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STABILITY OF EULER-LAGRANGE QUADRATIC FUNCTIONAL EQUATIONS IN NON-ARCHIMEDEAN NORMED SPACES

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Abstract

In this paper, we prove the stability of Euler-Lagrange quadratic mappings in the framework of non-Archimedean normed spaces. Our results in the setting of non-Archimedean normed spaces are different from the results in the setting of normed spaces.

Keywords: Generalized Hyers-Ulam stability, Euler-Lagrange functional equation. Non-Archimedean normed space, p-adic field.

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1. Introduction and preliminaries

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation $\mathcal{E}$ must be close to an exact solution of $\mathcal{E}$?" If there exists an affirmative answer, we say that the equation $\mathcal{E}$ is stable [4]. During the last decades several stability problems for various functional equations have been investigated by numerous mathematicians. We refer the reader to the survey articles [4, 5, 8, 19] and monographs [3, 6, 9, 10, 20], and references therein.

By a *non-Archimedean field* we mean a field K equipped with a function (valuation) $|\cdot|$ from K into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r|\,|s|$, and $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in K$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. By the trivial valuation we mean the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$.

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Let X be a vector space over a field K with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is called a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) For any $r \in K, x \in X$, $\|rx\| = |r|\|x\|$;
(iii) The strong triangle inequality (ultrametric); namely,

$$\|x+y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in X).$$

Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n-1\},\ (n > m)$$

holds, a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By a *complete non-Archimedean normed space* we mean one in which every Cauchy sequence is convergent.

For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x,y) = |x-y|_p$ is denoted by $\mathbb{Q}_p$, which is called the *p-adic number field*.

In [1], the authors investigated stability of approximate additive mappings $f : \mathbb{Q}_p \to \mathbb{R}$. In [11, 12, 13], the stability of Cauchy, quadratic and cubic functional equations were investigated in the context of non-Archimedean normed spaces.

In this paper, by following some ideas from [2, 12, 13, 16, 17, 18], we establish the stability of Euler-Lagrange equations in the setting of non-Archimedean normed spaces.

Throughout the paper, we assume that X is a vector space and Y is a complete non-Archimedean normed space.

2. Stability results

The functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ is called the *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. J. M. Rassias introduced the *Euler-Lagrange* quadratic mapping,

$$f(a_1x + a_2y) + f(a_2x - a_1y) = \left(a_1^2 + a_2^2\right)[f(x) + f(y)], \tag{2.1}$$

see [14, 15].

J. M. Rassias introduced the generalized pertinent *Euler-Lagrange* quadratic mappings via his paper [16] and investigated the stability problem for the following generalized functional equation

$$m_1m_2Q(a_1x + a_2y) + Q(m_2a_2x - m_1a_1y) = \left(m_1a_1^2 + m_2a_2^2\right)[m_2Q(x) + m_1Q(y)], \tag{2.2}$$

for all vectors $x, y \in X$, any fixed pair (a_1, a_2) of nonzero reals and any fixed pair (m_1, m_2) of positive reals. Consider a nonlinear mapping $Q : X \to Y$ satisfying the fundamental *Euler-Lagrange* functional equation

$$m_1^2m_2Q(a_1x) + m_1Q(m_2a_2x) = m_0^2m_2Q\Big(\frac{m_1}{m_0}a_1x\Big) + m_0^2m_1Q\Big(\frac{m_2}{m_0}a_2x\Big), \tag{2.3}$$

with $m_0 = \frac{m_1m_2+1}{m_1+m_2}$, and $m = \frac{(m_1+m_2)(m_1a_1^2+m_2a_2^2)}{m_1m_2+1}$ for all $x \in X$, any fixed nonzero reals a_1, a_2 and any fixed positive reals m_1, m_2.

