides with the class of Gorenstein injective right R-modules.*

Proof. By [1, Corollary 1.2], we know that if $\mathrm{wdim}(R) < \infty$ then

$$\mathrm{r.gldim}(R) = \mathrm{r.Ggldim}(R) < \infty.$$

Thus every right R-module has finite injective dimension. Hence by Theorem 2.6, the class of $\mathcal{Y}$-Gorenstein injective right R-modules coincides with the class of Gorenstein injective right R-modules. □

2.8. Lemma. *The following assertions are equivalent:*

(1) *M is a $\mathcal{Y}$-Gorenstein injective right R-module.*
(2) *M satisfies the following two assertions:*
    (a) $\mathrm{Ext}^i_R(H, M) = 0$ *for any right R-module $H \in \mathcal{Y}$ and any $i \geq 1$.*

(b) *There exists an exact sequence of right R-modules*

$$\cdots \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow M \longrightarrow 0,$$

where each E^{-i} is injective and $\mathrm{Hom}_R(H,-)$ leaves the sequence exact for any $H \in \mathcal{Y}$.

(3) *There exists a short exact sequence of right R-modules*

$$0 \longrightarrow G \longrightarrow I \longrightarrow M \longrightarrow 0,$$

where I is injective and G is $\mathcal{Y}$-Gorenstein injective.

Proof. By the definition of $\mathcal{Y}$-Gorenstein injective modules, we immediately get (1) $\Longleftrightarrow$ (2) and (1) $\Longrightarrow$ (3).

(3) $\Longrightarrow$ (1) The proof is similar to that of [2, Proposition 2.2]. □

By Lemma 2.8 and [2, Proposition 2.4], we have the following:

2.9. Proposition. *Every right R-module is $\mathcal{Y}$-Gorenstein injective if and only if every right R-module in $\mathcal{Y}$ is projective.*

In particular, when the above equivalent conditions are satisfied R is quasi-Frobenius. □

2.10. Proposition. *The class of $\mathcal{Y}$-Gorenstein injective modules is closed under extensions and cokernels of monomorphisms. Furthermore it is closed under direct products and direct summands.*

Proof. Consider the exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$. First, assume that M', M'' are $\mathcal{Y}$-Gorenstein injective modules. By a proof similar to that of [7, Lemma 8.2.1], we can construct an exact sequence of right R-modules

$$\cdots \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow M \longrightarrow 0,$$

where each E^{-i} is injective and $\mathrm{Hom}_R(H,-)$ leaves the sequence exact for any $H \in \mathcal{Y}$. Since M', M'' are $\mathcal{Y}$-Gorenstein injective modules, $\mathrm{Ext}^i_R(H,M') = \mathrm{Ext}^i_R(H,M'') = 0$ for all $i > 0$ and all $H \in \mathcal{Y}$. Using the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^i_R(H,M') \longrightarrow \mathrm{Ext}^i_R(H,M) \longrightarrow \mathrm{Ext}^i_R(H,M'') \longrightarrow \cdots,$$

we get that $\mathrm{Ext}^i_R(H,M) = 0$ for all $i > 0$ and all $H \in \mathcal{Y}$. By Lemma 2.8, we know that M is $\mathcal{Y}$-Gorenstein injective.

Next, assume that M', M are $\mathcal{Y}$-Gorenstein injective modules. By Lemma 2.8, there exists a short exact sequence of right R-modules

$$0 \longrightarrow G \longrightarrow I \longrightarrow M \longrightarrow 0,$$

where I is injective and G is $\mathcal{Y}$-Gorenstein injective. Consider the following pullback diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G & = & G & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F & \longrightarrow & I & \longrightarrow & M'' & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From the left exact column and the fact that the class of $\mathcal{Y}$-Gorenstein injective modules is closed under extensions, we know that F is $\mathcal{Y}$-Gorenstein injective. Thus we get an exact sequence $0 \longrightarrow F \longrightarrow I \longrightarrow M'' \longrightarrow 0$, where I is injective and F is $\mathcal{Y}$-Gorenstein injective. From Lemma 2.8, we know M'' is $\mathcal{Y}$-Gorenstein injective.

By the definition of $\mathcal{Y}$-Gorenstein injective modules, we know that $\mathcal{Y}$-Gorenstein injective modules are closed under direct products. Hence $\mathcal{Y}$-Gorenstein injective modules are closed under direct summands by [12, Proposition 1.4]. □

2.11. Definition. We will say that M has $\mathcal{Y}$-Gorenstein injective dimension less than or equal to n, denoted $\mathcal{Y}\text{-Gid}(M) \leq n$, if there exists an exact sequence

$$0 \longrightarrow M \longrightarrow G^0 \longrightarrow \ldots \longrightarrow G^{n-1} \longrightarrow G^n \longrightarrow 0$$

with every G^i being $\mathcal{Y}$-Gorenstein injective. If no such finite sequence exists, define $\mathcal{Y}\text{-Gid}(M) = \infty$; otherwise, if n is the least such integer, define $\mathcal{Y}\text{-Gid}(M) = n$.

Define $r\mathcal{Y}\text{-GID}(R) = \sup\{\mathcal{Y}\text{-Gid}(M) \mid M \text{ is any right } R\text{-module}\}$.

