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TRANSLATIONS OF FUZZY IDEALS IN BCK/BCI-ALGEBRAS

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Abstract

Based on fuzzy set theory, extensions, translations and multiplications of ideals in BCK/BCI-algebras are discussed. Relations among fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy ideals are investigated.

Keywords: Fuzzy ideal, Fuzzy translation, Fuzzy extension, Fuzzy multiplication.

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1. Introduction

The study of BCK-algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. For the general development of BCK/BCI-algebras, the ideal theory and its fuzzification play an important role. Jun (together with Kim, Meng, Song and Xin) studied fuzzy aspects of several notions in BCK/BCI-algebras (see [2, 3, 4, 7]).

Lee *et al.* [5] discussed fuzzy translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications of fuzzy subalgebras in BCK/BCI-algebras. They investigated relations among fuzzy translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications.

In this paper, we discuss fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy ideals in BCK/BCI-algebras. We investigate relations among fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy ideals in in BCK/BCI-algebras.

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2. Preliminaries

A BCK-algebra is an important class of logical algebras introduced by K. Iséki, and has been extensively investigated by several researchers.

An algebra $(X; *, \theta)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = \theta)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = \theta)$,
(III) $(\forall x \in X)\ (x * x = \theta)$,
(IV) $(\forall x, y \in X)\ (x * y = \theta, y * x = \theta \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (\theta * x = \theta)$,

then X is called a *BCK-algebra*. Any BCK-algebra X satisfies the following axioms:

(a1) $(\forall x \in X)\ (x * \theta = x)$,
(a2) $(\forall x, y, z \in X)\ (x * y = \theta \Rightarrow (x * z) * (y * z) = \theta,\ (z * y) * (z * x) = \theta)$,
(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ (((x * z) * (y * z)) * (x * y) = \theta)$.

A subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

A subset A of a BCK/BCI-algebra X is called an *ideal* of X, denoted by $A \lhd X$, if it satisfies:

(b1) $\theta \in A$,
(b2) $(\forall x, y \in X)\ (x * y \in A, y \in A \Rightarrow x \in A)$.

We refer the reader to the books [1] and [6] for further information regarding BCK/BCI-algebras.

A fuzzy subset μ of a BCK/BCI-algebra X is called a *fuzzy subalgebra* of X it it satisfies:

$$(\forall x, y \in X)(\mu(x * y) \geq \min\{\mu(x), \mu(y)\}).$$

A fuzzy subset μ of a BCK/BCI-algebra X is called a *fuzzy ideal* of X, denoted by $\mu \lhd_f X$, it it satisfies:

(b3) $(\forall x \in X)\ (\mu(\theta) \geq \mu(x))$,
(b4) $(\forall x, y \in X)\ (\mu(x) \geq \min\{\mu(x * y), \mu(y)\})$.

3. Fuzzy translations and fuzzy multiplications of fuzzy ideals

In what follows let $X = (X, *, \theta)$ denote a BCK/BCI-algebra, and for any fuzzy subset μ of X, we let

$$\top := 1 - \sup\{\mu(x) \mid x \in X\}$$

unless otherwise specified.

3.1. Definition. [5] Let μ be a fuzzy subset of X and let $\alpha \in [0, \top]$. A mapping $\mu_\alpha^T : X \to [0, 1]$ is called a *fuzzy α-translation* of μ if it satisfies:

$$(\forall x \in X)(\mu_\alpha^T(x) = \mu(x) + \alpha).$$

3.2. Theorem. *If μ is a fuzzy ideal of X, then the fuzzy α-translation μ_α^T of μ is a fuzzy ideal of X for all $\alpha \in [0, \top]$.*

