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QUADRATIC MODULES FOR LIE ALGEBRAS

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Abstract

In this work we give the notion of quadratic module for Lie algebras and explore the connections between this structure, 2-crossed modules and simplicial Lie algebras in terms of hypercrossed complex pairings.

Keywords: Simplicial Lie algebra, Quadratic module, 2-Crossed module.

2000 AMS Classification: 18 G 50, 18 G 55.

Introduction

Crossed modules of groups defined by Whitehead in [11] are algebraic models of (connected) homotopy 2-types. The Lie algebra analogue of crossed modules was introduced by Kassel and Loday in [9]. Simplicial groups first studied by Kan, [8], have a well structured homotopy theory and they model all homotopy types of connected spaces. Conduché in [4] defined an algebraic model for connected 3-types. His models, called 2-crossed modules, have very pleasant properties and these 2-crossed modules form a category equivalent to that of simplicial groups with Moore complex of length 2 (cf. [4]). Ellis in [7] captured the algebraic structure of a Moore complex of length 2 in his definition of a 2-crossed module of Lie algebras. This is the Lie algebraic version of a group theoretic notion defined by Conduché.

Within the homotopy theory of simplicial Lie algebras, analogues of Samelson and Whitehead products are given by sums over shuffles $(a; b)$ of Lie products. Akça and Arvasi in [1] explained the relationship of these shuffles to crossed modules and 2-crossed modules of Lie algebras, more precisely, by using the image of the higher order Peiffer elements in the Moore complex of a simplicial Lie algebra, they have constructed a functor from the category of simplicial Lie algebras to that of 2-crossed modules of Lie algebras.

Quadratic modules introduced by Baues [3] are algebraic models for homotopy connected 3-types. Baues in [3] constructed a quadratic module from a simplicial group. In this paper we will give the notion of quadratic module for Lie algebras and we give the

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connections between quadratic modules, 2-crossed modules and simplicial Lie algebras. In the connection between simplicial Lie algebras and quadratic modules, we use the image of the $M_{\alpha,\beta}$ functions given in [1].

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1. Preliminaries

All Lie algebras will be over a fixed commutative ring k. The category of Lie algebras will be denoted by $\mathfrak{LieAlg}$.

Simplicial Lie Algebras

A simplicial Lie algebra $\mathbf{L}$, (cf. [1], [5] and [7]) consists of a family of Lie algebras L_n together with Lie algebra homomorphisms $d_i^n : L_n \to L_{n-1}$, $0 \leqslant i \leqslant n$, $(n \neq 0)$ and $s_i^n : L_n \to L_{n+1}$, $0 \leqslant i \leqslant n$, called face and degeneracy maps, satisfying the usual simplicial identities. In fact, a simplicial Lie algebra can be completely described as a functor $\mathbf{L} : \Delta^{op} \to \mathfrak{LieAlg}$, where Δ is the category of finite ordinals. We denote the category of simplicial Lie algebras by $\mathfrak{SimpLieAlg}$. We obtain for each $k \geqslant 0$ a subcategory $\Delta_{\leqslant k}$ determined by the objects $[j]$ of Δ with $j \leqslant k$. A k-truncated simplicial Lie algebra is a functor from $\Delta^{op}_{\leqslant k}$ to $\mathfrak{LieAlg}$. We denote the category of k-truncated simplicial Lie algebras by $\mathfrak{Tr}_k\mathfrak{SimpLieAlg}$.

Given a simplicial Lie algebra $\mathbf{L}$, the Moore complex $(\mathbf{NL}, \partial)$ of $\mathbf{L}$, is the chain complex defined by:

$$NL_n = \bigcap_{i=0}^{n-1} \ker d_i^n,$$

with $\partial_n : NL_n \to NL_{n-1}$ induced from d_n^n by restriction. We say that the Moore complex $\mathbf{NL}$ of a simplicial Lie algebra is of length k if $NE_n = 0$ for all $n \geqslant k+1$, so that a Moore complex of length k is also of length l for $l \geqslant k$. The category of simplicial Lie algebras with Moore complex of length k will be denoted by $\mathfrak{SimpLieAlg}_{\leqslant k}$.

