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ON THE EXISTENCE INTERVAL FOR THE INITIAL VALUE PROBLEM OF A FRACTIONAL DIFFERENTIAL EQUATION

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Abstract

We compute via a comparison function technique, a new bound for the existence interval of the initial value problem for a fractional differential equation given by means of Caputo derivatives. We improve in this way the estimate of the existence interval obtained very recently in the literature.

Keywords: Fractional differential equation, Leray-Schauder alternative, Local existence, Estimate of existence interval.

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1. Introduction

Fractional calculus is a powerful tool which plays an important role in the modeling of multi-scale problems.

Fractional differential equations have been found appropriate to describe the dynamics of complex systems in several branches of science and engineering [30, 28, 15, 19]. In the paper [14] a detailed discussion of this topic is made. Various applications, like in the reaction kinetics of proteins, the anomalous electron transport in amorphous materials, the dielectrical or mechanical relaxation of polymers, the modeling of glass-forming liquids and others, are successfully performed in numerous papers. See the presentations from [22, 28].

Several recent advancements in the theory and applications of non-integer differentiation and integration are described in [29, 19, 8, 18, 10, 20, 21, 23, 24]. For instance,

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fractional Lagrangian and Hamiltonian treatments of the field and mechanical systems are proposed by Băleanu and Muslih in [29, pp. 115 and following]. Other results concerning the promising new theory of fractional variational principles can be found in the contributions [1, 3, 4, 5, 6, 16, 25].

Let us consider the initial value problem for a fractional differential equation (FIVP) below

$$\begin{cases} D^a_{0+}\left(x - x_0\right)(t) = f(t, x(t)), & t > 0, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where the nonlinearity $f : D = [0, T] \times [x_0 - b, x_0 + b] \to \mathbb{R}$ is assumed continuous and $T, b > 0$ and x_0 are fixed real numbers.

The differential operator D^a_{0+} in problem (1.1) is the Riemann-Liouville differential operator of order $0 < a < 1$, namely

$$D^a_{0+}(u)(t) = \frac{1}{\Gamma(1-a)} \cdot \frac{d}{dt}\left[\int_0^t \frac{u(s)}{(t-s)^a} ds\right],$$

where $\Gamma(1-a) = \int_0^{+\infty} e^{-t} t^{-a} dt$ is the Gamma function. See [15, p. 70].

The motivation for inserting the initial datum x_0 into the differential operator comes from the physical origin of such mathematical models, and the reader can find comprehensive details in this respect in [28, p. 80], [11, p. 230]. The Riemann-Liouville operator with an inserted datum is called a Caputo differential operator — see [7], [15, p. 91].

Assuming that the FIVP has a solution $x(t)$, the formulas $\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a}$ and

$$\int_0^t f(s, x(s))\, ds = \frac{\sin \pi a}{\pi} \int_0^t \frac{1}{(t-s)^a} \int_0^s \frac{f(\tau, x(\tau))}{(s-\tau)^{1-a}}\, d\tau\, ds,$$

see [2, p. 196], allow us to rewrite (1.1) via an integration as

$$\int_0^t \frac{1}{\Gamma(1-a)(t-s)^a}\left[x(s) - x_0 - \frac{1}{\Gamma(a)} \int_0^s \frac{f(\tau, x(\tau))}{(s-\tau)^{1-a}}\, d\tau\right] ds = 0, \tag{1.2}$$

where $t > 0$.

We can regard from now on, by means of (1.2), any solution of FIVP (1.1) as a (continuous) solution of the singular integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(a)} \int_0^t \frac{f(s, x(s))}{(t-s)^{1-a}}\, ds, \quad t \geq t_0, \tag{1.3}$$

and vice versa. This fact is established in a rigorous manner in [15, Theorem 3.24, pp. 199-200] (for $\gamma = r = 0$ in the original notation).

Given the generality of f, we can replace it with $\Gamma(a)^{-1} f$, which means that we shall be interested in solving the next integral equation

$$x(t) = x_0 + \int_0^t \frac{f(s, x(s))}{(t-s)^{1-a}}\, ds, \quad t \geq t_0. \tag{1.4}$$

It has been established in [11], see also the paper [17, Theorem 3.1] and the comprehensive monograph [12, Sections 6.1, 6.2], that the equation (1.4) possesses at least one continuous solution defined throughout the interval $[0, T^\star]$, where

$$T^\star = \min\left\{T, \left(\frac{ab}{M}\right)^{\frac{1}{a}}\right\}, \quad M = \sup_{(t,x) \in D} |f(t, x)|.$$

The same estimate was derived in [15, Theorem 3.26 (iii), p. 206] in the case of f being Lipschitzian with respect to the second variable. See also the authoritative paper [11] for a comprehensive discussion about existence and uniqueness results and several important numerical consequences.

