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SOME $\triangle$-CONVERGENCE THEOREMS IN CAT(0) SPACES

Mujahid Abbas[*†] and Safeer Hussain Khan[‡]

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Abstract

In this paper, we use an iteration process for approximating common fixed points of two nonexpansive mappings by $\triangle$- and strong convergence in CAT(0) spaces. This process is independent of and simpler than the Ishikawa type iteration process.

Keywords: Iteration Process, Nonexpansive mappings, Common fixed points, $\triangle$-convergence, CAT(0) space.

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1. Introduction and Preliminaries

The notion of $\triangle$-convergence in general metric spaces was introduced by Lim [14] in 1976. Kirk and Panyanak [11] specialized this concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. Dhompongsa and Panyanak [7] continued to work in this direction. Their results involved Mann and Ishikawa iteration schemes involving one mapping. In this paper, we explore common fixed points of two nonexpansive mapping by an iteration scheme which is both independent and simpler than the Ishikawa type iteration scheme.

For the sake of completeness, let us recall some definitions and known results in the existing literature on this subject :

1.1. Definition. A metric space X is called a CAT(0) space [10] if it is geodesically connected and every geodesic triangle in X is at least as "thin" as its comparison triangle in Euclidean plane. For a vigorous discussion, see Bridson and Haefliger [1], Bruhat and Tits [2], and Burago-Burago-Ivanov [3]. The complex Hilbert ball with a hyperbolic metric is a CAT(0) space, see [9] and [15].

*Department of Mathematics, Lahore University of Management Sciences, 54792- Lahore, Pakistan. E-mail: `mujahid@lums.edu.pk`

†Corresponding Author.

‡Current Address: Department of Mathematics, Statistics and Physics, Qatar University, Doha 2713, Qatar. E-mail: `safeer@qu.edu.qa` `safeerhussain5@yahoo.com`

1.2. Definition. Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from the closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0)=x, c(l)=y$, and $d(c(t),c(t'))=|t-t'|$ for all $t,t' \in [0,l]$. In particular, c is an isometry and $d(x,y)=l$. The image of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x,y]$. For further details, we refer to [7].

Following are some elementary facts about CAT(0) spaces, cf. Dhompongsa and Panyanak [7].

1.3. Lemma. *Let (X,d) be a CAT(0) space. Then*

(i) *(X,d) is uniquely geodesic.*

(ii) *Let p,x,y be points of X, $\alpha \in [0,1]$, and let m_1 and m_2 denote, respectively, the points of $[p,x]$ and $[p,y]$ satisfying $d(p,m_1)=\alpha d(p,x)$ and $d(p,m_2)=\alpha d(p,y)$. Then*

$$d(m_1,m_2) \leq \alpha d(x,y). \tag{1.1}$$

(iii) *Let $x,y \in X, x \neq y$ and $z,w \in [x,y]$ be such that $d(x,z)=d(x,w)$. Then $z=w$.*

(iv) *Let $x,y \in X$. For each $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that*

$$d(x,z)=td(x,y) \quad \text{and} \quad d(y,z)=(1-t)d(x,y). \tag{1.2}$$

□

For convenience, from now on we will use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (1.2).

Dhompongsa and Panyanak [7] studied the $\triangle$-convergence of Picard, Mann and Ishikawa iterates (Theorems 3.1, 3.2 and 3.3 respectively in [7]). While acknowledging their contribution, we note that their schemes involve one mapping. The case of two mappings in iteration processes has also remained under study since Das and Debata [4] gave and studied a two mappings scheme on the pattern of the Ishikawa scheme:

$$\begin{cases} x_{n+1}=(1-\alpha_n)x_n+\alpha_n Ty_n, \\ y_n=(1-\beta_n)x_n+\beta_n Sx_n, \ n \in \mathbb{N}. \end{cases} \tag{1.3}$$

Also see, for example, Takahashi and Tamura [17] and Khan and Takahashi [13]. This scheme reduces to Ishikawa's scheme when $S=T$, and to Mann's Iteration scheme when $S=I$. Note that the two mappings case, that is, approximating the common fixed points, has its own importance as it has a direct link with the minimization problem, see for example Takahashi [16].

