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# WARPED PRODUCT SEMI-INVARIANT SUBMANIFOLDS IN LOCALLY DECOMPOSABLE RIEMANNIAN MANIFOLDS

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**Abstract**

Warped product semi-invariant submanifolds were defined and studied in locally Riemannian product manifolds in (K. Matsumoto, *On submanifolds of locally product Riemannian manifolds*, TRU Math. **18** (2), 145–157, 1982). In this paper, we go on to study warped product semi-invariant submanifolds in locally decomposable Riemannian manifolds. We prove several fundamental properties of warped product semi-invariant submanifolds, and establish a general inequality for a warped product semi-invariant submanifold in a locally decomposable Riemannian manifold. After that, we investigate warped product semi-invariant submanifolds in a locally decomposable Riemannian manifold which satisfy the equality case of the inequality and obtain some new results.

**Keywords:** Semi-invariant submanifold, Warped product semi-invariant submanifold, Locally decomposable Riemannian manifold.

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## 1. Introduction

It is well known that the notation of warped product plays an important role in differential geometry as well as in physics. R L Bishop and B. O'Neill studied warped product manifolds from a differential geometric point of view.

Generalizing the geometry of invariant and anti-invariant submanifolds, A. Bejancu defined CR-submanifolds in almost Hermitian (Kaehlerian) manifolds, and defined semi-invariant submanifods of locally product Riemannian manifolds [2]. Similar definitions were applied to submanifolds of almost contact metric manifolds by many geometers [see references].

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In [4], B-Y. Chen introduced the notion of a CR-warped product submanifold in a Kaehler manifold. He investigated such submanifolds and established a sharp relationship between the warping function $f$ of a CR-warped product submanifold $M_1 \times_f M_2$ and the squared norm of the second fundamental form $\| h \|$.

Firstly, S. Tachibana [7] introduced and studied a class of locally Riemannian product manifolds. Later, A. Bejancu [2] and K. Matsumoto [5] defined and studied the geometry of semi-invariant submanifolds of locally Riemannian product manifolds.

In [1], we defined semi-invariant submanifolds of a Riemannian product manifold and studied the fundamental properties of these submanifolds. Necessary and sufficient conditions were given for a semi-invariant submanifold of a Riemannian product manifold to be a locally Riemannian product manifold. Moreover, the integrability of invariant distributions and anti-invariant distributions were investigated.

In this paper, we go on to study the geometry of warped product semi-invariant submanifolds of a locally decomposable Riemannian manifold. We obtain an inequality for the squared norm of the second fundamental form in terms of the warping function for a warped product semi-invariant submanifold in a locally decomposable Riemannian manifold. After that we consider warped product semi-invariant submanifolds in a locally decomposable Riemannian manifold which satisfy the equality case of the inequality, and some applications are derived.

## 2. Preliminaries

In this section, we give the definitions and terminology used throughout this paper. We recall some necessary facts and formulas from the theory of submanifolds in any Riemannian manifold. For an arbitrary submanifold $M$ of any Riemannian manifold $\bar{M}$, the Gauss and Weingarten formulas are, respectively, given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.1}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.2}$$

for any vector fields $X, Y$ tangent to $M$ and $V$ normal to $M$, where $\bar{\nabla}$, $\nabla$ denote the Levi-Civita connections on $\bar{M}$ and $M$, respectively. Moreover, $h : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM^{\perp})$ is the second fundamental form of $M$ in $\bar{M}$, where $\Gamma(TM)$ denotes the Lie algebra of vector fields on $M$. $\nabla^{\perp}$ is the normal connection on the normal bundle $\Gamma(TM^{\perp})$ and $A_V$ is the shape operator of $M$ with respect to $V$. Furthermore, $A_V$ and $h$ are related by the formula

$$g(A_V X, Y) = g(h(X,Y), V) \tag{2.3}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$, where $g$ denotes the Riemannian metric on $M$ as well as on $\bar{M}$.

Now, if we denote the Riemannian curvature tensors of the connections $\bar{\nabla}$ and $\nabla$ by $\bar{R}$ and $R$, respectively, then the equations of Gauss, Codazzi and Ricci are, respectively, given by formulas

$$\begin{aligned} g(\bar{R}(X,Y)Z,W) = g(R(X,Y)Z,W) &+ g(h(X,W),h(Y,Z)) \\ &- g(h(X,Z),h(Y,W)) \end{aligned} \tag{2.4}$$

$$g(\bar{R}(X,Y)\xi,\eta) = g(\bar{R}(X,Y)^{\perp}\xi,\eta) - g([A_\xi, A_\eta]X,Y) \tag{2.5}$$

and

$$\{\bar{R}(X,Y)Z\}^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) \tag{2.6}$$

for any vector fields $X, Y, Z, W$ tangent to $M$ and $\xi, \eta$ normal to $M$, where $\{\bar{R}(X,Y)Z\}^{\perp}$ denotes the normal component of $\bar{R}(X,Y)Z$ and the covariant derivative $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla_X^{\perp} h(Y,Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y)$$

for any vector fields $X, Y, Z$ tangent to $M$.