The reason for the second term in the estimate of $T^\star$ is a consequence of asking that $x(t)$ from (1.4) be in $[x_0 - b, x_0 + b]$ for any $t \geq 0$. Precisely,

$$|x(t) - x_0| \leq \int_0^t |f(t-s, x(t-s)| \frac{ds}{s^{1-a}} \leq M \cdot \frac{t^a}{a} \leq b,$$

which implies that the solution of (1.1) can last for at most

$$T^\star \leq \left(\frac{ab}{M}\right)^{\frac{1}{a}} \tag{1.5}$$

units of t.

In some circumstances, however, it is possible to improve significantly the estimate (1.5). We shall present here such a result, based on a comparison function technique. This procedure has been inspired by some recent attempts in the same direction for ordinary differential equations, see the contributions [26, 27]. Let us emphasize that similar estimates can be provided for initial value problems involving the plethora of fractional derivatives that are used nowadays to model various complex phenomena from science and technology. We have refrained from presenting the most general conclusion due to the unnecessary intricacies of such a task.

2. Estimate of $T^\star$

Let us set several quantities first. We introduce $p, q > 1$ such that $(1-a)q < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. We also define

$$\begin{aligned} c &= \frac{1}{1-(1-a)q} = \frac{1}{q} \cdot \frac{1}{\frac{1}{q} - 1 + a} = \frac{1}{q} \cdot \frac{1}{a - \frac{1}{p}} = \frac{p}{q} \cdot \frac{1}{ap-1} \\ &= \frac{p-1}{ap-1}, \end{aligned}$$

$C = c^{p-1}$, and

$$\begin{aligned} \lambda &= [1-(1-a)q]\frac{p}{q} = \frac{1-(1-a)q}{q-1} = \frac{\frac{1}{q} - 1 + a}{1 - \frac{1}{q}} = \frac{a - \frac{1}{p}}{\frac{1}{p}} \\ &= ap - 1. \end{aligned}$$

We introduce next the continuous functions $g : [0, T] \times [-b, b] \to \mathbb{R}$ and $w : [0, b] \to [0, +\infty)$ given by

$$g(t, y) = f(t, x_0 + y), \quad w(r) = \sup\{|g(s, y)| : s \in [0, T],\ |y| \leq r\}.$$

We notice that $w(0) = 0$ if and only if $f(\cdot, x_0)$ is identically null in $[0, T]$. By assuming that $w(0) > 0$, we introduce the continuous function $W : [0, b^p] \to [0, +\infty)$ via the formula

$$W(r) = \int_0^r \frac{dv}{\left[w\left(v^{\frac{1}{p}}\right)\right]^p} = p \int_0^{r^{\frac{1}{p}}} \left[\frac{\xi}{w(\xi)}\right]^p \frac{d\xi}{\xi}.$$

The proof of our estimate relies on the fixed point result known as the Leray-Schauder alternative, see [13, Theorem 5.3, pp. 61-62].

2.1. Theorem. *Let $P : N \to N$ be a completely continuous (compact) operator acting on the normed linear space N. Then, either there exists $y \in N$ such that*

$$y = P(y)$$

or the set

$$E(P) = \{y \in N : y = \eta P(y) \text{ for a certain } \eta \in (0,1)\}$$

is unbounded. □

We shall take $N = C([0, T^\star], \mathbb{R})$ with the usual *sup*-norm, where assuming $w(0) > 0$,

$$\begin{aligned} T^\star &= \min\left\{T, \left[\frac{W(b^p)}{C}\right]^{\frac{1}{1+\lambda}}\right\} \\ &= \min\left\{T, \left[\left(\frac{ap-1}{p-1}\right)^{p-1} \cdot p\int_0^b \frac{\xi^{p-1}}{[w(\xi)]^p} d\xi\right]^{\frac{1}{ap}}\right\}, \end{aligned} \tag{2.1}$$

and

$$P(y)(t) = \int_0^t \frac{g(s, y(s))}{(t-s)^{1-a}} ds, \quad t \in [0, T^\star],\ y \in N.$$

The compactness of the operator P is standard, see [9] – where Schauder's fixed point theorem has been employed to prove local existence of solution for the FIVP (1.1) – and [15, p. 139].

