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ON GENERALIZED DERIVATIONS AND COMMUTATIVITY OF PRIME AND SEMIPRIME RINGS

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Abstract

Let R be a prime ring and θ, ϕ endomorphisms of R. An additive mapping $F : R \longrightarrow R$ is called a generalized (θ, ϕ)-derivation on R if there exists a (θ, ϕ)-derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)\theta(y) + \phi(x)d(y)$ for all $x, y \in R$. Let S be a non-empty subset of R. In the present paper for various choices of S we study the commutativity of a semiprime (prime) ring R admitting a generalized (θ, ϕ)-derivation F satisfying any one of the properties: (i) $F(x)F(y) - xy \in Z(R)$, (ii) $F(x)F(y) + xy \in Z(R)$, (iii) $F(x)F(y) - yx \in Z(R)$, (iv) $F(x)F(y) + yx \in Z(R)$, (v) $F[x, y] - [x, y] \in Z(R)$, (vi) $F[x, y] + [x, y] \in Z(R)$, (vii) $F(x \circ y) - x \circ y \in Z(R)$, and (viii) $F(x \circ y) + x \circ y \in Z(R)$, for all $x, y \in S$.

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Introduction

Let R be an associative ring with centre $Z(R)$. A ring R is said to be prime (resp. semiprime) if $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$ (resp. $aRa = \{0\}$ implies that $a = 0$). For any $x, y \in R$ we shall write $[x, y] = xy - yx$ and $x \circ y = xy + yx$. An additive subgroup U of R is said to be a Lie ideal of R if $[x, u] \in U$ for all $x \in R$ and $u \in U$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let θ, ϕ be endomorphisms of R. An additive mapping $d : R \longrightarrow R$ is called

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a (θ,ϕ)-derivation if $d(xy) = d(x)\theta(y) + \phi(x)d(y)$ for all $x, y \in R$. An additive mapping $F : R \longrightarrow R$ is called a generalized (θ,ϕ)-derivation on R if there exists a (θ,ϕ)-derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)\theta(y) + \phi(x)d(y)$ for all $x, y \in R$.

We shall call a generalized (θ, I)-derivation a generalized θ-derivation, where I is the identity automorphism of R. Similarly a generalized (I,ϕ)-derivation will be called a generalized ϕ-derivation

1. Lie ideals and generalized derivations in prime rings

In order to prove our theorems, we will make extensive use of the following known results.

1.1. Lemma. [5, Lemma 4] *If $U \not\subseteq Z(R)$ is a Lie ideal of a 2-torsion free prime ring R, and $a, b \in R$ are such that $aUb = (0)$, then either $a = 0$ or $b = 0$.* □

1.2. Lemma. [3, Lemma 3.4] *Let R be a 2-torsion free prime ring and $U \not\subseteq Z(R)$ a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If the elements $a \in U$ and $b \in R$ are such that $axb + bxa = 0$, then $axb = bxa = 0$ for all $x \in U$.* □

1.3. Lemma. [2, Theorem 7] *Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R. If R admits a nonzero derivation d such that $[d(u), u] \in Z(R)$, for all $u \in U$, then $U \subseteq Z(R)$.* □

1.4. Theorem. *Let R be a 2-torsion free prime ring and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If R admits a generalized derivation F with associated nonzero derivation d such that $F([x, y]) - [d(x), d(y)] = 0$, for all $x, y \in U$, then $U \subseteq Z(R)$.*

Proof. Suppose that $U \not\subseteq Z(R)$. By assumption we have

(1.1) $$F[x, y] = [d(x), d(y)], \quad \text{for all } x, y \in U.$$

Replacing y by $2yx$ in (1.1) and using the fact that R is 2-torsion free, we get

(1.2) $$F([x, y])x+[x, y]d(x) = [d(x), d(y)]x+d(y)[d(x), x]+[d(x), y]d(x), \quad \text{for all } x, y \in U.$$

Comparing (1.1) and (1.2), we have

(1.3) $$[x, y]d(x) = d(y)[d(x), x] + [d(x), y]d(x), \quad \text{for all } x, y \in U.$$