2.12. Proposition. *Let M be a right R-module with finite $\mathcal{Y}$-Gorenstein injective dimension n. Then there exist exact sequences*

$$0 \longrightarrow M \longrightarrow I \longrightarrow F \longrightarrow 0$$

with I $\mathcal{Y}$-Gorenstein injective, $\text{id}(F) \leq n-1$, *and*

$$0 \longrightarrow I' \longrightarrow F' \longrightarrow M \longrightarrow 0$$

with I' $\mathcal{Y}$-Gorenstein injective, $\text{id}(F') \leq n$.

Proof. We will prove the desired result by induction on n. If $n = 0$, then M is $\mathcal{Y}$-Gorenstein injective, thus there exists an exact sequence

$$0 \longrightarrow H \longrightarrow E \longrightarrow M \longrightarrow 0$$

with E injective and H $\mathcal{Y}$-Gorenstein injective. We also have the exact sequence

$$0 \longrightarrow M \longrightarrow M \longrightarrow 0 \longrightarrow 0.$$

Now, let $n = 1$ and let $0 \longrightarrow M \longrightarrow I_0 \xrightarrow{d_0} I_1 \longrightarrow 0$ be an exact sequence with each I_i $\mathcal{Y}$-Gorenstein injective. By the case $n = 0$, we know there is an exact sequence $0 \longrightarrow H_0 \longrightarrow E_0 \longrightarrow I_1 \longrightarrow 0$ with E_0 injective and H_0 $\mathcal{Y}$-Gorenstein injective. Consider the following pullback diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H_0 & = & H_0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & G_0 & \longrightarrow & E_0 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & I_0 & \longrightarrow & I_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

From the exact middle column and the fact that $\mathcal{Y}$-Gorenstein injective modules are closed under extensions, we know that G_0 is $\mathcal{Y}$-Gorenstein injective. Thus we get the exact sequence

$$0 \longrightarrow M \longrightarrow G_0 \longrightarrow E_0 \longrightarrow 0,$$

where E_0 is injective and G_0 is $\mathcal{Y}$-Gorenstein injective. Since G_0 is $\mathcal{Y}$-Gorenstein injective, we get the exact sequence

$$0 \longrightarrow H_1 \longrightarrow E_1 \longrightarrow G_0 \longrightarrow 0,$$

where E_1 is injective and H_1 is $\mathcal{Y}$-Gorenstein injective. Consider the following pullback diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H_1 & = & H_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F_1 & \longrightarrow & E_1 & \longrightarrow & E_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & G_0 & \longrightarrow & E_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

From the exact middle row, we know that $id(F_1) \leq 1$. Thus we have the exact sequence

$$0 \longrightarrow H_1 \longrightarrow F_1 \longrightarrow M \longrightarrow 0$$

with $\mathrm{id}(F_1) \leq 1$ and H_1 $\mathcal{Y}$-Gorenstein injective.

Suppose $n > 1$. Then we have an exact sequence

$$0 \longrightarrow M \longrightarrow I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} I_n \longrightarrow 0$$

with each I_i $\mathcal{Y}$-Gorenstein injective. Let $K_{n-1} = \mathrm{im}(d_0)$, then we have exact sequences

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow K_{n-1} \longrightarrow 0,$$

$$0 \longrightarrow K_{n-1} \longrightarrow I_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} I_n \longrightarrow 0,$$

i.e. $\mathcal{Y}$-Gid$(K_{n-1}) = n-1$. By the induction hypothesis we know there is an exact sequence $0 \longrightarrow H_{n-1} \longrightarrow F_{n-1} \longrightarrow K_{n-1} \longrightarrow 0$ with $\mathrm{id}(F_{n-1}) \leq n-1$ and H_{n-1} $\mathcal{Y}$-Gorenstein injective. Consider the following pullback diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H_{n-1} & = & H_{n-1} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & G_{n-1} & \longrightarrow & F_{n-1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & I_0 & \longrightarrow & K_{n-1} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

From the exact middle column

$$0 \longrightarrow H_{n-1} \longrightarrow G_{n-1} \longrightarrow I_0 \longrightarrow 0,$$

we know that G_{n-1} is $\mathcal{Y}$-Gorenstein injective. Thus we get the exact sequence

$$0 \longrightarrow M \longrightarrow G_{n-1} \longrightarrow F_{n-1} \longrightarrow 0$$

with G_{n-1} $\mathcal{Y}$-Gorenstein injective and $\mathrm{id}(F_{n-1}) \leq n-1$. As in the previous case, since G_{n-1} is $\mathcal{Y}$-Gorenstein injective, there exists a short exact sequence

$$0 \longrightarrow G_n \longrightarrow J \longrightarrow G_{n-1} \longrightarrow 0,$$

where J is injective and G_n is $\mathcal{Y}$-Gorenstein injective. Consider the following pullback diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_n & = & G_n & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F_n & \longrightarrow & J & \longrightarrow & F_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & G_{n-1} & \longrightarrow & F_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since J is injective and $\mathrm{id}(F_{n-1}) \leq n-1$, we get $\mathrm{id}(F_n) \leq n$ by the middle row. Letting $I = G_{n-1},\ F = F_{n-1},\ I' = G_n,\ F' = F_n$, we get the desired result. □