A nonlinear mapping $Q : X \to Y$ is a called generalized *Euler-Lagrange* quadratic if it satisfies (2.2) and (2.3). It is said that the nonlinear mappings $\overline{Q} : X \to Y$, and

$\overline{\overline{Q}} : X \to Y$ are 2-dimensional *Euler-Lagrange* quadratic weights of the first form if we have

$$\overline{Q}(x) = \frac{m_0^2 m_2 Q\big(\frac{m_1}{m_0} a_1 x\big) + m_0^2 m_1 Q\big(\frac{m_2}{m_0} a_2 x\big)}{m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)},$$

and

$$\overline{\overline{Q}}(x) = \frac{m_1 m_2 Q(a_1 x) + Q(m_2 a_2 x)}{m_2 (m_1 a_1^2 + m_2 a_2^2)}$$

for all $x \in X$.

2.1. Lemma. ([16]) *Let $Q : X \to Y$ be a generalized* Euler-Lagrange *quadratic mapping satisfying (2.2). If $m \neq 1$, then we have*

$$Q(0) = 0, \ Q(m^n x) = m^{2n} Q(x)$$

for all $x \in X$ and all integers $n \in \mathbb{Z}$. □

Suppose that $f : X \to Y$ is a mapping. We define a generalized *Euler-Lagrange* difference operator $D_{m_1,m_2}^{a_1,a_2}$ of equation (2.2) as

$$D_{m_1,m_2}^{a_1,a_2} f(x,y) := m_1 m_2 f(a_1 x + a_2 y) + f(m_2 a_2 x - m_1 a_1 y) - (m_1 a_1^2 + m_2 a_2^2)[m_2 f(x) + m_1 f(y)].$$

In this section, we prove the stability of the generalized *Euler-Lagrange* quadratic functional equation in non-Archimedean normed spaces.

2.2. Theorem. *Let $\varphi : X \times X \to [0,\infty)$ and $\psi : X \to [0,\infty)$ be functions such that*

$$\lim_{n\to\infty} \frac{\varphi(m^n x, m^n y)}{|m|^{2n}} = 0 \quad (x, y \in X) \tag{2.4}$$

and

$$\lim_{n\to\infty} \frac{\psi(m^n x)}{|m|^{2n}} = 0 \quad (x \in X), \tag{2.5}$$

where, $m = \frac{(m_1+m_2)\left(m_1 a_1^2 + m_2 a_2^2\right)}{m_1 m_2 + 1}$ and $|m| > 1$ for any fixed pair $(a_1; a_2)$ of nonzero reals and any fixed pair $(m_1; m_2)$ of positive reals. Suppose that $f : X \to Y$ is a mapping satisfying

$$\|D_{m_1,m_2}^{a_1,a_2} f(x,y)\| \le \varphi(x,y), \tag{2.6}$$

for all $x, y \in X$, and

$$\|m_1^2 m_2 f(a_1 x) + m_1 f(m_2 a_2 x) - m_0^2 m_2 f\Big(\frac{m_1}{m_0} a_1 x\Big) - m_0^2 m_1 f\Big(\frac{m_2}{m_0} a_2 x\Big)\| \le \psi(x). \tag{2.7}$$

Then there exists a unique generalized Euler-Lagrange *mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \sup\left\{ \frac{\varphi(m^n x, 0)}{|m^{2n}||m_0 m m_2|}, \ \frac{|m_1|\|f(0)\|}{|m^{2n}||m_2|}, \frac{\psi(m^n x)}{|m^{2n}||m_0 m m_1 m_2|}, \frac{\varphi(\frac{m_1 a_1 m^n x}{m_0}, \frac{m_2 a_2 m^n x}{m_0})}{|m^{2n}||m^2 m_1 m_2|}, \ \frac{\|f(0)\|}{|m^{2n} m^2 m_1 m_2|} : n \in \mathbb{N} \cup \{0\} \right\}. \tag{2.8}$$

Proof. Observe that the functional inequality (2.7) can be written as follows:

$$\|\overline{\overline{f}}(x) - \overline{f}(x)\| \le \frac{\psi(x)}{|m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)|} = \frac{\psi(x)}{|m_0 m m_1 m_2|} \quad (x \in X). \tag{2.9}$$