Proof. Assume that $\mu \lhd_f X$ and let $\alpha \in [0, \top]$. Then

$$\mu^T_\alpha(\theta) = \mu(\theta) + \alpha \geq \mu(x) + \alpha = \mu^T_\alpha(x)$$

and

$$\begin{aligned}\mu^T_\alpha(x) &= \mu(x) + \alpha \geq \min\{\mu(x * y), \mu(y)\} + \alpha \\ &= \min\{\mu(x * y) + \alpha,\ \mu(y) + \alpha\} \\ &= \min\{\mu^T_\alpha(x * y),\ \mu^T_\alpha(y)\}\end{aligned}$$

for all $x, y \in X$. Hence $\mu^T_\alpha \lhd_f X$. □

3.3. Theorem. *Let μ be a fuzzy subset of X such that the fuzzy α-translation μ^T_α of μ is a fuzzy ideal of X for some $\alpha \in [0, \top]$. Then μ is a fuzzy ideal of X.*

Proof. Assume that μ^T_α is a fuzzy ideal of X for some $\alpha \in [0, \top]$. Let $x, y \in X$. Then

$$\mu(\theta) + \alpha = \mu^T_\alpha(\theta) \geq \mu^T_\alpha(x) = \mu(x) + \alpha,$$

and so $\mu(\theta) \geq \mu(x)$. Now, we have

$$\begin{aligned}\mu(x) + \alpha = \mu^T_\alpha(x) &\geq \min\{\mu^T_\alpha(x * y), \mu^T_\alpha(y)\} \\ &= \min\{\mu(x * y) + \alpha, \mu(y) + \alpha\} \\ &= \min\{\mu(x * y), \mu(y)\} + \alpha,\end{aligned}$$

which implies that $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. Hence μ is a fuzzy ideal of X. □

3.4. Theorem. *Let $\alpha \in [0, \top]$ and let μ be a fuzzy ideal of X. If X is a BCK-algebra, then the fuzzy α-translation μ^T_α of μ is a fuzzy subalgebra of X.*

Proof. Since $x * y \leq x$ for any $x, y \in X$ and any fuzzy ideal is order reversing, we have

$$\begin{aligned}\mu^T_\alpha(x * y) &= \mu(x * y) + \alpha \ \geq \ \mu(x) + \alpha \\ &\geq \min\{\mu(x * y), \mu(y)\} + \alpha \\ &\geq \min\{\mu(x), \mu(y)\} + \alpha \\ &= \min\{\mu(x) + \alpha, \mu(y) + \alpha\} \\ &= \min\{\mu^T_\alpha(x), \mu^T_\alpha(y)\}.\end{aligned}$$

Hence μ^T_α is a fuzzy subalgebra of X. □

The following example shows that if X is a BCI-algebra, then Theorem 3.4 is not true.

3.5. Example. Consider the direct product $X := Y \times \mathbb{Z}$ where $(Y, *, 0)$ is a BCI-algebra and $(\mathbb{Z}, -, 0)$ is the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers. Let $A = Y \times \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers.

Define a fuzzy subset μ of X as follows:

$$\mu : X \to [0, 1], \quad x \mapsto \begin{cases} 0.7 & \text{if } x \in A, \\ 0.2 & \text{otherwise.} \end{cases}$$

Then μ is a fuzzy ideal of X and $\top = 0.3$. For $\alpha \in [0, \top]$, we have

$$\begin{aligned}\mu^T_\alpha\big((0,0) * (0,1)\big) = \mu^T_\alpha\big((0,-1)\big) &= \mu\big((0,-1)\big) + \alpha = 0.2 + \alpha \\ &< 0.7 + \alpha \ = \ \min\{\mu\big((0,0)\big), \mu\big((0,1)\big)\} + \alpha \\ &= \min\{\mu\big((0,0)\big) + \alpha, \mu\big((0,1)\big) + \alpha\} \\ &= \min\{\mu^T_\alpha\big((0,0)\big), \mu^T_\alpha\big((0,1)\big)\}\end{aligned}$$

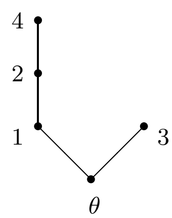
3.6. Definition. [5] Let μ_1 and μ_2 be fuzzy subsets of X. If $\mu_1(x) \leq \mu_2(x)$ for all $x \in X$, then we say that μ_2 is a *fuzzy extension* of μ_1.