2.13. Corollary. *Let* $0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow 0$ *be a short exact sequence of right R-modules, where I_0 and I_1 are $\mathcal{Y}$-Gorenstein injective modules and* $\mathrm{Ext}^1_R(I, M) = 0$ *for all injective right R-modules I. Then M is $\mathcal{Y}$-Gorenstein injective.*

Proof. From the short exact sequence $0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow 0$, we know that $\mathcal{Y}$-$\mathrm{Gid}(M) \leq 1$. By Proposition 2.12, there exists an exact sequence

$$0 \longrightarrow M \longrightarrow G \longrightarrow I \longrightarrow 0,$$

where G is $\mathcal{Y}$-Gorenstein injective and I is injective. By the assumption $\mathrm{Ext}^1_R(I, M) = 0$, this sequence splits, and hence M is $\mathcal{Y}$-Gorenstein injective. □

2.14. Lemma. *Let N be a right R-module, and consider two exact sequences of right R-modules:*

$$0 \longrightarrow N \longrightarrow G^0 \longrightarrow \ldots \longrightarrow G^{n-1} \longrightarrow G^n \longrightarrow 0,$$

$$0 \longrightarrow N \longrightarrow H^0 \longrightarrow \ldots \longrightarrow H^{n-1} \longrightarrow H^n \longrightarrow 0,$$

where $G^0, \ldots, G^{n-1}$ and $H^0, \ldots, H^{n-1}$ are $\mathcal{Y}$-Gorenstein injective. Then G^n is $\mathcal{Y}$-Gorenstein injective if and only if H^n is $\mathcal{Y}$-Gorenstein injective.

Proof. Using Proposition 2.10, the proof is similar to that of (i) $\Longrightarrow$ (iii) in [3, Theorem 1.2.7]. □

2.15. Proposition. *Let N be a right R-module with finite $\mathcal{Y}$-Gorenstein injective dimension. Then the following assertions are equivalent for a nonnegative integer n:*

1. $\mathcal{Y}$-$\mathrm{Gid}(N) \leq n$.
2. $\mathrm{Ext}^i_R(I,\ N) = 0$ *for all $i > n$ and all injective right R-modules I.*
3. $\mathrm{Ext}^i_R(E,\ N) = 0$ *for all $i > n$ and all right R-modules E of finite injective dimension.*
4. *For every exact sequence of right R-modules*

$$0 \longrightarrow N \longrightarrow G^0 \longrightarrow \ldots \longrightarrow G^{n-1} \longrightarrow K^n \longrightarrow 0,$$

where each G^i is $\mathcal{Y}$-Gorenstein injective, K^n is $\mathcal{Y}$-Gorenstein injective.

Furthermore, $\mathcal{Y}$-$Gid(N) = \sup\{i \in \mathbb{N} \mid \mathrm{Ext}^i_R(E,\ N) \neq 0$ *for some R-module E of finite injective dimension*$\} = \sup\{i \in \mathbb{N} \mid \mathrm{Ext}^i_R(I,\ N) \neq 0$ *for some injective R-module I*$\}$.

Proof. Follows from the proof of [12, Theorem 2.20] using Corollary 2.13 and Lemma 2.14. □

2.16. Corollary. *Let N be a right R-module with $\mathcal{Y}$-$Gid(N) < \infty$. Then* $\mathrm{Gid}(N) = \mathcal{Y}$-*$Gid(N)$.*

Proof. Since $\mathrm{Gid}(N) \leq \mathcal{Y}$-$\mathrm{Gid}(N)$, then from Proposition 2.15 and [12, Theroem 2.22] we know that $\mathrm{Gid}(N) = \mathcal{Y}$-$\mathrm{Gid}(N)$. □

2.17. Proposition. *Every right R-module with finite $\mathcal{Y}$-Gorenstein injective dimension has a special $\mathcal{Y}$-Gorenstein injective preenvelope.*

Proof. Let M be a right R-module with finite $\mathcal{Y}$-Gorenstein injective dimension. Then there exists an exact sequence

$$0 \longrightarrow M \longrightarrow I \longrightarrow F \longrightarrow 0$$

with I $\mathcal{Y}$-Gorenstein injective and $\mathrm{id}(F) \leq \mathcal{Y}\text{-Gid}(M) - 1$. Now if G' is a $\mathcal{Y}$-Gorenstein injective right R-module, then $\mathrm{Ext}^1_R(F, G') = 0$ which shows that $M \to I$ is a special $\mathcal{Y}$-Gorenstein injective preenvelope. □

Let $\mathcal{Y}$ be the class of all FP-injective right R-modules in Proposition 2.17. Then we have:

2.18. Corollary. *Every right R-module with finite Gorenstein FP-injective dimension has a Gorenstein FP-injective preenvelope.* □

If $\mathcal{Y}$ is the class of all injective right R-modules in Proposition 2.17, then we get

2.19. Corollary. [12, Theorem 2.15] *Every right R-module with finite Gorenstein injective dimension has a Gorenstein injective preenvelope.* □

Let $\mathcal{Y}\text{-}\mathcal{GI}(R)$ be the class of $\mathcal{Y}$-Gorenstein injective right R-modules.