Now substituting $2yx$ for y in (1.3) and using (1.3), we obtain

(1.4) $$d(x)y[d(x), x] + [d(x), x]yd(x) = 0, \quad \text{for all } x, y \in U.$$

Since $[d(x), x] \in U$, Lemma 1.2 yields that $d(x)y[d(x), x] = 0$, for all $x, y \in U$. That is $d(x)U[d(x), x] = (0)$ for all $x \in U$. Application of Lemma 1.1 yields that $d(x) = 0$ or $[d(x), x] = 0$, for all $x \in U$. Since d is a nonzero derivation, $[d(x), x] = 0$, for all $x \in U$. Thus Lemma 1.3 implies that $U \subseteq Z(R)$, which is a contradiction. Hence the theorem is proved. □

Using similar arguments to the above, we can prove the following:

1.5. Theorem. *Let R be a 2-torsion free prime ring and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If R admits a generalized derivation F with associated nonzero derivation d such that $F[x, y] + [d(x), d(y)] = 0$, for all $x, y \in U$, then $U \subseteq Z(R)$.* □

2. One sided ideals and generalized derivations in prime and semiprime rings

Daif and Bell [7] proved that if a semiprime ring R admits a derivation d such that either $d([x,y]) + [x,y] = 0$ or $d([x,y]) - [x,y] = 0$, for all x, y, in a nonzero ideal I of R, then R is necessarily commutative. Hongan [8] generalized the above result, considering R satisfying the conditions $d([x,y]) + [x,y] \in Z(R)$ and $d([x,y]) - [x,y] \in Z(R)$, for all $x, y \in I$. Motivated by the above observations, we explore the commutativity of a prime ring admitting a generalized derivation F satisfying any one of the following conditions:

(i) $F([x,y]) - [x,y] \in Z(R)$,
(ii) $F([x,y]) + [x,y] \in Z(R)$,
(iii) $F(x \circ y) - (x \circ y) \in Z(R)$, and
(iv) $F(x \circ y) + (x \circ y) \in Z(R)$,

for all x, y in some appropriate subsets of R.

2.1. Lemma. [9, Lemma 3] *If a prime ring R contains a nonzero commutative right ideal I, then R is commutative.* □

2.2. Theorem. *Let R be a prime ring and I a nonzero right ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d such that $d(Z(R)) \neq (0)$. If $F([x,y]) - [x,y] \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Proof. Since $d(Z(R)) \neq (0)$, there exists $c \in Z(R)$ such that $d(c) \neq 0$. Thus $d(c) \in Z(R)$. By assumption, we have

$$F([x,y]) - [x,y] \in Z(R), \quad \text{for all } x, y \in I. \tag{2.1}$$

Replacing y by yc in (2.1), we have

$$\{F([x,y]) - [x,y]\}c + [x,y]d(c) \in Z(R), \quad \text{for all } x, y \in I. \tag{2.2}$$

This implies that $[[x,y]d(c), r] = 0$, for all $x, y \in I$ and $r \in R$. That is, $[[x,y], r]d(c) = 0$, for all $x, y \in I$ and $r \in R$. Since R is prime and $d(c) \neq 0$, we find that $[[x,y], r] = 0$ for all $x, y \in I$ and $r \in R$. Replacing y by yx, we have

$$[x,y][x,r] + [[x,y], r]x = 0, \quad \text{for all } x, y \in I, \ r \in R \tag{2.3}$$

In view of the fact that $[[x,y], r] = 0$, relation (2.3) yields that $[x,y][x,r] = 0$ for all $x, y \in I$ and $r \in R$. Replace r by ry, to obtain $[x,y]r[x,y] = 0$ for all $x, y \in I$ and $r \in R$, that is, $[x,y]R[x,y] = (0)$ for all $x, y \in I$. The primeness of R yields that $[x,y] = 0$ for all $x, y \in I$, i.e. I is a commutative right ideal. Hence application of Lemma 2.1 completes the proof of the theorem. □

2.3. Theorem. *Let R be a prime ring and I a nonzero right ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d such that $d(Z(R)) \neq (0)$. If $F([x,y]) + [x,y] \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Proof. If R satisfies the assumption $F([x,y]) + [x,y] \in Z(R)$ for all $x, y \in I$, then the generalized derivation $(-F)$ also satisfies $(-F)([x,y]) - [x,y] \in Z(R)$ for all $x, y \in I$, and hence the proof follows from Theorem 2.2. □