Replacing x and y by 0 in (2.6) we have

$$\|m_1 m_2 f(0) + f(0) - m_0 m (m_1 + m_2) f(0)\| \le \varphi(0,0),$$

or

$$\|f(0)\| \leq \frac{\varphi(0,0)}{|(m_1+m_2+1)(m-1)|}. \tag{2.10}$$

Moreover substituting $y=0$ in (2.6), one concludes the functional inequality

$$\|m_1m_2f(a_1x)+f(m_2a_2x)-m_0m[m_2f(x)+m_1f(0)]\| \leq \varphi(x,0)$$

or

$$\|\overline{\overline{f}}(x)-f(x)-\frac{m_1}{m_2}f(0)\| \leq \frac{\varphi(x,0)}{|m_2(m_1a_1^2+m_2a_2^2)|} = \frac{\varphi(x,0)}{|m_0mm_2)|}.$$

Hence

$$\begin{aligned} \|\overline{\overline{f}}(x)-f(x)\| &\leq \max\left\{\left\|\overline{\overline{f}}(x)-f(x)-\frac{m_1}{m_2}f(0)\right\|,\ \left\|\frac{m_1}{m_2}f(0)\right\|\right\} \\ &\leq \max\left\{\frac{\varphi(x,0)}{|m_0mm_2|},\ \frac{|m_1|}{|m_2|}\|f(0)\|\right\}. \end{aligned} \tag{2.11}$$

In addition, replacing x, y in (2.6) by $\frac{m_1a_1x}{m_0}$ and $\frac{m_2a_2x}{m_0}$ respectively, one gets the functional inequality

$$\begin{aligned} &\left\|m_1m_2f(mx)+f(0)-m_0m\left[m_2f\Big(\frac{m_1a_1x}{m_0}\Big)+m_1f\Big(\frac{m_2a_2x}{m_0}\Big)\right]\right\| \\ &\qquad \leq \varphi\Big(\frac{m_1a_1}{m_0}x, \frac{m_2a_2}{m_0}x\Big), \end{aligned}$$

or

$$\left\|\frac{f(mx)}{m^2}+\frac{f(0)}{m^2m_1m_2}-\overline{f}(x)\right\| \leq \frac{1}{|m^2m_1m_2|}\varphi\left(\frac{m_1a_1}{m_0}x, \frac{m_2a_2}{m_0}x\right).$$

So

$$\left\|\frac{f(mx)}{m^2}-\overline{f}(x)\right\| \leq \max\left\{\frac{1}{|m^2m_1m_2|}\varphi\left(\frac{m_1a_1}{m_0}x, \frac{m_2a_2}{m_0}x\right), \frac{\|f(0)\|}{|m^2m_1m_2|}\right\}. \tag{2.12}$$

Using the functional inequalities (2.9), (2.11) and (2.12), and the triangle inequality, we have the basic inequality

$$\begin{aligned} \left\|\frac{f(mx)}{m^2}-f(x)\right\| &\leq \max\left\{\|f(x)-\overline{\overline{f}}(x)\|, \|\overline{\overline{f}}(x)-\overline{f}(x)\|, \left\|\overline{f}(x)-\frac{f(mx)}{m^2}\right\|\right\} \\ &\leq \max\left\{\frac{\varphi(x,0)}{|m_0mm_2|}, \frac{|m_1|}{|m_2|}\|f(0)\|, \frac{\psi(x)}{|m_0mm_1m_2|},\right. \\ &\qquad \left.\frac{\varphi(\frac{m_1a_1x}{m_0}, \frac{m_2a_2x}{m_0})}{|m^2m_1m_2|}, \frac{\|f(0)\|}{|m^2m_1m_2|}\right\}. \end{aligned} \tag{2.13}$$