We can now state and prove our result.

2.2. Theorem. *The following alternative is valid: either the FIVP* (1.1) *has the constant solution $x = x_0$ throughout $[0, T]$, or it has at least one* (*non-constant*) *solution $x(t)$ in $[0, T^\star]$, where $T^\star$ is given by* (2.1).

Proof. Assume that $w(0) > 0$. According to Theorem 2.1, it is enough to establish that the set $E(P)$ is bounded.

Take $y \in E(P)$. Then, we have the estimates

$$\begin{aligned} |y(t)| &\le \int_0^t \frac{|g(s, y(s))|}{(t-s)^{1-a}}\, ds \le \int_0^t \frac{w(|y(s)|)}{(t-s)^{1-a}}\, ds \\ &\le \left\{\int_0^t \left[\frac{1}{(t-s)^{1-a}}\right]^q ds\right\}^{\frac{1}{q}} \cdot \left\{\int_0^t [w(|y(s)|)]^p ds\right\}^{\frac{1}{p}} \\ &= \left[t^{1-(1-a)q} c\right]^{\frac{1}{q}} \cdot \left\{\int_0^t [w(|y(s)|)]^p ds\right\}^{\frac{1}{p}} \end{aligned}$$

and

$$\begin{aligned} |y(t)|^p &\le c^{p-1} t^\lambda \cdot \int_0^t [w(|y(s)|)]^p\, ds \\ &= C t^\lambda z(t),\ z(t) = \int_0^t [w(|y(s)|)]^p\, ds, \end{aligned} \tag{2.2}$$

for any $t \in [0, T^\star]$.

Fix now $t_0 \in (0, T^\star]$. We deduce that

$$z'(t) = [w(|y(t)|)]^p \le \left[w\left((C t_0^\lambda z(t))^{\frac{1}{p}}\right)\right]^p,\ t \in [0, t_0],$$

and

$$\frac{[\alpha z(t)]'}{\left[w\left((\alpha z(t))^{\frac{1}{p}}\right)\right]^p} \le \alpha, \quad \alpha = Ct_0^\lambda.$$

Integrating in $[0, t_0]$, we obtain

$$W(\alpha z(t)) \le \alpha t \tag{2.3}$$

and also, by taking $t = t_0$ in (2.3),

$$W(Ct^\lambda z(t)) \le Ct^{1+\lambda} \le C(T^\star)^{1+\lambda} \le W(b^p) \tag{2.4}$$

for any $t \in [0, T^\star]$.

Since the function $W : [0, b^p] \to [0, W(b^p)]$ is bijective, by combining the estimates (2.2), (2.4), we conclude that

$$||y||_N = \sup_{s\in[0,T^\star]} |y(s)| \le b, \quad y \in E(P).$$

We have established the boundedness of $E(P)$. □

3. Comparison of the estimates (1.5) and (2.1)

Set $x_0, L > 0$ and take

$$f(t,x) = Lx, \ (t,x) \in D.$$

Then, $M = L(x_0 + b)$ and $w(r) = L(x_0 + r)$ for any $r \in [0, b]$.

We have to compare the quantities

$$H_1(b) = \left[\frac{ab}{L(x_0+b)}\right]^{\frac{1}{a}}$$

and

$$H_2(b) = \left[\left(\frac{ap-1}{p-1}\right)^{p-1} \cdot \frac{p}{L^p}\int_0^b \frac{\xi^{p-1}}{(x_0+\xi)^p} d\xi\right]^{\frac{1}{ap}},$$

where $b \ge 0$.

Introduce the quantities $Q_i(b) = [H_i(b)]^{ap}$ for $b \ge 0$, where $i = 1, 2$. Then, we notice that

$$\lim_{b\to+\infty} Q_1(b) = \left(\frac{a}{L}\right)^p. \tag{3.1}$$

On the other hand, since for $b \ge x_0$ it is clear that

$$\int_{x_0}^b \frac{\xi^{p-1}}{(x_0+\xi)^p} d\xi \ge \left(\frac{1}{2}\right)^p \int_{x_0}^b \frac{d\xi}{\xi} = \left(\frac{1}{2}\right)^p \cdot \log\left(\frac{b}{x_0}\right),$$

we obtain that

$$\lim_{b\to+\infty} Q_2(b) = +\infty. \tag{3.2}$$

The formulas (3.1), (3.2) show that *the estimate* (2.1) *is much better than its classical counterpart* (1.5) *for large values of* b, similarly to the case of ordinary differential equations, see the paper [26] and its references.

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