2.4. Theorem. *Let R be a prime ring and I a nonzero right ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d such that $d(Z(R)) \neq (0)$. If $F(x \circ y) - (x \circ y) \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Proof. By assumption, we have

$$F(x \circ y) - (x \circ y) \in Z(R), \quad \text{for all } x, y \in I. \tag{2.4}$$

Since $d(Z(R)) \neq (0)$, there exists $c \in Z(R)$ such that $d(c) \neq 0$ and $d(c) \in Z(R)$. Replacing y by yc in (2.4), we have

$$\{F(x \circ y) - x \circ y\}c + (x \circ y)d(c) \in Z(R), \quad \text{for all } x, y \in I. \tag{2.5}$$

That is, $(x \circ y)d(c) \in Z(R)$, for all $x, y \in I$. Since $d(c) \neq 0$ and R is prime, it follows that $(x \circ y) \in Z(R)$ for all $x, y \in I$. Thus $[(x \circ y), r] = 0$ for all $x, y \in I$ and $r \in R$. Substituting yx for y, we obtain $(x \circ y)[x, r] = 0$ for all $x, y \in I$ and $r \in R$. Replacing r by sr, we find that $(x \circ y)R[x, r] = (0)$ for all $x, y \in I$ and $r \in R$. Now the primeness of R, for each $x \in I$, gives either $(x \circ y) = 0$ or $[r, x] = 0$ for all $y \in I$ and $r \in R$. Let $I_1 = \{x \in I \mid (x \circ y) = 0 \text{ for all } y \in I\}$ and $I_2 = \{x \in I \mid [r, x] = 0, \text{ for all } r \in R\}$. Then I_1 and I_2 are both additive subgroups of I whose union is I. Hence either $I_1 = I$ or $I_2 = I$.

If $I_1 = I$, then $(x \circ y) = 0$ for all $x, y \in I$. Now replace y by yz, to get $(x \circ yz) = (x \circ y)z - y[x, z] = 0$, which gives $y[x, z] = 0$ for all $x, y, z \in I$. Thus $yR[x, z] = 0$ for all $x, y, z \in I$. Since I is a nonzero right ideal of R, the primeness of R yields that $[x, z] = 0$ for all $x, z \in I$. Thus I is commutative and an application of Lemma 2.1 gives that R is commutative. On the other hand if $I_2 = I$, then $[r, x] = 0$ for all $r \in R$ and $x \in I$. Substituting xs for x, we get $x[r, s] = 0$ for all $x \in I$ and $r, s \in R$. Since I is a nonzero right ideal of R, $[r, s] = 0$ for all $r, s \in R$. Hence in both the cases R is commutative. □

Using the same techniques with the necessary variations, we get the following:

2.5. Theorem. *Let R be a prime ring and I a nonzero right ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d such that $d(Z(R)) \neq (0)$. If $F(x \circ y) + (x \circ y) \in Z(R)$ for all $x, y \in I$, then R is commutative.* □

The following example demonstrates that the above results do not hold for arbitrary rings.

2.6. Example. Consider S as any ring. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| \; a, b \in S \right\}$ and let $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \middle| \; b \in S \right\}$ be an ideal of R. Define $F : R \longrightarrow R$ by $F(x) = 2e_{11}x - xe_{11}$. Then F is a generalized derivation with associated derivation d given by $d(x) = e_{11}x - xe_{11}$. It can be easily seen that R satisfies the properties (i) $F([x, y]) - [x, y] \in Z(R)$, (ii) $F([x, y]) + [x, y] \in Z(R)$, (iii) $F(x \circ y) - (x \circ y) \in Z(R)$ and (iv) $F(x \circ y) + (x \circ y) \in Z(R)$ for all $x, y \in I$. However, R is not commutative.

The following Lemmas are generalizations of a result of Mayne [9] and a result of Bresar [6, Lemma 4], respectively.