Replacing x by m^nx in (2.13) we obtain

$$\begin{aligned} &\left\|\frac{f(m^nx)}{m^{2n}}-\frac{f(m^{n+1}x)}{m^{2(n+1)}}\right\| \\ &\quad \leq \max\left\{\frac{\varphi(m^nx,0)}{|m^{2n}m_0mm_2|}, \frac{|m_1|}{|m^{2n}m_2|}\|f(0)\|, \frac{\psi(m^nx)}{|m^{2n}m_0mm_1m_2|},\right. \\ &\qquad \left.\frac{\varphi(\frac{m_1a_1m^nx}{m_0}, \frac{m_2a_2m^nx}{m_0})}{|m^{2n}m^2m_1m_2|}, \frac{\|f(0)\|}{|m^{2n}m^2m_1m_2|}\right\}. \end{aligned} \tag{2.14}$$

It follows from (2.4), (2.5) and $|m| > 1$ that the sequence $\{\frac{f(m^n x)}{m^{2n}}\}$ is Cauchy. Since Y is complete, we conclude that $\{\frac{f(m^n x)}{m^{2n}}\}$ is convergent. Set $Q(x) = \lim_{n\to\infty}\{\frac{f(m^n x)}{m^{2n}}\}$, Now, from the inequalities (2.13) and (2.14), one gets the inequalities

$$\begin{aligned}\left\|\frac{f(m^n x)}{m^{2n}} - f(x)\right\| &\le \max\left\{\|\frac{f(m^j x)}{m^{2j}} - \frac{f(m^{j+1}x)}{m^{2(j+1)}}\| : 0 \le j \le n-1\right\}\\ &\le \max\left\{\frac{\varphi(m^j x, 0)}{|m^{2j} m_0 m m_2|}, \frac{|m_1|}{|m^{2j} m_2|}\|f(0)\|, \frac{\psi(m^j x)}{|m^{2j} m_0 m m_1 m_2|},\right.\\ &\left.\frac{\varphi(\frac{m_1 a_1 m^j x}{m_0}, \frac{m_2 a_2 m^j x}{m_0})}{|m^{2j} m^2 m_1 m_2|}, \frac{\|f(0)\|}{|m^{2j} m^2 m_1 m_2|} : 0 \le j \le n-1\right\}.\end{aligned} \tag{2.15}$$

Taking the limit as $n \to \infty$ in (2.15) we find that the mapping Q satisfies the inequality (2.8).

Besides, we claim that the mapping Q satisfies the generalized Euler-Lagrange equation. In fact, it is clear from (2.6) that the following inequality

$$\|D^{a_1,a_2}_{m_1,m_2} f(m^n x, m^n y)\| \le \frac{1}{|m^{2n}|}\varphi(m^n x, m^n y) \tag{2.16}$$

holds for all $x, y \in X$ and $n \in \mathbb{N}$. Taking the limit $n \to \infty$ we obtain from (2.4) that $D^{a_1,a_2}_{m_1,m_2} Q(x, y) = 0$.

Now let $\widehat{Q} : X \to X$ be another generalized Euler-Lagrange mapping satisfying the equation $D^{a_1,a_2}_{m_1,m_2}\widehat{Q}(x, y) = 0$ and the inequality

$$\begin{aligned}\|f(x) - \widehat{Q}(x)\| \le \sup&\left\{\frac{\varphi(m^n x, 0)}{|m^{2n}||m_0 m m_2|}, \frac{|m_1|\|f(0)\|}{|m^{2n}||m_2|}, \frac{\psi(m^n x)}{|m^{2n}||m_0 m m_1 m_2|},\right.\\ &\left.\frac{\varphi(\frac{m_1 a_1 m^n x}{m_0}, \frac{m_2 a_2 m^n x}{m_0})}{|m^{2n}||m^2 m_1 m_2|}, \frac{\|f(0)\|}{|m^{2n} m^2 m_1 m_2|} : n \in \mathbb{N}\right\}.\end{aligned}$$