3.7. Definition. Let μ_1 and μ_2 be fuzzy subsets of X. Then μ_2 is called a *fuzzy ideal extension* of μ_1 if the following assertions are valid:

(b5) μ_2 is a fuzzy extension of μ_1.
(b6) $\mu_1 \lhd_f X \Rightarrow \mu_2 \lhd_f X$.

3.8. Example. Consider a set $X = \{\theta, 1, 2, 3, 4\}$. The Hasse diagram (Figure 1)

Figure 1. Hasse diagram



makes X into a BCK-algebra, where the BCK-operation $*$ on X is given as follows:

$$x * y := \begin{cases} \theta & \text{if } x \leq y, \\ x & \text{if } y = \theta,\ y < x \text{ or } x \text{ and } y \text{ are incomparable} \end{cases}$$

for every $x, y \in X$. Let μ_1 be a fuzzy subset of X defined by

$$\mu_1 = \begin{pmatrix} \theta & 1 & 2 & 3 & 4 \\ 0.7 & 0.5 & 0.4 & 0.6 & 0.3 \end{pmatrix}.$$

It is routine to check that μ_1 is a fuzzy ideal of X. Let μ_2 be a fuzzy subset of X given by

$$\mu_2 = \begin{pmatrix} \theta & 1 & 2 & 3 & 4 \\ 0.75 & 0.53 & 0.47 & 0.61 & 0.38 \end{pmatrix}.$$

Note that $\mu_2(x) \geq \mu_1(x)$ for all $x \in X$, that is, μ_2 is a fuzzy extension of μ_1, and μ_2 is a fuzzy ideal of X. Hence μ_2 is a fuzzy ideal extension of μ_1.

For a fuzzy subset μ of X, $\alpha \in [0, \top]$ and $t \in [0, 1]$ with $t \geq \alpha$, let

$$U_\alpha(\mu; t) := \{x \in X \mid \mu(x) \geq t - \alpha\}.$$

It is clear that if $\mu \lhd_f X$, then $U_\alpha(\mu; t) \lhd X$ for all $t \in \operatorname{Im}(\mu)$ with $t \geq \alpha$. But, if we do not give a condition that μ is a fuzzy ideal of X then $U_\alpha(\mu; t)$ may not be an ideal of X, as seen in the following example.

3.9. Example. Consider the BCK-algebra X which is given in Example 3.8. Let μ be a fuzzy subset of X defined by

$$\mu = \begin{pmatrix} \theta & 1 & 2 & 3 & 4 \\ 0.8 & 0.5 & 0.7 & 0.3 & 0.2 \end{pmatrix}.$$

Since $\mu(1) = 0.5 < 0.7 = \min\{\mu(1 * 2), \mu(2)\}$, we know that μ is not a fuzzy ideal of X. Also, $U_{0.1}(\mu; 0.63) = \{0, 2\}$ is not an ideal of X since $1 * 2 = 0 \in \{0, 2\}$, but $1 \notin \{0, 2\}$.

3.10. Theorem. *For $\alpha \in [0, \top]$, let μ_α^T be the fuzzy α-translation of μ. Then the following are equivalent:*

(1) $\mu_\alpha^T \lhd_f X$.
(2) $(\forall t \in \operatorname{Im}(\mu))\ (t > \alpha \Rightarrow U_\alpha(\mu; t) \lhd X)$.

Proof. Assume that $\mu_\alpha^T \lhd_f X$ and let $t \in \mathrm{Im}(\mu)$ be such that $t > \alpha$. Since $\mu_\alpha^T(\theta) \geq \mu_\alpha^T(x)$ for all $x \in X$, we have

$$\mu(\theta) + \alpha = \mu_\alpha^T(\theta) \geq \mu_\alpha^T(x) = \mu(x) + \alpha \geq t$$

for $x \in U_\alpha(\mu; t)$. Hence $\theta \in U_\alpha(\mu; t)$. Let $x, y \in X$ be such that $x * y \in U_\alpha(\mu; t)$ and $y \in U_\alpha(\mu; t)$. Then $\mu(x * y) \geq t - \alpha$ and $\mu(y) \geq t - \alpha$, i.e., $\mu_\alpha^T(x * y) = \mu(x * y) + \alpha \geq t$ and $\mu_\alpha^T(y) = \mu(y) + \alpha \geq t$. Since $\mu_\alpha^T \lhd_f X$, it follows that

$$\mu(x) + \alpha = \mu_\alpha^T(x) \geq \min\{\mu_\alpha^T(x * y), \mu_\alpha^T(y)\} \geq t,$$

that is, $\mu(x) \geq t - \alpha$ so that $x \in U_\alpha(\mu; t)$. Therefore $U_\alpha(\mu; t) \lhd X$.