2.20. Theorem. *If $r\mathcal{Y}\text{-GID}(R) < \infty$, then $\big({}^{\perp}(\mathcal{Y}\text{-}\mathcal{GI}(R)), \mathcal{Y}\text{-}\mathcal{GI}(R)\big)$ is a complete hereditary cotorsion theory.*

Proof. We first prove that $\big({}^{\perp}(\mathcal{Y}\text{-}\mathcal{GI}(R)), \mathcal{Y}\text{-}\mathcal{GI}(R)\big)$ is a cotorsion theory.

Obviously, $\mathcal{Y}\text{-}\mathcal{GI}(R) \subseteq \big({}^{\perp}(\mathcal{Y}\text{-}\mathcal{GI}(R))\big)^{\perp}$. So we only need to prove that $\big({}^{\perp}(\mathcal{Y}\text{-}\mathcal{GI}(R))\big)^{\perp} \subseteq \mathcal{Y}\text{-}\mathcal{GI}(R)$. Let M be any module in $\big({}^{\perp}(\mathcal{Y}\text{-}\mathcal{GI}(R))\big)^{\perp}$, then $\mathrm{Ext}^1_R(N, M) = 0$ for any $N \in {}^{\perp}\big(\mathcal{Y}\text{-}\mathcal{GI}(R)\big)$. Since $r\mathcal{Y}\text{-GID}(R) < \infty$, M has a finite $\mathcal{Y}$-Gorenstein injective dimension. Then by Proposition 2.17, there exists a special $\mathcal{Y}$-Gorenstein injective preenvelope

$$0 \longrightarrow M \longrightarrow I \longrightarrow F \longrightarrow 0$$

with I $\mathcal{Y}$-Gorenstein injective and $\mathrm{id}(F) \leq \mathcal{Y}\text{-Gid}(M) - 1$. Then we have a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(F, M) \longrightarrow \mathrm{Hom}_R(F, I) \longrightarrow \mathrm{Hom}_R(F, F) \longrightarrow \mathrm{Ext}^1_R(F, M) \longrightarrow \cdots.$$

Since $F \in {}^{\perp}\big(\mathcal{Y}\text{-}\mathcal{GI}(R)\big)$, $\mathrm{Ext}^1_R(F, M) = 0$. Thus M is a direct summand of I, hence M is $\mathcal{Y}$-Gorenstein injective by Proposition 2.10. Hence $\big({}^{\perp}(\mathcal{Y}\text{-}\mathcal{GI}(R))\big)^{\perp} = \mathcal{Y}\text{-}\mathcal{GI}(R)$, and so $\big({}^{\perp}(\mathcal{Y}\text{-}\mathcal{GI}(R)), \mathcal{Y}\text{-}\mathcal{GI}(R)\big)$ is a complete cotorsion theory.

By Proposition 2.10, we know that the class of all $\mathcal{Y}$-Gorenstein injective modules is closed under cokernels of monomorphisms. Thus $\big({}^{\perp}(\mathcal{Y}\text{-}\mathcal{GI}(R)), \mathcal{Y}\text{-}\mathcal{GI}(R)\big)$ is a complete hereditary cotorsion theory. □

3. $\mathcal{X}$-Gorenstein projective modules

3.1. Definition. [2, Definition 2.1] Let $\mathcal{X}$ be a class of right R-modules that contains all projective right R-modules. A right R-module M is called $\mathcal{X}$-*Gorenstein projective* if there exists an exact sequence

$$\mathcal{P} \equiv \cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^{0} \longrightarrow P^{1} \longrightarrow \cdots$$

of projective right R-modules such that $M = \ker(P^0 \to P^1)$ and $\mathrm{Hom}_R(\mathcal{P}, F)$ is exact whenever $F \in \mathcal{X}$.

The sequence $\mathcal{P}$ is called an $\mathcal{X}$-*complete projective resolution*.

3.2. Lemma. [2, Theorem 2.3] *1. Let* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *be a short exact sequence of R-modules, where C is* $\mathcal{X}$*-Gorenstein projective. Then, A is* $\mathcal{X}$*-Gorenstein projective if and only if B is* $\mathcal{X}$*-Gorenstein projective.*

2. Let $(M_i)_{i\in I}$ *be a family of R-modules. Then,* $\bigoplus_{i\in I} M_i$ *is* $\mathcal{X}$*-Gorenstein projective if and only if* M_i *is* $\mathcal{X}$*-Gorenstein projective for every* $i \in I$. □

3.3. Remark. (1) Obviously, we have the following implications:

projective modules $\Longrightarrow$ $\mathcal{X}$-Gorenstein projective modules $\Longrightarrow$ Gorenstein projective modules.

(2) Let $\mathcal{X}$ be the class of projective right R-modules. Then the class of $\mathcal{X}$-Gorenstein projective right R-modules coincides with the class of Gorenstein projective right R-modules.

(3) Let $\mathcal{X}$ be the class of flat right R-modules. Then the class of $\mathcal{X}$-Gorenstein projective right R-modules coincides with the class of strongly Gorenstein flat right R-modules [4].

(4) If $\mathcal{X}$ is the class of Gorenstein projective right R-modules, then every $\mathcal{X}$-Gorenstein projective right R-module is projective.