2.7. Lemma. [4, Theorem 3] *Let R be a semiprime ring and I a nonzero left ideal of R. If R admits a derivation d such that $d(I) \neq (0)$ and $[d(x), x] \in Z(R)$ for all $x \in I$, then $I \subseteq Z(R)$.* □

2.8. Lemma. [3, Lemma 2.6] *Let R be a 2-torsion free semiprime ring and I a nonzero left ideal of R. If $a, b \in R$ and $axb + bxa = 0$ for all $x \in I$, then $axb = bxa = 0$ for all $x \in I$.* □

2.9. Theorem. *Let R be a 2-torsion free semiprime ring and I a nonzero left ideal of R such that $A_r(I) = 0$, the right annihilator of I. If R admits a generalized derivation F with associated nonzero derivation d such that $F[x, y] - [d(x), d(y)] = 0$ for all $x, y \in I$, then $I \subseteq Z(R)$.*

Proof. By assumption, we have

(2.6) $F[x,y]-[d(x),d(y)]=0, \quad \text{for all } x,y\in I.$

Replacing y by yx in (2.6), we get

(2.7) $F([x,y])x+[x,y]d(x)=[d(x),d(y)]x+d(y)[d(x),x]+[d(x),y]d(x), \quad \text{for all } x,y\in I.$

Comparing (2.6) and (2.7), we have

(2.8) $[x,y]d(x)=d(y)[d(x),x]+[d(x),y]d(x), \quad \text{for all } x,y\in I.$

Now substituting xy for y in (2.8) and using (2.8), we obtain

(2.9) $d(x)y[d(x),x]+[d(x),x]yd(x)=0, \quad \text{for all } x,y\in I.$

Application of Lemma 2.8 yields that $d(x)y[d(x),x]=0$, for all $x,y\in I$. This implies that $[d(x),x]y[d(x),x]=0$, for all $x,y\in I$. Thus, we have $[d(x),x]I[d(x),x]=(0)$, i.e. $(I[d(x),x])^2=(0)$. Hence $I[d(x),x]$ is a nilpotent left ideal of R. Since R is semiprime, $I[d(x),x]=(0)$, for all $x\in I$. By our hypothesis $[d(x),x]=0$ for all $x\in I$. Hence by Lemma 2.7, we have $I\subseteq Z(R)$. □

Using similar arguments to the above we can prove the following:

2.10. Theorem. *Let R be a 2-torsion free semiprime ring and I a left ideal of R such that $A_r(I)=0$, the right annihilator of I. If R admits a generalized derivation F with associated nonzero derivation d such that $F[x,y]+[d(x),d(y)]=0$ for all $x,y\in I$, then $I\subseteq Z(R)$.* □

3. Ideals and generalized (θ,ϕ)-derivations in prime rings

3.1. Theorem. *Let R be a prime ring and I a nonzero ideal of R. Suppose that ϕ is an automorphism of R. If R admits a generalized ϕ-derivation F with associated ϕ-derivation d such that $F(xy)-xy\in Z(R)$ for all $x,y\in I$, then either $d=0$ or R is commutative.*

Proof. By assumption, we have $F(xy)-xy\in Z(R)$ for all $x,y\in I$. This can be written as $F(x)y+\phi(x)d(y)-xy\in Z(R)$. Replacing y by yz, we obtain

(3.1) $F(x)yz+\phi(x)d(y)z+\phi(x)\phi(y)d(z)-xyz\in Z(R), \text{ for all } x,y,z\in I.$

Thus, in particular

(3.2) $[(F(x)y+\phi(x)d(y)-xy)z+\phi(x)\phi(y)d(z),z]=0, \text{ for all } x,y,z\in I.$

Using (3.1) and (3.2), we get

(3.3) $[\phi(x)\phi(y)d(z),z]=0, \text{ for all } x,y,z\in I.$

Replacing x by rx in the above expression we obtain $[\phi(r),z]\phi(x)\phi(y)d(z)=0$ for all $x,y,z\in I$ and $r\in R$. Now replace y by yr, to get $[\phi(r),z]\phi(x)\phi(r)\phi(y)(d(z))=0$ for all $x,y,z\in I$. That is, $[\phi(r),z]\phi(x)R\phi(y)(d(z))=(0)$ for all $x,y,z\in I$. Thus, the primeness of R yields that for each $z\in I$, either $[\phi(r),z]\phi(x)=0$ or $\phi(y)d(z)=0$.

Let $I_1=\{z\in I \mid [\phi(r),z]\phi(x)=0, \text{ for all } x\in I \text{ and } r\in R\}$ and $I_2=\{z\in I \mid \phi(y)d(z)=0, \text{ for all } x\in I\}$. Then I_1 and I_2 are two additive subgroups of I whose union is I. Therefore either $I_1=I$ or $I_2=I$.