**2.1. Definition.** Let $M$ be an $n$-dimensional submanifold of a Riemannian manifold $\bar{M}$. The *mean-curvature vector field $H$ of $M$* is defined by the formula

$$H = \frac{1}{n}\sum_{j=1}^{n} h(e_j, e_j),$$

where $\{e_j\}$, $1 \leq j \leq n$, is a locally orthonormal basis of $\Gamma(TM)$. If a submanifold $M$ satisfies one of the conditions;

$$h = 0, \quad H = 0, \quad h(X,Y) = g(X,Y)H$$

then it is said to be a *totally geodesic*, *minimal* and *totally-umbilical* submanifold, respectively [3].

Furthermore, the norm of $h$ is defined by

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j), h(e_i,e_j)). \tag{2.7}$$

## 3. Locally decomposable Riemannian manifolds

let $\bar{M}$ be an $m$-dimensional manifold with a tensor field $F$ of type $(1,1)$ such that

$$F^2 = I, \quad F \neq \pm I. \tag{3.1}$$

Then we say that $\bar{M}$ is an *almost product manifold with almost structure $F$*. If we set

$$P = \frac{1}{2}(F + I), \quad Q = \frac{1}{2}(I - F),$$

then we can easily see that

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q \ \ \textit{and} \ \ F = P - Q.$$

Thus $P$ and $Q$ define two orthogonal complementary distributions denoted by $D$ and $D^{\perp}$ on $\bar{M}$. If an almost product manifold $\bar{M}$ admits a Riemannian metric $g$ such that

$$g(FX, FY) = g(X,Y) \tag{3.2}$$

for any vector fields $X$ and $Y$ on $\bar{M}$, then $\bar{M}$ is said to be an *almost product Riemannian manifold*. The covariant derivative of almost product structure $F$ is defined by

$$(\bar{\nabla}_X F)Y = \bar{\nabla}_X FY - F(\bar{\nabla}_X Y)$$

for any vector fields $X$ and $Y$ on $\bar{M}$, where $\bar{\nabla}$ denotes the Riemannian connection on $\bar{M}$. If $(\bar{\nabla}_X F)Y = 0$, the almost product Riemannian manifold $\bar{M}$ is said to be a *locally decomposable Riemannian manifold* [6].

Since $F^2 = I$, we can easily see that the eigenvalues of $F$ are 1 or $-1$. An eigenvector corresponding to the eigenvalue 1 is in $P$ and an eigenvector corresponding to $-1$ is in $Q$.

**3.1. Definition.** Let $M$ be a submanifold of a locally decomposable Riemannian manifold $\bar{M}$.

1.) A submanifold $M$ is said to be an *invariant submanifold* if $F$ preserves any tangent space of $M$, i.e., $F(T_xM) \subset T_xM$ for each $x \in M$.

2.) A submanifold $M$ is said to be an *anti-invariant submanifold* if $F$ maps any tangent space of $M$ into its normal space, i.e., $F(T_xM) \subset T_xM^{\perp}$ for each $x \in M$.
3.) As a general case, a submanifold $M$ is said to be a *semi-invariant submanifold* if it admits an invariant distribution $D$ whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant, that is, $T_xM = D_x \oplus D_x^{\perp}$ and $F(D_x^{\perp}) \subset T_xM^{\perp}$ for each $x \in M$.

In this paper we are concerned with Case 3.) as a general case. Now, if we denote the orthogonal complementary distribution of $F(D^{\perp})$ in $TM^{\perp}$ by $\nu$, then we have the direct sum

$$TM^{\perp} = F(D^{\perp}) \oplus \nu. \tag{3.3}$$

It is easily seen that $\nu$ is an invariant subbundle with respect to $F$.