3.4. Proposition. *A right R-module M is projective if and only if M belongs to* $\mathcal{X}$ *and M is* $\mathcal{X}$*-Gorenstein projective.*

Proof. The proof is similar to that of Proposition 2.3. □

3.5. Corollary. *The following statements are equivalent for a ring R:*

(1) $\mathcal{X}$ *is the class of projective right R-modules.*
(2) *Every* $X \in \mathcal{X}$ *is* $\mathcal{X}$*-Gorenstein projective.*

Proof. (1) $\Longrightarrow$ (2) Trivial by Remark 3.3 (1).

(2) $\Longrightarrow$ (1) By Proposition 3.4, we know that $\mathcal{X}$ is the class of projective right R-modules. □

Let $\mathcal{X}$ be the class of flat right R-modules in Corollary 3.5, then we have

3.6. Corollary. [4, Proposition 2.15] *The following statements are equivalent for a ring R:*

(1) *R is right perfect.*
(2) *Every flat right R-module is strongly Gorenstein flat.* □

3.7. Theorem. *Let R be a ring with* $\mathrm{r.Ggldim}(R) < \infty$*. Then the following statements are equivalent:*

(1) *The class of* $\mathcal{X}$*-Gorenstein projective right R-modules coincides with the class of Gorenstein projective right R-modules.*
(2) *Every module in* $\mathcal{X}$ *has finite projective dimension.*

Proof. Using [15, Proposition 2.5], the proof is similar to that of Theorem 2.6. □

Let $\mathcal{X}$ be the class of flat right R-modules in Theorem 3.7. Then we have the following result which generalizes [4, Corollary 2.8]:

3.8. Corollary. *Let R be a ring with* $\mathrm{r.Ggldim}(R) < \infty$*. Then the class of strongly Gorenstein flat modules coincides with the class of Gorenstein projective modules.*

Proof. By [1, Corollary 2.7], we know that if $\mathrm{r.Ggldim}(R) < \infty$, then $\mathrm{pd}(F) < \infty$ for any flat right R-module F. Thus by Theorem 3.7 we get the desired results. □

3.9. Corollary. *Let R be a ring with* $\text{r.Ggldim}(R) < \infty$ *and* $\text{wdim}(R) < \infty$*. Then the class of $\mathcal{X}$-Gorenstein projective right R-modules coincides with the class of Gorenstein projective right R-modules.*

Proof. Using Theorem 3.7, the proof is similar to that of Corollary 2.7. □

3.10. Definition. We will say that M has $\mathcal{X}$*-Gorenstein projective dimension less than or equal to* n, denoted $\mathcal{X}\text{-Gpd}(M) \leq n$, if there exists an exact sequence

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with every P_i being $\mathcal{X}$-Gorenstein projective. If no such finite sequence exists, define $\mathcal{X}\text{-Gpd}(M) = \infty$; otherwise, if n is the least such integer, define $\mathcal{X}\text{-Gpd}(M) = n$.

Define $r\mathcal{X}\text{-GPD}(R) = \sup\{\mathcal{X}\text{-Gpd}(M) \mid M \text{ is any right } R\text{-module}\}$.

3.11. Proposition. *Let M be a right R-module with finite $\mathcal{X}$-Gorenstein projective dimension n, then there exist exact sequences*

$$0 \longrightarrow H \longrightarrow G \longrightarrow M \longrightarrow 0$$

with G $\mathcal{X}$-Gorenstein projective and $pd(H) \leq n-1$ and

$$0 \longrightarrow M \longrightarrow H' \longrightarrow G' \longrightarrow 0$$

with G' $\mathcal{X}$-Gorenstein projective and $\text{pd}(H') \leq n$.

Proof. Using the fact that the class of $\mathcal{X}$-Gorenstein projective modules is closed under extensions, the proof is similar to that of Proposition 2.12. □

3.12. Corollary. *Let*

$$0 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

be a short exact sequence of right R-modules, where G_0 and G_1 are $\mathcal{X}$-Gorenstein projective modules and $\text{Ext}^1_R(M,Q) = 0$ *for all projective right R-modules Q. Then M is $\mathcal{X}$-Gorenstein projective.* □

3.13. Lemma. *Let M be a right R-module, and consider two exact sequences of right R-modules:*

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \ldots \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$

$$0 \longrightarrow H_n \longrightarrow H_{n-1} \longrightarrow \ldots \longrightarrow H_0 \longrightarrow M \longrightarrow 0,$$

where $G_0, \ldots, G_{n-1}$ and $H_0, \ldots, H_{n-1}$ are $\mathcal{X}$-Gorenstein projective, then G_n is $\mathcal{X}$-Gorenstein projective if and only if H_n is $\mathcal{X}$-Gorenstein projective.

Proof. Using Lemma 3.2, the proof is similar to that of (i) $\Longrightarrow$ (iii) in [3, Theorem 1.2.7]. □

3.14. Proposition. *Let M be a right R-module with finite $\mathcal{X}$-Gorenstein projective dimension, then the following assertions are equivalent for a nonnegative integer n:*

(1) $\mathcal{X}\text{-Gpd}(M) \leq n$.
(2) $\text{Ext}^i_R(M,P) = 0$ *for all $i > n$ and all projective right R-modules P.*
(3) $\text{Ext}^i_R(M,Q) = 0$ *for all $i > n$ and all right R-modules Q of finite projective dimension.*
(4) *For every exact sequence of right R-modules*

$$0 \longrightarrow K_n \longrightarrow G_{n-1} \longrightarrow \ldots \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$

where each G_i is $\mathcal{X}$-Gorenstein projective, K_n is $\mathcal{X}$-Gorenstein projective.