If $I_2=I$ then $\phi(y)d(z)=0$ for all $y,z\in I$. Replace y by $[y,r]$ to get $[\phi(y),\phi(r)]d(z)=0$ for all $y,z\in I$ and $r\in R$. Now replace r by sr to get $[\phi(y),\phi(s)]\phi(r)d(z)=0$ for all $y,z\in I$ and $r,s\in R$ i.e., $[\phi(y),\phi(s)]Rd(z)=(0)$, for all $y,z\in I$ and $s\in R$. Again the primeness of R gives that either $[\phi(y),\phi(s)]=0$ or $d(z)=0$ for all $y\in I$ and $s\in R$. If $[\phi(y),\phi(s)]=0$, for all $y\in I$ and $s\in R$, then $[y,s]=0$ i.e., I is commutative. Hence R

is commutative by Lemma 2.1. On the other hand if $d(z) = 0$ for all $z \in I$, this implies that $d = 0$ on R.

Now assume the remaining possibility, i.e. $I_1 = I$. Now we have $[\phi(r), z]\phi(x) = 0$ for all $x, z \in I$ and $r \in R$. That is, $\phi^{-1}[\phi(r), z]RI = (0)$ for all $z \in I$. The primeness of R implies that $[\phi(r), z] = 0$ for all $z \in I$ and $r \in R$, and hence we get the required result. □

One can note that if R admits a generalized ϕ-derivation F satisfying $F(xy) + xy \in Z(R)$ for all $x, y \in I$, then the generalized ϕ-derivation $(-F)$ also satisfies $(-F)(xy) - xy \in Z(R)$ for all $x, y \in I$. Hence in view of Theorem 3.1 we conclude the following:

3.2. Theorem. *Let R be a prime ring and I a nonzero ideal of R. Suppose ϕ is an automorphism of R. If R admits a generalized ϕ-derivation F with associated ϕ-derivation d such that $F(xy)+xy \in Z(R)$ for all $x, y \in I$, then either $d = 0$ or R is commutative.* □

3.3. Theorem. *Let R be a prime ring and I a nonzero ideal of R. Suppose ϕ is an automorphism of R. If F is a generalized ϕ-derivation with associated ϕ-derivation d such that $F(xy) - yx \in Z(R)$ for all $x, y \in I$, then either $d = 0$ or R is commutative.*

Proof. For any $x, y \in I$ we have $F(xy) - yx \in Z(R)$. This can be written as $F(x)y + \phi(x)d(y) - yx \in Z(R)$ for all $x, y \in I$. Substituting xy for x, we obtain

$$F(x)yy + \phi(x)d(y)y + \phi(x)\phi(y)d(y) - yxy \in Z(R), \text{ for all } x, y \in I. \tag{3.4}$$

In particular

$$[(F(x)y + \phi(x)d(y) - yx)y + \phi(x)\phi(y)d(y), y] = 0, \text{ for all } x, y \in I. \tag{3.5}$$

An application of (3.4) and (3.5) gives $[\phi(x)\phi(y)d(y), y] = 0$ for all $x, y \in I$, i.e.

$$\phi(x)\phi(y)[d(y), y] + \phi(x)[\phi(y), y]d(y) + [\phi(x), y]\phi(y)d(y), \text{ for all } x, y \in I. \tag{3.6}$$

Replacing x by zx in (3.6) and using (3.6), we find that

$$[\phi(z), y]\phi(x)\phi(y)d(y) = 0, \text{ for all } x, y, z \in I. \tag{3.7}$$

Replacing x by xr in (3.7), we get $[\phi(z), y]\phi(x)\phi(r)\phi(y)d(y) = 0$ for all $x, y, z \in I$, $r \in R$, i.e. $[\phi(z), y]\phi(x)R\phi(y)d(y) = (0)$ for all $x, y, z \in I$. Thus the primeness of R gives that for each $y \in I$, either $[\phi(z), y]\phi(x) = 0$ or $\phi(y)d(y) = 0$, for all $y \in I$. The sets $y \in I$ for which these two properties hold, are additive subgroups of I whose union is I. Then either $[\phi(z), y]\phi(x) = 0$ or $\phi(y)d(y) = 0$, for all $x, y, z \in I$. If $\phi(y)d(y) = 0$, for all $y \in I$, then linearization gives

$$\phi(x)d(y) + \phi(y)d(x) = 0, \text{ for all } x, y \in I. \tag{3.8}$$

Replace y by zy to get

$$\phi(x)d(z)y + \phi(x)\phi(z)d(y) + \phi(z)\phi(y)d(x) = 0, \text{ for all } x, y \in I. \tag{3.9}$$