If $\bar{M}_1(c_1)$ is a real space form with sectional curvature $c_1$, and $\bar{M}_2(c_2)$ is a real space form with sectional curvature $c_2$, then the Riemannian curvature tensor $\bar{R}$ of the locally Riemannian product manifold $\bar{M} = \bar{M}_1(c_1) \times \bar{M}_2(c_2)$ is given by

$$\begin{aligned} \bar{R}(X,Y)Z = \frac{1}{4}(c_1 + c_2)\{g(Y,Z)X - g(X,Z)Y + g(FY,Z)FX \\ - g(FX,Z)FY\} + \frac{1}{4}(c_1 - c_2)\{g(FY,Z)X - g(FX,Z)Y \\ + g(Y,Z)FX - g(X,Z)FY\} \end{aligned} \tag{3.4}$$

for any vector fields $X, Y$ and $Z$ tangent to $\bar{M}$ [6].

## 4. Warped product semi-invariant submanifolds in locally decomposable Riemannian manifolds

Let $(M_1, g_1)$ and $(M_2, g_2)$ be two Riemannian manifolds and $f$ a positive definite differentiable function on $M_1$. The *warped product* of manifolds $M_1$ and $M_2$ is the Riemannian manifold

$$M = M_1 \times_f M_2 = (M_1 \times M_2, g), \tag{4.1}$$

where $g = g_1 + f^2 g_2$ and $f$ is called the *warping function*. If the warping function is constant, then it is said to be a *usual Riemannian product*. A warped product manifold $M = M_1 \times_f M_2$ is characterized by the fact that $M_1$ and $M_2$ are totally geodesic and totally umbilical submanifolds of $M$, respectively.

We recall the general formulae on a warped product

$$\nabla_Z X = \nabla_X Z = (X \ln f)Z \tag{4.2}$$

for any $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$, where $\nabla$ denotes the Levi-Civita connection on $M$ [3].

In [1] it was proved that there is no warped product semi-invariant submanifold such that totally geodesic and totally umbilical submanifolds of a warped product are invariant and anti-invariant submanifolds of a locally Riemannian product manifold, respectively. In [1] it was shown that a warped product semi-invariant submanifold exists which has totally umbilical and totally geodesic submanifolds of a warped product which are existent invariant and anti-invariant submanifolds of a locally Riemannian product manifold, that is, warped product semi-invariant submanifolds in a locally decomposable Riemannian manifold, are of the form $M_{\perp} \times_f M_T$, where $M_{\perp}$ and $M_T$ are anti-invariant and invariant submanifolds of $\bar{M}$, respectively (see example 4.1).

This paper deals with the cases of inequality and equality between the squared norm of the second fundamental form, and the norm of the gradient of the warping function.

**4.1. Theorem.** *Let $M = M_\perp \times_f M_T$ be a warped product semi-invariant submanifold of a locally decomposable Riemannian manifold $\bar{M}$. Then we have*

i.) *The squared norm of the second fundamental form of $M$ satisfies*

$$\|h\|^2 \geq q\|\nabla \ln f\|^2, \tag{4.3}$$

*where $\nabla \ln f$ is the gradient of $\ln f$ and $q = \dim(M_T)$.*

ii.) *If the equality sign of (4.3) holds identically, then $M_\perp$ and $M_T$ are totally geodesic anti-invariant and totally umbilical invariant submanifolds of $\bar{M}$, respectively.*

*Proof.* Let $M = M_\perp \times_f M_T$ be a warped product semi-invariant submanifold of a locally decomposable Riemannian manifold $\bar{M}$. Then by using (2.1) and (4.2) we have

$$\begin{aligned} g(h(FZ,W),FX) &= g(\bar{\nabla}_W FZ, FX) = g(F\bar{\nabla}_W Z, FX) = g(\bar{\nabla}_W Z, X) \\ &= -g(\nabla_X W, Z) = -X(\ln f)g(Z,W), \end{aligned} \tag{4.4}$$

for any $X \in \Gamma(TM_\perp)$ and $Z, W \in \Gamma(TM_T)$. On the other hand, we denote by $h_T$ the second fundamental form of $M_T$ in $\bar{M}$, then we have

$$\begin{aligned} g(h_T(Z,W),X) &= g(\nabla_Z W, X) = -g(\nabla_Z X, W) \\ &= -(X \ln f)g(Z,W), \end{aligned} \tag{4.5}$$

for any $X \in \Gamma(TM_\perp)$ and $Z, W \in \Gamma(TM_T)$, that is,

$$h_T(Z,W) = -\nabla \ln f g(Z,W). \tag{4.6}$$

Now, let $\{e_1, e_2, \ldots, e_p, e^1, e^2, \ldots, e^q\}$ be a locally orthonormal frame of $\Gamma(TM)$ such that $\{e_i\}$ and $\{e^j\}$, $1 \leq i \leq p$, $1 \leq j \leq q$, are tangent to $M_\perp$ and $M_T$, respectively. Moreover, let $\{Fe_1, Fe_2, \ldots, Fe_p, N_1, N_2, \ldots, N_s\}$ be a locally orthonormal basis of $\Gamma(TM^\perp)$ such that $\{Fe_i\}$ and $\{N_r\}$, $1 \leq i \leq p$, $1 \leq r \leq s$, are tangent to $\Gamma(F(TM_\perp))$ and $\Gamma(\nu)$, respectively.