We, in this paper, obtain common fixed points of nonexpansive mappings using the following iteration scheme: Let x_0 be an arbitrary point in C and

$$x_{n+1}=c_n x_n \oplus (1-c_n)[\frac{a_n}{1-c_n}Tx_n \oplus \frac{b_n}{1-c_n}Sx_n], \ n \in \mathbb{N}, \tag{1.4}$$

where $\mathbb{N}$ stands for the set of natural numbers, $\{a_n\},\{b_n\},\{c_n\}$ are sequences in $(0,1)$ with $a_n+b_n+c_n=1$. Note that, $\frac{a_n}{1-c_n}Tx_n \oplus \frac{b_n}{1-c_n}Sx_n = u$ (say) is a point on the segment $[Tx_n, Sx_n]$, and $x_{n+1}=c_n x_n \oplus (1-c_n)u$ is a point on the segment $[u,x_n]$.

Also, note that this scheme reduces to Mann's iteration scheme when either $S=T$ or $S=I$ or $T=I$, and is independent of and simpler than both the Ishikawa scheme and (1.3).

The proofs of the following two lemmas can be found in Dhompongsa and Panyanak [7].

1.4. Lemma. [7, Lemma 2.4] *Let X be a CAT(0) space. Then* $d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z)$ *for all* $x, y, z \in X$ *and* $t \in [0,1]$. □

1.5. Lemma. [7, Lemma 2.5] *Let X be a CAT(0) space. Then*

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2$$

for all $x, y, z \in X$ *and* $t \in [0,1]$.

1.6. Remark. Let X be a CAT(0) space. Let $x, y, z, w \in X$ and $a, b, c \in [0,1]$ with $a+b+c = 1$ and $c \neq 1$. Then $cz \oplus (1-c)\left(\frac{a}{1-c}x \oplus \frac{b}{1-c}y\right)$ is a point on the segment $[u, c]$, where $u = \frac{a}{1-c}x \oplus \frac{b}{1-c}y$ is a point on the segment $[x, y]$. Also,

$$d(cz \oplus (1-c)\left(\frac{a}{1-c}x \oplus \frac{b}{1-c}y\right), w) \leq ad(x,w) + bd(y,w) + cd(z,w).$$

As:

$$\begin{aligned} d(cz \oplus (1-c)&\left(\frac{a}{1-c}x \oplus \frac{b}{1-c}y\right), w) \\ &= d(cz \oplus (1-c)u, w) \\ &\leq (1-c)d(u,w) + cd(z,w) \\ &= (1-c)d\left(\frac{a}{1-c}x \oplus \frac{b}{1-c}y, w\right) + cd(z,w) \\ &\leq (1-c)\left(\frac{a}{1-c}d(x,w) + \frac{b}{1-c}d(y,w)\right) + cd(z,w) \\ &= ad(x,w) + bd(y,w) + cd(z,w). \end{aligned}$$

1.7. Lemma. *Let X be a CAT(0) space. Then for all $x, y, z, w \in X$ and $a, b, c \in [0,1]$ with $a+b+c = 1$, the following holds.*

$$\begin{aligned} d((1-c)&\left(\frac{a}{1-c}x \oplus \frac{b}{1-c}y\right) \oplus cz, w)^2 \\ &\leq ad(x,w)^2 + bd(y,w)^2 + cd(z,w)^2 - \frac{ab}{1-c}d(x,y)^2. \end{aligned}$$

Proof. Let $x, y, z, w \in X$ and $a, b, c \in [0,1]$ with $a+b+c = 1$. As in the above lemma, to avoid division by zero, we take $c \neq 1$. We use Lemma 1.5 twice in the following.

$$\begin{aligned} d&\left[(1-c)\left(\frac{a}{1-c}x \oplus \frac{b}{1-c}y\right) \oplus cz, w)\right]^2 \\ &\leq (1-c)d\left(\frac{a}{1-c}x \oplus \frac{b}{1-c}y, w\right)^2 + cd(z,w)^2 \\ &- c(1-c)d\left(\frac{a}{1-c}x \oplus \frac{b}{1-c}y, z\right)^2 \\ &\leq (1-c)d\left(\frac{a}{1-c}x \oplus \frac{b}{1-c}y, w\right)^2 + cd(z,w)^2 \\ &\leq (1-c)\left[\frac{a}{1-c}d(x,w)^2 + \frac{b}{1-c}d(y,w)^2 - \frac{ab}{(1-c)^2}d(x,y)^2\right] + cd(z,w)^2 \\ &= ad(x,w)^2 + bd(y,w)^2 + cd(z,w)^2 - \frac{ab}{1-c}d(x,y)^2. \end{aligned}$$