1.1. Simplicial Lie Algebras and 2-Crossed Modules. Let M and N be two Lie algebras. By an action of N on M we mean a k-bilinear map $N \times M \to M$, $(n, m) \mapsto n \cdot m$, satisfying

$$[n, n'] \cdot m = n \cdot (n' \cdot m) - n'(n \cdot m)$$
$$n \cdot [m, m'] = [n \cdot m, m'] + [m, n \cdot m']$$

for all $m, m' \in M$ and $n, n' \in N$.

Recall that a *crossed module of Lie algebras* is a Lie homomorphism $\partial : M \to N$ together with an action of N on M, denoted by $n \cdot m$ for $n \in N$ and $m \in M$, such that the following conditions are satisfied:

$$CM1) \quad \partial(n \cdot m) = [n, \partial m] \qquad CM2) \quad \partial m \cdot m' = [m, m']$$

for all $m, m' \in M$, $n \in N$.

The following definition is due to Ellis (cf. [7]).

A *2-crossed module of Lie algebras* is a complex

$$M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0$$

of Lie algebras together with an action of M_0 on M_1, and a function

$$\{-, -\} : M_1 \times M_1 \longrightarrow M_2,$$

called a Peiffer lifting, which satisfies the following axioms:

$$\begin{aligned}
1.\quad & \partial_2\{y_0,y_1\} = \partial_1 y_0 \cdot y_1 - [y_0,y_1],\\
2.\quad & \{\partial_2(x_1),\partial_2(x_2)\} = [x_1,x_2],\\
3.\quad & \{\partial_2 x, y\} = \partial_1 y \cdot x - y \cdot x,\\
4.\quad & \{y,\partial_2 x\} = y \cdot x,\\
5.\quad & z\cdot\{y_0,y_1\} = \{z\cdot y_0, y_1\} + \{y_0, z\cdot y_1\},\\
6.\quad & \{y_0,[y_1,y_2]\} = \partial_1(y_1)\cdot\{y_0,y_2\} - \partial_1(y_2)\cdot\{y_0,y_1\}\\
& \qquad - \{y_1, \partial_1(y_0)\cdot y_2 - [y_0,y_2]\} + \{y_2, \partial_1(y_0)\cdot y_1 - [y_0,y_1]\}\\
7.\quad & \{[y_0,y_1],y_2\} = \partial_1(y_0)\cdot\{y_1,y_2\} + \{y_0,[y_1,y_2]\}\\
& \qquad - \partial_1(y_1)\cdot\{y_0,y_2\} - \{y_1,[y_0,y_2]\},
\end{aligned}$$

for all $x, x_1, x_2 \in M_2$, $y, y_0, y_1, y_2 \in M_1$ and $z \in M_0$.

The category of 2-crossed modules of Lie algebras will be denoted by $\mathfrak{X}_2\mathfrak{LMod}$. The following theorem was proved by Ellis in [7].

1.1. Theorem. *The category of 2-crossed modules of Lie algebras is equivalent to that of simplicial Lie algebras with Moore complex of length 2.* □

Akça and Arvasi, [1], studied simplicial Lie algebras and their properties. They considered a simplicial Lie algebra **L** which is 2-truncated, i.e., its Moore complex looks like:

$$NL_2 \xrightarrow{\partial_2} NL_1 \xrightarrow{\partial_1} NL_0 \ ,$$

and they showed that this complex has a 2-crossed module structure of Lie algebras in terms of hypercrossed complex pairings. This result is an analogue, for Lie algebras, of Arvasi and Porter's result in the case of commutative algebras (cf. [2]).