Comparing (3.8) and (3.9), we get $\phi(x)d(z)y + \phi(x)\phi(z)d(y) - \phi(z)\phi(x)d(y) = 0$ for all $x, y, z \in I$. That is,

$$\phi(x)d(z)yr + [\phi(x), \phi(z)]d(y)r + [\phi(x), \phi(z)]\phi(y)d(r) = 0, \text{ for all } x, y, z \in I,\ r \in R. \tag{3.10}$$

An application of (3.9) in (3.10) yields that $[\phi(x), \phi(z)]\phi(y)d(r) = 0$ for all $x, y, z \in I$ and $r \in R$. Now replace y by ys to get $[\phi(x), \phi(z)]\phi(y)\phi(s)d(r) = 0$ for all $x, y, z \in I$ and $r, s \in R$, i.e. $[\phi(x), \phi(z)]\phi(y)Rd(r) = (0)$ for all $x, y, z \in I$ and $r \in R$. Thus the primeness of R implies that either $[\phi(x), \phi(z)]\phi(y) = 0$ or $d(r) = 0$, for all $x, y, z \in I$ and $r \in R$. Assume $[x, z]y = 0$. Then $[x, z] = 0$ for all $x, z \in I$. Since I is a nonzero ideal of a prime ring R, then R is commutative by Lemma 2.1. On the other hand we have $[\phi(z), y]\phi(x) = 0$ for all $x, y, z \in I$. Substituting x for rx we get $[\phi(z), y]\phi(r)\phi(x) = 0$ for all $x, y, z \in I$ and $r \in R$. That is, $[\phi(z), y]R\phi(x) = (0)$ for all $x, y, z \in I$. Since I is a nonzero ideal and R is prime, $[\phi(z), y] = 0$ for all $y, z \in I$. Again I is commutative so R is commutative by Lemma 2.1. Hence the theorem is completely proved. □

Arguing as above we can prove the following:

3.4. Theorem. *Let R be a prime ring and I a nonzero ideal of R. Suppose ϕ is an automorphism of R. If F is a generalized ϕ-derivation with associated ϕ-derivation d is such that $F(xy) + yx \in Z(R)$ for all $x, y \in I$, then either $d = 0$ or R is commutative.* □

3.5. Theorem. *Let R be a prime ring and I a nonzero ideal of R. Suppose ϕ is an automorphism of R. If R admits a generalized ϕ-derivation F with associated nonzero ϕ-derivation d such that $F(x)F(y) - xy \in Z(R)$ for all $x, y \in I$, then either $d = 0$ or R is commutative.*

Proof. By assumption we have $F(x)F(y) - xy \in Z(R)$ for all $x, y \in I$. Replacing y by yr, we find that

$$(F(x)F(y) - xy)r + F(x)\phi(y)d(r) \in Z(R), \text{ for all } x, y \in I, r \in R. \tag{3.11}$$

This implies that

$$[F(x)\phi(y)d(r), r] = 0, \text{ for all } x, y \in I, \ r \in R. \tag{3.12}$$

This can be rewritten as

$$F(x)[\phi(y)d(r), r] + [F(x), r]\phi(y)d(r) = 0, \text{ for all } x, y \in I, \ r \in R. \tag{3.13}$$

Substituting $(\phi^{-1}(F(x)))y$ for y in (3.14) and using (3.14), we find that

$$[F(x), r]F(x)\phi(y)d(r) = 0, \text{ for all } x, y \in I, \ r \in R. \tag{3.14}$$

That is, $[F(x), r]F(x))R\phi(y)d(r) = (0)$. Thus for each $r \in R$ the primeness of R forces that either $[F(x), r]F(x) = 0$ or $\phi(y)d(r) = 0$. The sets of all $r \in R$ for which these two properties hold form additive subgroups of R whose union is I. Hence either $[F(x), r]F(x) = 0$ or $\phi(y)d(r) = 0$ for all $x, y \in I$ and $r \in R$. If $\phi(y)d(r) = 0$ then replace y by ys, to obtain $\phi(y)\phi(s)d(r) = 0$ for all $y \in I$ and $r, s \in R$, i.e. $\phi(y)Rd(r) = (0)$ for all $r \in R$ and $y \in I$.