Furthermore, $\mathcal{X}$-Gpd$(M) = \sup\{i \in \mathbb{N} \mid \mathrm{Ext}_R{}^i(M,Q) \neq 0$ *for some* R*-module* Q *of finite projective dimension* $\} = \sup\{i \in \mathbb{N} \mid \mathrm{Ext}_R{}^i(M,P) \neq 0$ *for some projective* R*-module* $P\}$.

Proof. Using Corollary 3.12 and Lemma 3.13, the proof is similar to that of Proposition 2.15. □

3.15. Corollary. *Let* M *be a right* R*-module with* $\mathcal{X}$*-Gpd*$(M) < \infty$*, then* $\mathrm{Gpd}(M) = \mathcal{X}$*-Gpd*$(M)$.

Proof. Since $\mathrm{Gpd}(M) \leq \mathcal{X}\text{-Gpd}(M)$, then from Proposition 3.14 and [12, Theroem 2.20] we know that $\mathrm{Gpd}(M) = \mathcal{X}\text{-Gpd}(M)$. □

3.16. Proposition. *Every right* R*-module with finite* $\mathcal{X}$*-Gorenstein projective dimension has a special* $\mathcal{X}$*-Gorenstein projective precover.*

Proof. Let M be a right R-module with finite $\mathcal{X}$-Gorenstein projective dimension. Then there exists an exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow M \longrightarrow 0$$

with G $\mathcal{X}$-Gorenstein projective and $\mathrm{pd}(H) \leq \mathcal{X}\text{-Gpd}(M) - 1$. Now if G' is an $\mathcal{X}$-Gorenstein projective right R-module, then $\mathrm{Ext}^1_R(G', H) = 0$, which shows that $G \to M$ is a special $\mathcal{X}$-Gorenstein projective precover. □

Let $\mathcal{X}$ be the class of flat right R-modules in Proposition 3.16, then we have

3.17. Corollary. *Every right* R*-module with finite strongly Gorenstein flat dimension has a strongly Gorenstein flat precover.*

If $\mathcal{X}$ is the class of projective right R-modules in Proposition 3.16, then we get

3.18. Corollary. [12, Theorem 2.10] *Every right* R*-module with finite Gorenstein projective dimension has a Gorenstein projective precover.* □

Let $\mathcal{X}$-$\mathcal{GP}(R)$ be the class of $\mathcal{X}$-Gorenstein projective right R-modules.

3.19. Theorem. *If* $r\mathcal{X}$*-GPD*$(R) < \infty$*, then* $\big(\mathcal{X}\text{-}\mathcal{GP}(R), (\mathcal{X}\text{-}\mathcal{GP}(R))^{\perp}\big)$ *is a complete hereditary cotorsion theory.*

Proof. The proof is similar to that of Theorem 2.20. □

4. $\mathcal{Y}$-Gorenstein flat modules

4.1. Definition. Let $\mathcal{Y}$ be a class of right R-modules that contains all injective right R-modules. A left R-module M is called $\mathcal{Y}$*-Gorenstein flat* if there exists an exact sequence

$$\mathcal{F} \equiv \ \cdots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^{0} \longrightarrow F^{1} \longrightarrow \cdots$$

of flat left R-modules such that $M = \ker(F^0 \to F^1)$, which remains exact whenever $G \otimes_R -$ is applied for any $G \in \mathcal{Y}$.

The sequence $\mathcal{F}$ is called a $\mathcal{Y}$*-complete flat resolution*.

4.2. Proposition. *Let* $\mathcal{X}$ *be a class of right* R*-modules that contains all projective right* R*-modules, and* $\mathcal{Y}$ *a class of left* R*-modules that contains all injective left* R*-modules. If* $\mathcal{Y}^+ \subseteq \mathcal{X}$*, then every* $\mathcal{X}$*-Gorenstein projective right* R*-module is* $\mathcal{Y}$*-Gorenstein flat.*

Proof. Let M be any $\mathcal{X}$-Gorenstein projective right R-module, then there exists an exact sequence

$$\mathcal{P} \equiv \ \cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^{0} \longrightarrow P^{1} \longrightarrow \cdots$$

of projective right R-modules such that $M = \ker(P^0 \to P^1)$ and $\mathrm{Hom}_R(\mathcal{P}, F)$ is exact whenever $F \in \mathcal{X}$.