2. Quadratic and 2-crossed modules of Lie algebras

Baues, in [3], defined the notion of quadratic module of groups as an algebraic model for connected homotopy 3-types. In this section, we define a Lie algebra version of this structure, and we define a functor from the category of 2-crossed modules to that of quadratic modules of Lie algebras.

Let $\partial : C \to R$ be a pre-crossed module, and $P_1(\partial) = C$ and $P_2(\partial)$ be the Peiffer Lie ideal of C generated by elements of the form

$$\langle x, y\rangle = (\partial x)\cdot y - [x,y],$$

which is called the *Peiffer element* for $x, y \in C$.

A *nil(2)-module* is a pre-crossed module $\partial : C \to R$ with an additional "nilpotency" condition. This condition is $P_3(\partial) = 0$, where $P_3(\partial)$ is the ideal of the Lie algebra C generated by the Peiffer elements $\langle x_1, x_2, x_3\rangle$ of length 3.

If $\partial : C \to R$ is a pre-crossed module, then the homomorphism

$$\partial^{cr} : C^{cr} = C/P_2(\partial) \longrightarrow R$$

is a crossed module since

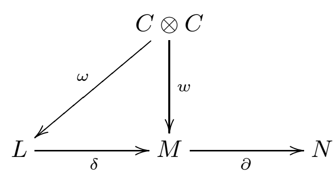
$$\begin{aligned}\langle [x],[y]\rangle &= \partial^{cr}(x+P_2(\partial))\cdot(y+P_2(\partial)) - [(x+P_2(\partial)),(y+P_2(\partial))]\\ &= \partial^{cr}(x+P_2(\partial))\cdot(y+P_2(\partial)) - ([x,y]+P_2(\partial))\\ &= (\partial(x)\cdot y - [x,y]) + P_2(\partial)\\ &= P_2(\partial) \quad (\because\ \langle x,y\rangle \in P_2(\partial)\ )\\ &= [0]\end{aligned}$$

for $[x]=x+P_2(\partial)$, $[y]=y+P_2(\partial)\in C^{cr}$. The homomorphism ∂^{cr} is called the *crossed module* associated with the pre-crossed module ∂. Similarly, if $\partial : C \to R$ is a pre-crossed module, then the homomorphism

$$\partial^{nil} : C^{nil} = C/P_3(\partial) \longrightarrow R$$

is a nil(2)-module associated with the pre-crossed module ∂. Clearly, a nil(1)-module is a crossed module.

2.1. Definition. A *quadratic module* (ω,δ,∂) of Lie algebras is a diagram



of homomorphisms of Lie algebras such that the following axioms are satisfied.

$\mathfrak{QM}1$) The homomorphism $\partial : M \to N$ is a nil(2)-module and the quotient map $M \twoheadrightarrow C = M^{cr}/[(M^{cr}),(M^{cr})]$ is given by $x \mapsto [x]$, where $[x]\in C$ denotes the class represented by $x\in M$. The map w is defined by Peiffer multiplication, i.e., $w([x]\otimes[y]) = \partial(x)\cdot y - [x,y]$ for $x,y\in M$.

$\mathfrak{QM}2$) The homomorphisms δ and ∂ satisfy $\delta\partial = 0$ and the quadratic map ω is a lift of the map w, that is $\delta\omega = w$ or equivalently

$$\delta\omega([x]\otimes[y]) = w([x]\otimes[y]) = \partial(x)\cdot y - [x,y]$$

for $x,y\in M$.

$\mathfrak{QM}3$) L is a Lie N-algebra and all homomorphisms of the diagram are equivariant with respect to the action of N. Moreover, the action of N on L satisfies the following equality

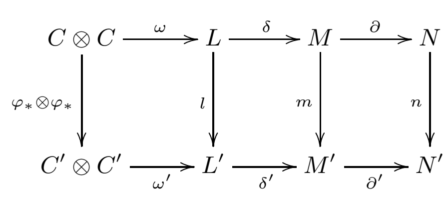
$$\partial(x)\cdot a = \omega([\delta a]\otimes[x] + [x]\otimes[\delta a])$$

for $a\in L, x\in N$.