Since $Q(x) = \frac{Q(m^n x)}{m^{2n}}$, $\widehat{Q}(x) = \frac{\widehat{Q}(m^n x)}{m^{2n}}$ for all $x \in G$ and all $n \in \mathbb{N}$. Thus we have

$$\begin{aligned}\|Q(x) - \widehat{Q}(x)\| &= \left\|\frac{Q(m^k x)}{m^{2k}} - \frac{\widehat{Q}(m^k x)}{m^{2k}}\right\|\\ &\le \max\left\{\frac{1}{|m^{2k}|}\|Q(m^k x) - f(m^k x)\|, \frac{1}{|m^{2k}|}\|\widehat{Q}(m^k x) - f(m^k x)\|\right\}\\ &\le \sup\left\{\frac{\varphi(m^{n+k} x, 0)}{|m^{2(n+k)}||m_0 m m_2|}, \frac{|m_1|\|f(0)\|}{|m^{2(n+k)}||m_2|}, \frac{\psi(m^{n+k} x)}{|m^{2(n+k)}||m_0 m m_1 m_2|},\right.\\ &\left.\frac{\varphi(\frac{m_1 a_1 m^{n+k} x}{m_0}, \frac{m_2 a_2 m^{n+k} x}{m_0})}{|m^{2(n+k)}||m^2 m_1 m_2|}, \frac{\|f(0)\|}{|m^{2(n+k)} m^2 m_1 m_2|} : n \in \mathbb{N}\right\}\\ &= \sup\left\{\frac{\varphi(m^j x, 0)}{|m^{2j}||m_0 m m_2|}, \frac{|m_1|\|f(0)\|}{|m^{2j}|m_2|}, \frac{\psi(m^j x)}{|m^{2j}||m_0 m m_1 m_2|},\right.\\ &\left.\frac{\varphi(\frac{m_1 a_1 m^j x}{m_0}, \frac{m_2 a_2 m^j x}{m_0})}{|m^{2n}||m^2 m_1 m_2|}, \frac{\|f(0)\|}{|m^{2j} m^2 m_1 m_2|} : j \ge k\right\}.\end{aligned}$$

If $k \to \infty$ we have $Q = \widehat{Q}$. □

2.3. Theorem. *Let $\varphi : X \times X \to [0,\infty)$ and $\psi : X \to [0,\infty)$ be two functions such that*

$$\lim_{n\to\infty} |m|^{2n} \varphi\Big(\frac{x}{m^n}, \frac{y}{m^n}\Big) = 0 \quad (x, y \in X) \tag{2.17}$$

and

$$\lim_{n\to\infty} |m|^{2n} \psi\Big(\frac{x}{m^n}\Big) = 0 \quad (x \in X), \tag{2.18}$$

where, $m = \frac{(m_1+m_2)(m_1a_1^2+m_2a_2^2)}{m_1m_2+1}$ and $0 < |m| < 1$ for any fixed pair $(a_1; a_2)$ of nonzero reals and any fixed pair $(m_1; m_2)$ of positive reals, and $f : X \to Y$ is a mapping satisfying

$$\|D^{a_1,a_2}_{m_1,m_2} f(x, y)\| \le \varphi(x, y) \tag{2.19}$$

and

$$\Big\| m_1^2 m_2 f(a_1x) + m_1 f(m_2a_2x) - m_0^2 m_2 f\Big(\frac{m_1}{m_0} a_1 x\Big) - m_0^2 m_1 f\Big(\frac{m_2}{m_0} a_2 x\Big)\Big\| \le \psi(x) \tag{2.20}$$

for all $x, y \in X$. Then there exists a unique generalized Euler-Lagrange mapping $Q : X \to Y$ such that

$$\begin{aligned}\|f(x) - Q(x)\| \le \sup\Bigg\{ &\frac{|m^{2n+1}|\varphi(\frac{x}{m^n}, 0)}{|m_0mm_2|}, \frac{|m^{2n}||m_1|\|f(0)\|}{|m_2|}, \frac{|m^{2n+1}|\psi(\frac{x}{m^n})}{|m_0mm_1m_2|},\\ &\frac{|m^{2n}|\varphi(\frac{m_1a_1x}{m^nm_0}, \frac{m_2a_2x}{m^nm_0})}{|m^2m_1m_2|}, \frac{|m^{2n+2}\|f(0)\|}{|m_1m_2|} : n \in \mathbb{N}\Bigg\}.\end{aligned} \tag{2.21}$$

Proof. Using the same method as in Theorem 2.2, we conclude that

$$Q(x) = \lim_{n\to\infty} \Big\{ m^{2n} f\Big(\frac{x}{m^n}\Big)\Big\}$$

is the unique Euler-Lagrange mapping satisfying (2.21). □

In the next theorem we consider the case that $m = 1$.