Here, we note that $\{Fe^j\}_{j=1}^q$ are also orthonormal basis vectors of $\Gamma(TM_T)$ for $M_T$ is an invariant submanifold with respect to $F$. So we have

$$\begin{aligned} \|h\|^2 = \sum_{i,k=1}^{p} g(h(e_i,e_k),h(e_i,e_k)) + 2\sum_{i=1}^{p}\sum_{j=1}^{q} g(h(e_i,e^j),h(e_i,e^j)) \\ + \sum_{j,\ell=1}^{q} g(h(e^j,e^\ell),h(e^j,e^\ell)). \end{aligned} \tag{4.7}$$

Furthermore, by the rules of linear algebra, we know that

$$h(FZ,W) = \sum_{i=1}^{p} g(h(FZ,W),Fe_i)Fe_i + \sum_{r=1}^{s} g(h(FZ,W),N_r)N_r,$$

for any $Z, W \in \Gamma(TM_T)$. Thus we have

$$\|h(FZ,W)\|^2 = \sum_{i=1}^{p} g(h(FZ,W),Fe_i)^2 + \sum_{r=1}^{s} g(h(FZ,W),N_r)^2. \tag{4.8}$$

By using (4.4) and (4.8), we obtain

$$\begin{aligned} \sum_{k,\ell=1}^{2} \|h(Fe^k,e^\ell)\|^2 &= \sum_{i=1}^{p}\sum_{k,\ell=1}^{q} (e_i(\ln f))^2 g(e^k,e^\ell) \\ &= \sum_{i=1}^{p}\sum_{\ell=1}^{q} (e_i(\ln f))^2 g(e^\ell,e^\ell) = q\sum_{i=1}^{p} g(e_i,\nabla \ln f)^2. \end{aligned} \tag{4.9}$$

So we have

$$\|h\|^2 \geq q\sum_{i=1}^{p}(e_i \ln f)^2 = q\sum_{i=1}^{p} g(\nabla \ln f, e_i)^2 = q\|\nabla \ln f\|^2, \tag{4.10}$$

which proves our assertion.

If the equality sign in (4.3) holds identically, we have

$$h(D^{\perp}, D^{\perp}) = 0, \quad h(D, D) \in \Gamma(F(D^{\perp})) \text{ and } h(D, D^{\perp}) \in \Gamma(F(D^{\perp})). \tag{4.11}$$

The first condition in (4.11) implies that $M_{\perp}$ is totally geodesic in $\bar{M}$ because $M_{\perp}$ is a totally geodesic submanifold in $M$. Since $M_T$ is a totally umbilical submanifold in $M$, the second condition in (4.11) implies that $M_T$ is a totally umbilical submanifold in $\bar{M}$. □

**4.2. Theorem.** *Let $M = M_{\perp} \times_f M_T$ be a warped product semi-invariant submanifold satisfying $\|h\|^2 = q\|\nabla \ln f\|^2$ in a locally decomposable Riemannian manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$. Then we have*

- i.) *$M_{\perp}$ is a totally geodesic anti-invariant submanifold of $M = M_1(c_1) \times M_2(c_2)$. Hence $M_{\perp}$ is a real space form of constant sectional curvature $\frac{1}{4}(c_1 + c_2)$.*
- ii.) *$M_T$ is a totally umbilical invariant submanifold of $\bar{M} = M_1(c_1) \times M_2(c_2)$. Moreover, $M_T$ is a totally umbilical invariant submanifold satisfying $g(FZ, W) = 0$, for any $Z, W \in \Gamma(TM_T)$, then $M_T$ is a real space form of constant sectional curvature $\epsilon = \frac{1}{4}(c_1 + c_2) + \|\nabla \ln f\|^2$.*
- iii.) *If $q > 1$, then the warping function $f$ satisfies $\|\nabla f\|^2 = f^2(4\epsilon - c_1 - c_2)$.*

*Proof.* If we denote the Riemannian curvature tensor of $M_{\perp}$ by $R_{\perp}$, then by using (2.4), (3.4) and taking into account that $M_T$ is totally geodesic and anti-invariant in $\bar{M}$, we have

$$R_{\perp}(X, Y, V, U) = \frac{1}{4}(c_1 + c_2) \tag{4.12}$$

for any $X, Y, U, V \in \Gamma(TM_{\perp})$, which gives i.).