□

Before we go to our main results, we need to recall the definitions and related concepts about $\triangle$-convergence. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\} = \inf\{r(x, \{x_n\} : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known (see, e.g. [6, Proposition 7]) that in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

A sequence $\{x_n\}$ in X is said to $\triangle$-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\triangle-\lim_n x_n = x$, and call x the $\triangle$-limit of $\{x_n\}$, see [11, 14].

The following lemma can be found, for example, in [7].

1.8. Lemma. [7, Lemma 2.7]

(i) *Every bounded sequence in a complete CAT(0) space X has a $\triangle$-convergent subsequence.*
(ii) *If C is a closed convex subset of X and $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*
(iii) *If C is a closed convex subset of X and $f : C \to X$ is a nonexpansive mapping, then the conditions, $\{x_n\}$ $\triangle$-converges to x and $d(x_n, f(x_n)) \to 0$, imply $x \in C$ and $f(x) = x$.* □

2. Main results

Now we are all set to prove our main results. We start with proving a key lemma for later use.

2.1. Lemma. *Let C be a nonempty closed convex subset of a complete CAT(0) space X and F the set of all common fixed points of two nonexpansive mappings T and S of C. Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some ε in $(0, 1)$ with $a_n + b_n + c_n = 1$, and let $\{x_n\}$ be defined by the iteration process (1.4). If $F \neq \phi$ then*

(i) $\lim_{n\to\infty} d(x_n, q)$ *exists for all $q \in F$.*
(ii) $\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, Sx_n) = 0.$

Proof. Let $q \in F$. Then by Lemma 1.7,

$$\begin{aligned} d(x_{n+1}, q)^2 &= d(c_n x_n \oplus (1-c_n)[\frac{a_n}{1-c_n} Tx_n \oplus \frac{b_n}{1-c_n} Sx_n], q)^2 \\ &\leq a_n d(x_n, q)^2 + b_n d(Tx_n, q)^2 + c_n d(Sx_n, q)^2 - \frac{a_n b_n}{1-c_n} d(x_n, Tx_n)^2 \\ &\leq d(x_n, q)^2 - \frac{a_n b_n}{1-c_n} d(x_n, Tx_n)^2. \end{aligned}$$

It follows that

$$d(x_{n+1}, q)^2 \leq d(x_n, q)^2 \tag{2.1}$$

and

$$\frac{a_n b_n}{1-c_n} d(x_n, Tx_n)^2 \leq d(x_n, q)^2 - d(x_{n+1}, q)^2. \tag{2.2}$$

The inequality (2.1) shows that $\{d(x_n, q)\}$ is decreasing, and this proves part (i).

Now (2.2) implies that

$$\begin{aligned} d(x, Tx_n)^2 &\le \frac{1-c_n}{a_n b_n}\left[d(x_n, q)^2 - d(x_{n+1}, q)^2\right] \\ &\le \frac{1-\varepsilon}{\varepsilon^2}\left[d(x_n, q)^2 - d(x_{n+1}, q)^2\right]. \end{aligned}$$

This gives $\limsup_{n\to\infty} d(x_n, Tx_n)^2 \le 0$ by part (i), so that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Since T and S play symmetric roles in the iteration process, using Lemma 1.7 and similar arguments to those given above, we obtain $\lim_{n\to\infty} d(x_n, Sx_n) = 0$. □

2.2. Theorem. *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let T and S be two nonexpansive mappings of C. Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some ε in $\left(0, \frac{1}{2}\right)$ with $a_n + b_n + c_n = 1$. If $F \neq \phi$, then $\{x_n\}$ defined by the iteration process (1.4) △-converges to a common fixed point of T and S.*