$\mathfrak{QM}4$) For $a,b\in L$,

$$\omega([\delta a]\otimes[\delta b]) = [a,b].$$

A map $\varphi : (\omega,\delta,\partial)\to(\omega',\delta',\partial')$ between quadratic modules is given by a commutative diagram, $\varphi = (l,m,n)$



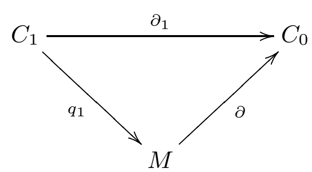
where (m, n) is a map between pre-crossed modules which induces $\varphi_* : C \to C'$ and where l is an n-equivariant homomorphism. Let $\mathfrak{QM}$ be the category of quadratic modules and of maps as in the above diagram.

Now, we construct a functor from the category of 2-crossed modules to the category of quadratic modules of Lie algebras.

Let

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

be a 2-crossed module of Lie algebras. Let P_3 be the ideal of C_1 generated by elements of the form $\langle\langle x, y\rangle, z\rangle$ and $\langle x, \langle y, z\rangle\rangle$ for $x, y, z \in C_1$. Let $q_1 : C_1 \to C_1/P_3$ be the quotient map and $M = C_1/P_3$. Since ∂_1 is a pre-crossed module, we obtain $\partial_1(\langle\langle x, y\rangle, z\rangle) = 0$ and $\partial_1(\langle x, \langle y, z\rangle\rangle) = 0$. That is, we have $\partial_1(P_3) = 0$. Thus, the map $\partial : M \to C_0$ given by $\partial(x + P_3) = \partial_1(x)$, for all $x \in C_1$, is a well defined homomorphism. Therefore, we have a commutative diagram



Let P_3' be the ideal of C_2 generated by elements of the form

$$\{x, \langle y, z\rangle\} \text{ and } \{\langle x, y\rangle, z\},$$

where $\{-, -\}$ is the Peiffer lifting map. We have

$$L = C_2/P_3'.$$

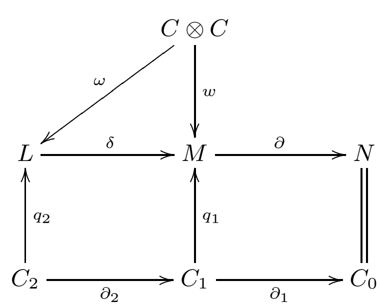
We can write from the first axiom of 2-crossed modules

$$\begin{aligned} \partial_2\{x, \langle y, z\rangle\} &= (\partial_1 x \cdot \langle y, z\rangle) - [x, \langle y, z\rangle] \\ &= \langle x, \langle y, z\rangle\rangle, \\ \partial_2\{\langle x, y\rangle\, z\} &= \partial_1(\langle x, y\rangle) \cdot z - [\langle x, y\rangle, z] \\ &= \langle\langle x, y\rangle, z\rangle, \end{aligned}$$

and thus we obtain $\partial_2(P_3') = P_3$. Then, $\delta : L \longrightarrow M$ given by $\delta(l + P_3') = \partial_2 l + P_3$ is a well defined homomorphism. Let

$$C = \frac{M^{cr}}{[M^{cr}, M^{cr}]}.$$

Thus we get the following commutative diagram;



where q_1 and q_2 are the quotient maps. The quadratic map

$$\omega : C \otimes C \longrightarrow L$$

is given by the Peiffer lifting map, namely

$$\omega\left([q_1x] \otimes [q_1y]\right) = q_2\{x, y\}$$

for all $x, y \in C_1$, $q_1x, q_1y \in M$ and $[q_1x] \otimes [q_1y] \in C \otimes C$.

2.2. Proposition. *The diagram*

$$\begin{array}{ccccc} & & C \otimes C & & \\ & \swarrow^{\omega} & \downarrow^{w} & & \\ L & \xrightarrow[\delta]{} & M & \xrightarrow[\partial]{} & N \end{array}$$

is a quadratic module of Lie algebras.