Since I is a nonzero ideal of R and R is prime, the above relation yields that $d(r) = 0$ for all $r \in R$. Therefore we assume the remaining possibility that $[F(x), r]F(x) = 0$ for all $x \in I$ and $r \in R$. Substituting r by sr and using this we find that $[F(x), r]RF(x) = (0)$ for all $x \in I$ and $r \in R$. The primeness of R implies that for each $x \in I$, either $F(x) = 0$ or $[F(x), r] = 0$. Thus in each case we have $[F(x), r] = 0$ for all $x \in I$ and $r \in R$. Replacing x by xr and using this we find that

$$[\phi(x), r]d(r) + \phi(x)[d(r), r] = 0, \text{ for all } x \in I, \ r \in R. \tag{3.15}$$

Now again replace x by sx in (3.15) to get

$$\phi(s)[\phi(x), r]d(r) + [\phi(s), r]\phi(x)d(r) + \phi(s)\phi(x)[d(r), r] = 0, \text{ for all } x \in I, \ r \in R. \tag{3.16}$$

Comparing (3.15) and (3.16), we get $[\phi(s), r]\phi(x)d(r) = 0$ for all $x \in I$ and $r, s \in R$. That is, $[\phi(s), r]\phi(x)Rd(r) = (0)$ for all $x \in I$ and $r, s \in R$. Thus, the primeness of R gives either $[\phi(s), r]\phi(x) = 0$ or $d(r) = 0$.

If $[\phi(s), r]\phi(x) = 0$ for all $r, s \in R$ and $x \in I$, we have $[\phi(s), r] = 0$ for all $r, s \in R$. Hence, using Lemma 2.1, we get the required result. □

Using the same arguments we can prove the following:

3.6. Theorem. *Let R be a prime ring and I a nonzero ideal of R. Suppose ϕ is an automorphism of R. If R admits a generalized ϕ-derivation F with associated ϕ-derivation d such that $F(x)F(y)+xy \in Z(R)$ for all $x, y \in I$, then either $d = 0$ or R is commutative.* □

3.7. Theorem. *Let R be a prime ring and I a nonzero ideal of R. Suppose ϕ is an automorphism of R. Then the following conditions are equivalent:*

(i) *R admits a generalized ϕ-derivation F with associated nonzero ϕ-derivation d such that $F(xy) - xy \in Z(R)$ or $F(xy) + xy \in Z(R)$, for all $x, y \in I$.*
(ii) *R admits a generalized ϕ-derivation F with associated nonzero ϕ-derivation d such that $F(xy) - yx \in Z(R)$ or $F(xy) + yx \in Z(R)$, for all $x, y \in I$.*
(iii) *R admits a generalized ϕ-derivation F with associated nonzero ϕ-derivation d such that $F(x)F(y) - xy \in Z(R)$ or $F(x)F(y) + xy \in Z(R)$, for all $x, y \in I$.*
(iv) *R is commutative.*

Proof. Obviously, (iv) $\Longrightarrow$ (i), (ii) and (iii). Now, we show that (i) $\Longrightarrow$ (iv). For each $x \in I$ we set $I_1 = \{y \in I \mid F(xy) - xy \in Z(R)\}$ and $I_2 = \{y \in I \mid F(xy) + xy \in Z(R)\}$. Then I_1 and I_2 are additive subgroups of I whose union is I. Thus by Brauer's trick, either $I_1 = I$ or $I_2 = I$. Therefore, R is commutative by Theorem 3.1 and Theorem 3.2.

(ii) $\Longrightarrow$ (iv) For each $x \in I$, set $I_1 = \{y \in I \mid F(xy) - yx \in Z(R)\}$ and $I_2 = \{y \in I \mid F(xy) + yx \in Z(R)\}$. Arguing as above and using Theorem 3.3 and Theorem 3.4, R is commutative.

It remains to prove that (iii) $\Longrightarrow$ (iv) Now for each $x \in I$, set $I_1 = \{y \in I \mid F(x)F(y) - xy \in Z(R)\}$ and $I_2 = \{y \in I \mid F(x)F(y) + xy \in Z(R)\}$. Then using similar arguments, R is commutative by Theorem 3.5 and Theorem 3.6 □

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