For any $E \in \mathcal{Y}$, since $\mathcal{Y}^+ \subseteq \mathcal{X}$, we get that $E^+ \in \mathcal{X}$. Applying the functor $\mathrm{Hom}_R(-, E^+)$ to the exact sequence $\mathcal{P}$ gives an exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_R(P^1, E^+) \longrightarrow \mathrm{Hom}_R(P^0, E^+) \longrightarrow \mathrm{Hom}_R(P^{-1}, E^+) \longrightarrow \cdots .$$

But the above sequence is naturally isomorphic to

$$\cdots \longrightarrow (P^1 \otimes_R E)^+ \longrightarrow (P^0 \otimes_R E)^+ \longrightarrow (P^{-1} \otimes_R E)^+ \longrightarrow \cdots .$$

Therefore we have an exact sequence

$$\cdots \longrightarrow P^{-1} \otimes_R E \longrightarrow P^0 \otimes_R E \longrightarrow P^1 \otimes_R E \longrightarrow \cdots .$$

Thus M is $\mathcal{Y}$-Gorenstein flat. □

4.3. Corollary. [4, Proposition 2.3] *Let R be a left coherent ring. Then every strongly Gorenstein flat right R-module is Gorenstein flat.*

Proof. Let $\mathcal{X}$ be the class of flat right R-modules and $\mathcal{Y}$ the class of FP-injective left R-modules. If R is a left coherent ring, then $\mathcal{Y}^+ \subseteq \mathcal{X}$ [10, Theorem 2.2]. From Proposition 4.2, we get the desired results. □

4.4. Proposition. *For any left R-module M, we consider the following conditions.*

(1) *M is a $\mathcal{Y}$-Gorenstein flat left R-module.*
(2) *M^+ is a $\mathcal{Y}$-Gorenstein injective right R-module.*

Then (1) $\Longrightarrow$ (2). If R is right coherent, then also (2) $\Longrightarrow$ (1).

Proof. The proof is similar to that of [12, Theorem 3.6]. □

4.5. Proposition. *If R is right coherent, then the class of $\mathcal{Y}$-Gorenstein flat left R-modules is closed under extensions, kernels of epimorphisms, direct sums and direct summands.*

Proof. From Proposition 2.10 and the equivalence in Proposition 4.4, we get that the class of $\mathcal{Y}$-Gorenstein flat left R-modules is closed under extensions and kernels of epimorphisms. By the definition of $\mathcal{Y}$-Gorenstein flat left R-modules, we easily get that the class of $\mathcal{Y}$-Gorenstein flat left R-modules is closed under arbitrary direct sums. Now, comparing this fact with [12, Proposition 1.4], we get that the class of $\mathcal{Y}$-Gorenstein flat left R-modules is closed under direct summands. □

4.6. Proposition. *Let R be a right coherent ring and*

$$0 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

a short exact sequence, where G_0 and G_1 are $\mathcal{Y}$-Gorenstein flat left R-modules and $\mathrm{Tor}_1^R(I, M) = 0$ for all injective right R-modules I. Then M is a $\mathcal{Y}$-Gorenstein flat left R-module.

Proof. Let $H_0 = G_0^+$ and $H_1 = G_1^+$. Then from Proposition 4.4 we know that H_0 and H_1 are $\mathcal{Y}$-Gorenstein injective right R-modules. Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ to the exact sequence

$$0 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$

we get a short exact sequence

$$0 \longrightarrow M^+ \longrightarrow H_0 \longrightarrow H_1 \longrightarrow 0,$$

where H_0 and H_1 are $\mathcal{Y}$-Gorenstein injective right R-modules. By [7, Theorem 3.2.1], we have an isomorphism

$$\mathrm{Ext}_R^1(I, M^+) \cong \mathrm{Tor}_1^R(I, M)^+ = 0$$

for all injective right R-modules I. Thus from Corollary 2.13 we get that M^+ is a $\mathcal{Y}$-Gorenstein injective right R-module. Hence M is a $\mathcal{Y}$-Gorenstein flat left R-module by Proposition 4.4. □

4.7. Definition. We will say that M has *$\mathcal{Y}$-Gorenstein flat dimension less than or equal to n*, denoted by $\mathcal{Y}$-Gfd$(M) \leq n$, if there exists an exact sequence

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with every F_i being $\mathcal{Y}$-Gorenstein flat. If no such finite sequence exists, define $\mathcal{Y}$-Gfd$(M) = \infty$; otherwise, if n is the least such integer, define $\mathcal{Y}$-Gfd$(M) = n$.

Define $l\mathcal{Y}$-GFD$(R) = \sup\{\mathcal{Y}\text{-Gfd}(M) \mid M \text{ is any left } R\text{-module}\}$.

4.8. Proposition. *Let R be a right coherent ring and M a left R-module with finite $\mathcal{Y}$-Gorenstein flat dimension n. Then there exist exact sequences*

$$0 \longrightarrow H \longrightarrow G \longrightarrow M \longrightarrow 0$$

with G $\mathcal{Y}$-Gorenstein flat, $\mathrm{fd}(H) \leq n-1$ *and*

$$0 \longrightarrow M \longrightarrow H' \longrightarrow G' \longrightarrow 0$$

with G' $\mathcal{Y}$-Gorenstein flat and $\mathrm{fd}(H') \leq n$.