2.4. Theorem. *Assume that $f : X \to Y$ and $\phi : X \times X \to [0,\infty)$ are two mappings for which*

$$\|D^{a_1,a_2}_{m_1,m_2} f(x, y)\| \le \phi(x, y) \tag{2.22}$$

holds for all $x, y \in X$. Suppose that $m := \frac{(m_1+m_2)(m_1a_1^2+m_2a_2^2)}{m_1m_2+1} = 1$, $m_2a_2 = m_1a_1$, and if $|l| > 1$ then

$$\lim_{n\to\infty} \frac{\phi(l^n x, l^n x)}{|l|^{2n}} = 0$$

(if $|l| < 1$, then $\lim_{n\to\infty} |l|^{2n}\phi(\frac{x}{l^n}, \frac{x}{l^n}) = 0$), where $l := a_1 + a_2$ is given with $|l| \neq 0, 1$. Then there exists a unique generalized Euler-Lagrange quadratic mapping $Q : X \to Y$ satisfying $D^{a_1,a_2}_{m_1,m_2} Q(x, y) = 0$ and

$$\|f(x) - Q(x)\| \le \begin{cases} \sup\Big\{ \frac{\phi(l^n x, l^n x)}{|l|^{2n+2}|m_1m_2|}, \frac{\|f(0)\|}{|l|^{2n+2}|m_1m_2|} : n \in \mathbb{N}\Big\} & \text{if } |l| > 1,\\ \sup\Big\{ \frac{|l|^{2n+2}}{|m_1m_2|}\phi\Big(\frac{x}{l^n}, \frac{x}{l^n}\Big), \frac{|l|^{2n+2}\|f(0)\|}{m_1m_2|} : n \in \mathbb{N}\Big\} & \text{if } |l| < 1.\end{cases}$$

Moreover, if there exists a mapping $\psi : X \to [0,\infty)$, then the function f satisfies approximately the following fundamental functional equation

$$\Big\| m_1^2 m_2 f(a_1x) + m_1 f(m_2a_2x) - m_0^2 m_2 f\Big(\frac{m_1}{m_0} a_1 x\Big) - m_0^2 m_1 f\Big(\frac{m_2}{m_0} a_2 x\Big)\Big\| \le \psi(x),$$

and if $|l| > 1$*, then*

$$\lim_{n\to\infty} \frac{\psi(l^n x)}{|l|^{2n}} = 0 \tag{2.23}$$

(*if* $|l| < 1$*, then* $\lim_{n\to\infty} |l|^{2n}\psi\big(\frac{x}{|l|^n}\big) = 0$) *holds for all* $x \in X$.

Proof. From the fact that $m := \frac{(m_1+m_2)(m_1a_1^2+m_2a_2^2)}{m_1m_2+1} = 1$ and $m_2a_2 = m_1a_1$, we have

$$\frac{m_1m_2+1}{m_1m_2} = \frac{m_1^2a_1+a_2}{m_1^2a_1} = (a_1+a_2)^2.$$

Replacing y by x in (2.22), we obtain

$$\|f(lx) - l^2 f(x)\| \le \max\left\{\frac{\phi(x,x)}{|m_1m_2|}, \frac{\|f(0)\|}{|m_1m_2|}\right\}$$

and so

$$\left\|\frac{f(l^{n+1}x)}{l^{2n+2}} - \frac{f(l^n x)}{l^{2n}}\right\| \le \max\left\{\frac{\phi(l^{2n}x, l^{2n}x)}{|l|^{2n+2}|m_1m_2|}, \frac{\|f(0)\|}{|l|^{2n+2}|m_1m_2|}\right\} \text{ if } |l| > 1$$

and

$$\left\|l^{2n+2} f\Big(\frac{x}{l^{n+1}}\Big) - l^{2n} f\Big(\frac{x}{l^n}\Big)\right\| \le \max\left\{|l|^{2n+2}\frac{\phi\big(\frac{x}{l^{2n}}, \frac{x}{l^{2n}}\big)}{|m_1m_2|}, |l|^{2n+2}\frac{\|f(0)\|}{|m_1m_2|}\right\} \quad \text{if } |l| < 1$$

for all $x \in X$ and nonnegative integers n.