Conversely, suppose that $U_\alpha(\mu; t) \lhd X$ for every $t \in \mathrm{Im}(\mu)$ with $t > \alpha$. If there exists $a \in X$ such that $\mu_\alpha^T(\theta) < \beta \leq \mu_\alpha^T(a)$, then $\mu(a) \geq \beta - \alpha$ but $\mu(\theta) < \beta - \alpha$. This shows that $a \in U_\alpha(\mu; t)$ and $\theta \notin U_\alpha(\mu; t)$. This is a contradiction, and so $\mu_\alpha^T(\theta) \geq \mu_\alpha^T(x)$ for all $x \in X$.

Now assume that there exist $a, b \in X$ such that $\mu_\alpha^T(a) < \gamma \leq \min\{\mu_\alpha^T(a * b), \mu_\alpha^T(b)\}$. Then $\mu(a * b) \geq \gamma - \alpha$ and $\mu(b) \geq \gamma - \alpha$, but $\mu(a) < \gamma - \alpha$. Hence $a * b \in U_\alpha(\mu; \gamma)$ and $b \in U_\alpha(\mu; \gamma)$, but $a \notin U_\alpha(\mu; \gamma)$. This is impossible, and therefore

$$\mu_\alpha^T(x) \geq \min\{\mu_\alpha^T(x * y), \mu_\alpha^T(y)\}$$

for all $x, y \in X$. Consequently, $\mu_\alpha^T \lhd_f X$. □

In Theorem 3.10(2), if $t \leq \alpha$, then $U_\alpha(\mu; t) = X$.

3.11. Theorem. *Let $\mu \lhd_f X$ and $\alpha, \beta \in [0, \top]$. If $\alpha \geq \beta$, then the fuzzy α-translation μ_α^T of μ is a fuzzy ideal extension of the fuzzy β-translation μ_β^T of μ.*

Proof. Straightforward. □

For every fuzzy ideal μ of X and $\beta \in [0, \top]$, the fuzzy β-translation μ_β^T of μ is a fuzzy ideal of X. If ν is a fuzzy ideal extension of μ_β^T, then there exists $\alpha \in [0, \top]$ such that $\alpha \geq \beta$ and $\nu(x) \geq \mu_\alpha^T(x)$ for all $x \in X$. Hence we have the following theorem.

3.12. Theorem. *Let μ be a fuzzy ideal of X and $\beta \in [0, \top]$. For every fuzzy ideal extension ν of the fuzzy β-translation μ_β^T of μ, there exists $\alpha \in [0, \top]$ such that $\alpha \geq \beta$ and ν is a fuzzy ideal extension of the fuzzy α-translation μ_α^T of μ.* □

The following example illustrates Theorem 3.12.

3.13. Example. Consider a BCI-algebra $X = \{\theta, 1, 2, a, b\}$ where the $*$-multiplication is defined by Table 1.

Table 1. $*$**-multiplication table for** X

$*$	θ	1	2	a	b
θ	θ	θ	θ	b	a
1	1	θ	1	b	a
2	2	2	θ	b	a
a	a	a	a	θ	b
b	b	b	b	a	θ

Let μ be a fuzzy subset of X defined by

$$\mu = \begin{pmatrix} \theta & 1 & 2 & a & b \\ 0.7 & 0.4 & 0.5 & 0.3 & 0.3 \end{pmatrix}.$$

Then μ is a fuzzy ideal of X, and $\top = 0.3$. If we take $\beta = 0.15$, then the fuzzy β-translation μ_β^T of μ is as follows:

$$\mu_\beta^T = \begin{pmatrix} \theta & 1 & 2 & a & b \\ 0.85 & 0.55 & 0.65 & 0.45 & 0.45 \end{pmatrix}.$$