Proof. Using the fact that the class of $\mathcal{Y}$-Gorenstein flat left R-modules is closed under extensions over a right coherent ring R, the proof is similar to that of Proposition 2.12. □

4.9. Proposition. *Let R be a right coherent ring and N a left R-module with finite $\mathcal{Y}$-Gorenstein flat dimension, then the following assertions are equivalent for a nonnegative integer n:*

(1) $\mathcal{Y}$-Gfd$(N) \leq n$.
(2) $\mathrm{Tor}_i^R(I,\ N) = 0$ *for all $i > n$ and all injective right R-modules I.*
(3) $\mathrm{Tor}_i^R(E,\ N) = 0$ *for all $i > n$ and all right R-modules E of finite injective dimension.*
(4) *For every exact sequence of left R-modules*

$$0 \longrightarrow K_n \longrightarrow G_{n-1} \longrightarrow \dots \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$

where each G_i is $\mathcal{Y}$-Gorenstein flat, K_n is $\mathcal{Y}$-Gorenstein flat.

Furthermore, $\mathcal{Y}$-Gfd$(N) = \sup\{i \in \mathbb{N} \mid \mathrm{Tor}_i^R(E,\ N) \neq 0$ *for some R-module E of finite injective dimension*$\} = \sup\{i \in \mathbb{N} \mid \mathrm{Tor}_i^R(I,\ N) \neq 0$ *for some injective R-module I*$\}$.

Proof. From Propositions 4.4 and 2.15, we know that $\mathcal{Y}$-Gfd$(N) = \mathcal{Y}$-Gid(N^+). Using Proposition 2.15 and the adjointness isomorphism

$$\mathrm{Ext}^1_R(L, M^+) \cong \mathrm{Tor}^R_1(L, M)^+$$

for all right R-modules L, we can easily get the desired results. □

Let $\mathcal{Y}$-$\mathcal{GF}(R)$ be the class of $\mathcal{Y}$-Gorenstein flat left R-modules.

4.10. Corollary. *Let R be a right coherent ring. If $l\mathcal{Y}$-GFD$(R) < \infty$, then $\mathcal{Y}$-$\mathcal{GF}(R)$ is closed under direct limits.* □

4.11. Lemma. [9, Proposition 3.2.2] *Let R be any ring, $\aleph_\beta$ an infinite cardinal number such that* $\mathrm{Card}(R) \leq \aleph_\beta$*, and M any R-module. Then, for any submodule $A \leq M$ with* $\mathrm{Card}(A) \leq \aleph_\beta$*, there exists a pure submodule $S \leq M$ such that $A \leq S$ and* $\mathrm{Card}(S) \leq \aleph_\beta$. □

4.12. Proposition. *Let R be a right coherent ring and S a pure submodule of $F \in \mathcal{Y}$-$\mathcal{GF}(R)$. Then $S \in \mathcal{Y}$-$\mathcal{GF}(R)$ and $F/S \in \mathcal{Y}$-$\mathcal{GF}(R)$.*

Proof. From the pure exact sequence $0 \longrightarrow S \longrightarrow F \longrightarrow F/S \longrightarrow 0$, we get a split exact sequence $0 \longrightarrow (F/S)^+ \longrightarrow F^+ \longrightarrow S^+ \longrightarrow 0$. By Proposition 4.4, we know F^+ is a $\mathcal{Y}$-Gorenstein injective right R-module. From the split exact sequence, we know that both S^+ and $(F/S)^+$ are direct summands of F^+. Thus by Proposition 2.10, S^+ and $(F/S)^+$ are $\mathcal{Y}$-Gorenstein injective right R-modules. By Proposition 4.4 again, $S \in \mathcal{Y}$-$\mathcal{GF}(R)$ and $F/S \in \mathcal{Y}$-$\mathcal{GF}(R)$. □

4.13. Theorem. *Let R be a right coherent ring and $l\mathcal{Y}$-GFD$(R) < \infty$. Then*

$$\big(\mathcal{Y}\text{-}\mathcal{GF}(R), \mathcal{Y}\text{-}\mathcal{GF}(R)^\perp\big)$$

is a perfect complete hereditary cotorsion theory.

Proof. Let $\mathrm{Card}(R) = \aleph_\beta$ and let X be a set of representatives of all modules $G \in \mathcal{Y}$-$\mathcal{GF}(R)$ with $\mathrm{Card}(G) \leq \aleph_\beta$. Then by a proof analogous to that of [9, Theroem 3.2.3], we get that $\mathcal{Y}$-$\mathcal{GF}(R)^\perp = X^\perp$. Thus $\big(\mathcal{Y}\text{-}\mathcal{GF}(R), \mathcal{Y}\text{-}\mathcal{GF}(R)^\perp\big)$ is cogenerated by a set. From Proposition 4.5, Corollary 4.10 and the fact that $\mathcal{Y}$-$\mathcal{GF}(R)$ contains all projective modules, we know that $\big(\mathcal{Y}\text{-}\mathcal{GF}(R), \mathcal{Y}\text{-}\mathcal{GF}(R)^\perp\big)$ is a perfect complete cotorsion theory by [9, Corollary 3.1.11 and Proposition 3.1.13]. By Proposition 4.5, we know that if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence with $B,\ C \in \mathcal{Y}$-$\mathcal{GF}(R)$, then $A \in \mathcal{Y}$-$\mathcal{GF}(R)$. Thus

$$\big(\mathcal{Y}\text{-}\mathcal{GF}(R), \mathcal{Y}\text{-}\mathcal{GF}(R)^\perp\big)$$

is a hereditary cotorsion theory. □

Acknowledgements

This work was partially supported by the National Science Foundation of China (Grant No. 11071111).

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