Now by a similar process to the proof of our previous theorems we may find that the sequences $\{\frac{f(l^n x)}{l^{2n}}\}$, when $|l| > 1$, and $\{l^{2n} f(\frac{x}{l^n})\}$, when $|l| > 1$, are Cauchy and so are convergent, and

$$Q(x) := \begin{cases} \lim_{n\to\infty} \dfrac{f(l^n x)}{l^{2n}} & \text{if } |l| > 1, \\ \lim_{n\to\infty} l^{2n} f(\frac{x}{l^n}) & \text{if } |l| < 1, \end{cases}$$

has our desired properties. □

2.5. Corollary. *Let* $|m| \neq 1$*, and* $\rho : [0,\infty) \to [0,\infty)$ *be defined by*

$$\rho(t) = \begin{cases} \frac{|m|^{2n}}{n+1} & \text{if } t = |m|^n r,\ |m| > 1,\ r < |m|,\ n \in \mathbb{N}\cup\{0\},\ r > 0, \\ \frac{1}{|m|^{2n}(n+1)} & \text{if } t = \frac{r}{|m|^n},\ 0 < |m| < 1,\ r < |m|,\ n \in \mathbb{N}\cup\{0\},\ r > 0, \\ t & \text{otherwise.} \end{cases}$$

Suppose that $\delta_1, \delta_2 > 0$*,* X *is a normed space and* $f : X \to Y$ *fulfills the inequalities*

$$\|D^{a_1,a_2}_{m_1,m_2} f(x,y)\| \le \delta_1\left(\rho(\|x\|) + \rho(\|y\|)\right) \quad (x, y \in X)$$

and

$$\left\|m_1^2m_2 f(a_1x) + m_1 f(m_2a_2x) - m_0^2m_2 f\Big(\frac{m_1}{m_0}a_1x\Big) - m_0^2m_1 f\Big(\frac{m_2}{m_0}a_2x\Big)\right\| \le \delta_2\rho(\|x\|). \tag{2.24}$$

Then there exists a unique generalized Euler-Lagrange mapping $Q : G \to X$ *such that if* $|m| > 1$*, then*

$$\|f(x) - Q(x)\| \le \sup\left\{\frac{\varphi(x,0)}{|m_0mm_2|}, \frac{|m_1|\|f(0)\|}{|m_2|}, \frac{\psi(x)}{|m_0mm_1m_2|}, \frac{\varphi(\frac{m_1a_1x}{m_0}, \frac{m_2a_2x}{m_0})}{|m^2m_1m_2|}, \frac{\|f(0)\|}{|m^2m_1m_2|}\right\},$$

and if $0 < |m| < 1$, *then*

$$\|f(x) - Q(x)\| \leq \sup \left\{ \frac{|m|\varphi(\frac{x}{m^n}, 0)}{|m_0 m m_2|}, \frac{|m_1| \|f(0)\|}{m_2|}, \frac{|m|\psi(x)}{|m_0 m m_1 m_2|}, \frac{|\varphi(\frac{m_1 a_1 x}{m_0}, \frac{m_2 a_2 x}{m_0})}{|m^2 m_1 m_2|}, \frac{|m^2 \|f(0)\|}{|m_1 m_2|} \right\}.$$

2.6. Remark. The hypotheses in Corollary 2.5 give us an example for which the crucial assumption $\Sigma_{i=0}^{\infty} \frac{\varphi(m^i x, m^i y)}{m^{2i}} < \infty$ in the main theorem of [17] does not hold on balls of X of the radius $r > 0$. Hence our results in the setting of non-Archimedean normed spaces are different from those of [17].

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