Proof. Let $q \in F$. Then by Lemma 2.1, $\lim\limits_{n\to\infty} d(x_n, q)$ exists for all $q \in F$. Thus $\{x_n\}$ is bounded. Therefore $\{x_n\}$ has a △-convergent subsequence. We now prove that every △-convergent subsequence of $\{x_n\}$ has a unique △-limit in F. For, let u and v be two △-limits of the subsequences $\{u_n\}$ and $\{v_n\}$ of $\{x_n\}$, respectively. By definition $A(\{u_n\}) = \{u\}$ and $A(\{v_n\}) = \{v\}$. By Lemma 2.1, $\lim\limits_{n\to\infty} d(u_n, Tu_n) = 0 = \lim\limits_{n\to\infty} d(u_n, Su_n)$. Now using the △-convergence of $\{u_n\}$ to u and the nonexpansiveness of T and S, we obtain $u \in F$ by a repeated application of Lemma 1.8 on T and S. Again in the same fashion, we can prove that $v \in F$.

Next, we prove the uniqueness. To this end, if u and v are distinct then by the uniqueness of asymptotic centers,

$$\begin{aligned} \lim_{n\to\infty} d(x_n, u) &= \limsup_{n\to\infty} d(u_n, u) \\ &< \limsup_{n\to\infty} d(u_n, v) \\ &= \limsup_{n\to\infty} d(x_n, v) \\ &= \limsup_{n\to\infty} d(v_n, v) \\ &< \limsup_{n\to\infty} d(v_n, u) \\ &= \limsup_{n\to\infty} d(x_n, u) \\ &= \lim_{n\to\infty} d(x_n, u) \end{aligned}$$

This is again a contradiction thereby completing the proof. □

2.3. Theorem. *Let X be a complete CAT(0) space and C, $\{x_n\}$, S and T be as in Lemma 2.1. If $F \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{d(x, p) : p \in F\}$.*

Proof. Necessity is obvious.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. As proved in Lemma 2.1, we have

$$d(x_{n+1}, p) \le d(x_n, p),$$

for all $p \in F$. This implies that

$$d(x_{n+1}, F) \le d(x_n, F),$$

so that $\lim_{n\to\infty} d(x_n, F)$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in C. Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists a positive integer n_0 such that

$$d(x_n, F) < \frac{\epsilon}{4}, \quad \forall n \geq n_0.$$

In particular, $\inf\{d(x_{n_0}, p) : p \in F\} < \frac{\epsilon}{4}$. Thus there must exist $p^* \in F$ such that

$$d(x_{n_0}, p^*) < \frac{\epsilon}{2}.$$

Now, for all $m, n \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(p^*, x_n) \\ &\leq 2d(x_{n_0}, p^*) \\ &< 2\left(\frac{\epsilon}{2}\right) = \epsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset C of a complete $CAT(0)$ space, and so it must converge to a point q in C. Now, $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(q, F) = 0$. Since F is closed, so we have $q \in F$. □

Khan and Fukhar-ud-din [12], introduced the so-called condition (A') and gave a slightly improved version of it in [8] as follows:

Two mappings $S, T : C \to C$ are said to satisfy the condition (A') if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that either $d(x, Tx) \geq f(d(x, F)$ or $d(x, Sx) \geq f(d(x, F)$ for all $x \in C$.

We use the condition (A') to study strong convergence of $\{x_n\}$ defined in (1.4). It is worth noting that, in the case of nonexpansive mappings $S, T : C \to C$, the condition (A') is weaker than the compactness of C.

2.4. Theorem. *Let X be a complete $CAT(0)$ space, C and $\{x_n\}$ be as in Lemma 2.1. Let $S, T : C \to C$ be two nonexpansive mappings satisfying the condition (A'). If $F \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. By Lemma 2.1, $\lim\limits_{n\to\infty} d(x_n, x^*)$ exists for all $x^* \in F$. Now, $d(x_{n+1}, x^*) \leq d(x_n, x^*)$ gives that

$$\inf_{x^*\in F} d(x_{n+1}, x^*) \leq \inf_{x^*\in F} d(x_n, x^*),$$

which means that $d(x_{n+1}, F) \leq d(x_n, F)$ and so $\lim_{n\to\infty} d(x_n, F)$ exists. By using the condition (A'), either

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} d(x_n, Sx_n) = 0.$$

In both cases, we have

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. The rest of the proof follows the pattern of the above theorem, and is therefore omitted. □

2.5. Remark. Theorems 2.2,2.3 and 2.4 contain the corresponding theorems proved for Mann's iteration process when any one of $T = S$, $T = I$ or $S = I$ holds.

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