If the equality (4.3) is satisfied, then from Theorem 4.1, we know that $M_T$ is a totally umbilical invariant submanifold of $\bar{M} = M_{\perp} \times_f M_T$. If $g(FZ, W) = 0$, then by using (2.4), (3.4) and (4.6), we obtain

$$g(R_T(Z, W)U, V) = \frac{1}{4}(c_1 + c_2 + \|\nabla \ln f\|^2)\{g(W, U)g(Z, V) - g(W, V)g(Z, U)\},$$

for any $Z, W, U, V \in \Gamma(TM_T)$, where $R_T$ denotes the Riemannian curvature tensor of $M_T$. In this case, the warping function $f$ satisfies $\|\nabla f\|^2 = f^2(4\epsilon - c_1 - c_2)$.

Here, we note that the condition $g(FZ, W) = 0$, for any $Z, W \in \Gamma(TM_T)$, is not important to differential geometry. Since $M_T$ is an invariant submanifold of a locally decomposable Riemannian manifold $\bar{M}$, it is also locally decomposable Riemannian manifold. Choosing vector fields $PZ$ and $QZ$ which have the same length, then from (3.1) and (3.2), $FZ$ and $Z$ are orthogonal, which proves our assertion. Thus the proof is complete. □

Next, we will give an example for warped product semi-invariant submanifolds in a locally decomposable Riemannian manifold to illustrate our results.

**4.3. Example.** Let $M$ be a submanifold in $\mathbb{R}^4$ with coordinates $(x_1, x_2, y_1, y_2)$ given by

$$x_1 = u\cos\theta, \quad x_2 = u\sin\theta, \quad y_1 = u\cos\beta, \quad \text{and} \quad y_2 = u\sin\beta,$$

where $u > 0$, $\theta$ and $\beta$ denote arbitrary parameters.

It is easy to check that the tangent bundle of $M$ is spanned by the vectors

$$Z_1 = \cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2} + \cos\beta\frac{\partial}{\partial y_1} + \sin\beta\frac{\partial}{\partial y_2},$$
$$Z_2 = -u\sin\theta\frac{\partial}{\partial x_1} + u\cos\theta\frac{\partial}{\partial x_2}$$
$$Z_3 = -u\sin\beta\frac{\partial}{\partial y_1} + u\cos\beta\frac{\partial}{\partial y_2}.$$

We have defined the almost Riemannian product structure of $\mathbb{R}^4$ by

$$F\Big(\frac{\partial}{\partial x_i}\Big) = -\frac{\partial}{\partial x_i} \quad\text{and}\quad F\Big(\frac{\partial}{\partial y_i}\Big) = \frac{\partial}{\partial y_i},\ i = 1,2.$$

Then the space $F(TM)$ becomes

$$FZ_1 = -\cos\theta\frac{\partial}{\partial x_1} - \sin\theta\frac{\partial}{\partial x_2} + \cos\beta\frac{\partial}{\partial y_1} + \sin\beta\frac{\partial}{\partial y_2},\ FZ_2 = u\sin\theta\frac{\partial}{\partial x_1} - u\cos\theta\frac{\partial}{\partial x_2}$$
$$FZ_3 = -u\sin\beta\frac{\partial}{\partial y_1} + u\cos\beta\frac{\partial}{\partial y_2}.$$

Since $FZ_1$ is orthogonal to $TM$, $FZ_2$ and $FZ_3$ are tangent to $TM$, $TM_\perp$ and $TM_T$ can be chosen as the subspaces $\mathrm{sp}\{Z_1\}$ and $\mathrm{sp}\{Z_2, Z_3\}$, respectively. Furthermore the Riemannian metric tensor of $M = M_\perp \times_f M_T$ is given by

$$g = 2du^2 + u^2(d\theta^2 + d\beta^2) = g_{M_\perp} \times_{u^2} g_{M_T}.$$

Thus $M$ is a 3-dimensional warped product semi-invariant submanifold of the locally decomposable Riemannian manifold $\mathbb{R}^4$ with the warping function $f = u$.

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