Let ν be a fuzzy subset of X defined by

$$\nu = \begin{pmatrix} \theta & 1 & 2 & a & b \\ 0.88 & 0.57 & 0.69 & 0.47 & 0.47 \end{pmatrix}.$$

Then ν is a fuzzy ideal extension of the fuzzy β-translation μ_β^T of μ. But ν is not a fuzzy α-translation of μ for all $\alpha \in [0, \top]$. If we take $\alpha = 0.17$, then $\alpha = 0.17 > 0.15 = \beta$ and the fuzzy α-translation μ_α^T of μ is given as follows:

$$\mu_\alpha^T = \begin{pmatrix} \theta & 1 & 2 & a & b \\ 0.87 & 0.57 & 0.67 & 0.37 & 0.37 \end{pmatrix}.$$

Note that $\nu(x) \geq \mu_\alpha^T(x)$ for all $x \in X$, and hence ν is a fuzzy ideal extension of the fuzzy α-translation μ_α^T of μ.

By means of the definition of fuzzy α-translation, we know that $\mu_\alpha^T(x) \geq \mu(x)$ for all $x \in X$. Hence we have the following theorem.

3.14. Theorem. *Let μ be a fuzzy ideal of X and $\alpha \in [0, \top]$. Then the fuzzy α-translation μ_α^T of μ is a fuzzy ideal extension of μ.* □

A fuzzy ideal extension of a fuzzy ideal μ may not be represented as a fuzzy α-translation of μ, that is, the converse of Theorem 3.14 is not true in general as shown by the following example.

3.15. Example. (1) In Example 3.8, μ_2 cannot be represented as a fuzzy α-translation of μ_1 for all $\alpha \in [0, \top]$.

(2) Consider a BCI-algebra $X = \{\theta, 1, a, b, c\}$ where the $*$-multiplication is defined by Table 2.

Table 2. $*$-multiplication table for X

$*$	θ	1	a	b	c
θ	θ	θ	a	b	c
1	1	θ	a	b	c
a	a	a	θ	c	b
b	b	b	c	θ	a
c	c	c	b	a	θ

Let μ be a fuzzy subset of X defined by

$$\mu = \begin{pmatrix} \theta & 1 & a & b & c \\ 0.9 & 0.6 & 0.3 & 0.3 & 0.5 \end{pmatrix}.$$

Then μ is a fuzzy ideal of X. Let ν be a fuzzy subset of X given by

$$\nu = \begin{pmatrix} \theta & 1 & a & b & c \\ 0.94 & 0.66 & 0.38 & 0.38 & 0.57 \end{pmatrix}.$$

Then ν is a fuzzy ideal extension of μ. But it is not the fuzzy α-translation μ_α^T of μ for all $\alpha \in [0, \top]$.

Clearly, the intersection of fuzzy ideal extensions of a fuzzy subset μ of X is a fuzzy ideal extension of μ. But the union of fuzzy ideal extensions of a fuzzy subset μ of X is not a fuzzy ideal extension of μ as seen in the following example.

3.16. Example. Consider a BCI-algebra $X = \{\theta, a, b, c\}$ with Cayley table (Table 3).

Table 3. Cayley table

$*$	θ	a	b	c
θ	θ	a	b	c
a	a	θ	c	b
b	b	c	θ	a
c	c	b	a	θ

Let μ, ν and δ be fuzzy subsets of X defined by

$$\mu = \begin{pmatrix} \theta & a & b & c \\ 0.7 & 0.3 & 0.5 & 0.3 \end{pmatrix},$$

$$\nu = \begin{pmatrix} \theta & a & b & c \\ 0.8 & 0.6 & 0.5 & 0.5 \end{pmatrix},$$

and

$$\delta = \begin{pmatrix} \theta & a & b & c \\ 0.9 & 0.4 & 0.6 & 0.4 \end{pmatrix},$$

respectively. Then ν and δ are fuzzy ideal extensions of μ. Obviously, the union $\nu \cup \delta$ is a fuzzy extension of μ, but it is not a fuzzy ideal extension of μ since

$$(\nu \cup \delta)(c) = 0.5 \ngeq 0.6 = \min\{(\nu \cup \delta)(c * b), (\nu \cup \delta)(b)\}.$$

3.17. Definition. Let μ be a fuzzy subset of X and $\gamma \in [0, 1]$. A *fuzzy γ-multiplication* of μ, denoted by μ_γ^m, is defined to be a mapping

$$\mu_\gamma^m : X \to [0, 1],\ x \mapsto \mu(x) \cdot \gamma.$$

For any fuzzy subset μ of X, a fuzzy 0-multiplication μ_0^m of μ is clearly a fuzzy ideal of X.

3.18. Theorem. *If μ is a fuzzy ideal of X, then the fuzzy γ-multiplication of μ is a fuzzy ideal of X for all $\gamma \in [0, 1]$.*

Proof. Straightforward. □

3.19. Theorem. *For any fuzzy subset μ of X, the following are equivalent:*

(1) *μ is a fuzzy ideal of X.*
(2) *$(\forall \gamma \in (0, 1])$ (μ_γ^m is a fuzzy ideal of X).*

Proof. Necessity follows from Theorem 3.18. Let $\gamma \in (0, 1]$ be such that μ_γ^m is a fuzzy ideal of X. Then $\mu(\theta) \cdot \gamma = \mu_\gamma^m(\theta) \geq \mu_\gamma^m(x) = \mu(x) \cdot \gamma$ and

$$\begin{aligned} \mu(x) \cdot \gamma = \mu_\gamma^m(x) &\geq \min\{\mu_\gamma^m(x * y), \mu_\gamma^m(y)\} \\ &= \min\{\mu(x * y) \cdot \gamma, \mu(y) \cdot \gamma\} \\ &= \min\{\mu(x * y), \mu(y)\} \cdot \gamma \end{aligned}$$

for all $x, y \in X$. Since $\gamma \neq 0$, it follows that $\mu(\theta) \geq \mu(x)$ and

$$\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$$

for all $x, y \in X$. Hence μ is a fuzzy ideal of X. □

3.20. Theorem. *Let μ be a fuzzy subset of X, $\alpha \in [0, \top]$ and $\gamma \in (0, 1]$. Then every fuzzy α-translation μ_α^T of μ is a fuzzy ideal extension of the fuzzy γ-multiplication μ_γ^m of μ.*

Proof. For every $x \in X$, we have

$$\mu_\alpha^T(x) = \mu(x) + \alpha \geq \mu(x) \geq \mu(x) \cdot \gamma = \mu_\gamma^m(x),$$

and so μ_α^T is a fuzzy extension of μ_γ^m.

Assume that μ_γ^m is a fuzzy ideal of X. Then μ is a fuzzy ideal of X by Theorem 3.19. It follows from Theorem 3.2 that the fuzzy α-translation μ_α^T of μ is a fuzzy ideal of X for all $\alpha \in [0, \top]$. Therefore every fuzzy α-translation μ_α^T is a fuzzy ideal extension of the fuzzy γ-multiplication μ_γ^m of μ. □

The following example shows that Theorem 3.20 is not valid for $\gamma = 0$.

3.21. Example. Consider a BCI-algebra $(\mathbb{Z}, *, 0)$ where $\mathbb{Z}$ is the set of all integers and $*$ is the minus operation. Define a fuzzy subset $\mu : \mathbb{Z} \to [0, 1]$ by

$$\mu(x) := \begin{cases} \frac{1}{5} & \text{if } x > 3, \\ \frac{1}{3} & \text{if } x \leq 3. \end{cases}$$

Obviously μ_0^m is a fuzzy ideal of $\mathbb{Z}$. But

$$\begin{aligned} \mu_\alpha^T(4) = \mu(4) + \alpha \ &= \ \tfrac{1}{5} + \alpha \ < \ \tfrac{1}{3} + \alpha \\ &= \min\{\mu(4 * 1), \mu(1)\} + \alpha \\ &= \min\{\mu(4 * 1) + \alpha, \mu(1) + \alpha\} \\ &= \min\{\mu_\alpha^T(4 * 1), \mu_\alpha^T(1)\} \end{aligned}$$

for all $\alpha \in [0, \frac{2}{3}]$, which shows that μ_α^T is not a fuzzy ideal of X. Hence μ_α^T is not a fuzzy ideal extension of μ_0^m for all $\alpha \in [0, \frac{2}{3}]$.

The following examples illustrate Theorem 3.20.

3.22. Example. Consider a BCK-algebra $X = \{\theta, a, b, c, d\}$ where the $*$-multiplication is defined by Table 4.

Table 4. $*$-multiplication table for X

$*$	θ	a	b	c	d
θ	θ	θ	θ	θ	θ
a	a	θ	θ	a	a
b	b	b	θ	b	b
c	c	c	c	θ	c
d	d	d	d	d	θ

Let μ be a fuzzy subset of X defined by

$$\mu = \begin{pmatrix} \theta & a & b & c & d \\ 0.7 & 0.4 & 0.2 & 0.5 & 0.1 \end{pmatrix}.$$

Then μ is a fuzzy ideal of X. If we take $\gamma = 0.2$, then the 0.2-multiplication $\mu^m_{0.2}$ of μ is given by

$$\mu^m_{0.2} = \begin{pmatrix} \theta & a & b & c & d \\ 0.14 & 0.08 & 0.04 & 0.10 & 0.02 \end{pmatrix}.$$

Clearly, $\mu^m_{0.2}$ is a fuzzy ideal of X. Also, for any $\alpha \in [0, 0.3]$, the fuzzy α-translation μ^T_α of μ is given as follows:

$$\mu^T_\alpha = \begin{pmatrix} \theta & a & b & c & d \\ 0.7+\alpha & 0.4+\alpha & 0.2+\alpha & 0.5+\alpha & 0.1+\alpha \end{pmatrix}.$$

Then μ^T_α is a fuzzy extension of $\mu^m_{0.2}$ and μ^T_α is always a fuzzy ideal of X for all $\alpha \in [0, 0.3]$. Therefore μ^T_α is a fuzzy ideal extension of $\mu^m_{0.2}$ for all $\alpha \in [0, 0.3]$.

3.23. Example. Consider a BCI-algebra $X = \{\theta, 1, 2, a, b\}$, where the $*$-multiplication is defined by Table 5.

Table 5. $*$-multiplication table for X

$*$	θ	1	2	a	b
θ	θ	θ	θ	a	a
1	1	θ	1	b	a
2	2	2	θ	a	a
a	a	a	a	θ	θ
b	b	a	b	1	θ

Let μ be a fuzzy subset of X defined by

$$\mu = \begin{pmatrix} \theta & 1 & 2 & a & b \\ 0.6 & 0.1 & 0.5 & 0.3 & 0.1 \end{pmatrix}.$$

Then μ is a fuzzy ideal of X. If we take $\gamma = 0.1$, then the 0.1-multiplication $\mu^m_{0.1}$ of μ is given by

$$\mu^m_{0.1} = \begin{pmatrix} \theta & 1 & 2 & a & b \\ 0.06 & 0.01 & 0.05 & 0.03 & 0.01 \end{pmatrix}.$$

Clearly, $\mu^m_{0.1}$ is a fuzzy ideal of X. Also, for any $\alpha \in [0, 0.4]$, the fuzzy α-translation μ^T_α of μ is given as follows:

$$\mu^T_\alpha = \begin{pmatrix} \theta & 1 & 2 & a & b \\ 0.6+\alpha & 0.1+\alpha & 0.5+\alpha & 0.3+\alpha & 0.1+\alpha \end{pmatrix}.$$

Then μ^T_α is a fuzzy extension of $\mu^m_{0.1}$ and μ^T_α is always a fuzzy ideal of X for all $\alpha \in [0, 0.4]$. Therefore μ^T_α is a fuzzy ideal extension of $\mu^m_{0.1}$ for all $\alpha \in [0, 0.4]$.

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