Proof. We leave the verification of the axioms of quadratic module to the reader. □

Thus we have defined a functor from the category of 2-crossed modules to that of quadratic modules of Lie algebras

$$\mathfrak{X}_2\mathfrak{Mod} \longrightarrow \mathfrak{QM}.$$

3. Quadratic modules and simplicial Lie algebras

In this section we will construct a functor from the simplicial Lie algebras to the quadratic modules of Lie algebras by using the $M_{\alpha,\beta}$ functions defined by Akça and Arvasi in [1] in order to see the role of the hypercrossed complex pairings in the structure. We will use the images of the $M_{\alpha,\beta}$ functions in verifying the axioms of quadratic module.

Now we recall a brief description of the $M_{\alpha,\beta}$ functions from [1].

For the ordered set $[n] = \{0 < 1 < \cdots < n\}$, let $\sigma_i^n : [n+1] \to [n]$ be the increasing surjective map given by

$$\sigma_i^n(j) = \begin{cases} j & \text{if } j \leqslant i, \\ j-1 & \text{if } j > i, \end{cases}$$

as used in Appendix B of Loday's book [10].

Let $S(n, n-r)$ be the set of all monotone increasing surjective maps from $[n]$ to $[n-r]$. This can be generated from the various σ_i^n by composition. The composition of these generating maps is subject to the following rule: $\sigma_j\sigma_i = \sigma_{i-1}\sigma_j$, $j < i$. This implies that every element $\sigma \in S(n, n-r)$ has a unique expression as $\sigma = \sigma_{i_1} \circ \sigma_{i_2} \circ \cdots \circ \sigma_{i_r}$ with $0 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n-1$, where the indices i_k are elements of $[n]$ such that $\{i_1, \ldots, i_r\} = \{i : \sigma(i) = \sigma(i+1)\}$. We thus can identify $S(n, n-r)$ with the set $\{(i_r, \ldots, i_1) : \ 0 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n-1\}$. In particular, the single element of $S(n, n)$, defined by the identity map on $[n]$, corresponds to the empty 0-tuple (), denoted by $\emptyset_n$. Similarly the only element of $S(n, 0)$ is $(n-1, n-2, \ldots, 0)$.

For all $n \geqslant 0$, let

$$S(n) = \bigcup_{0 \leqslant r \leqslant n} S(n, n-r).$$

We say that $\alpha = (i_r, \ldots, i_1) < \beta = (j_s, \ldots, j_1)$ in $S(n)$ if $i_1 = j_1, \ldots, i_k = j_k$ but $i_{k+1} > j_{k+1}, (k \geqslant 0)$ or if $i_1 = j_1, \ldots, i_r = j_r$ and $r < s$. This makes $S(n)$ an ordered set.

We recall briefly from Akça and Arvasi (cf. [1]) the construction of a family of k-bilinear morphisms. We define a set $P(n)$ consisting of pairs of elements (α,β) from $S(n)$ with $\alpha \cap \beta = \emptyset$ and $\beta < \alpha$, where $\alpha = (i_r,\ldots,i_1)$, $\beta = (j_s,\ldots,j_1) \in S(n)$. The k-linear morphisms that we will need,

$$\{M_{\alpha,\beta} : NL_{n-\#\alpha} \times NL_{n-\#\beta} \to NL_n : (\alpha,\beta) \in P(n), n \geqslant 0\},$$

are given as composites:

$$\begin{aligned} M_{\alpha,\beta}(x_\alpha, y_\beta) &= p[-,-](s_\alpha \times s_\beta)(x_\alpha, y_\beta) \\ &= p([s_\alpha(x_\alpha), s_\beta(y_\beta)]) \\ &= (1 - s_{n-1}d_{n-1}) \cdots (1 - s_0 d_0)([s_\alpha(x_\alpha), s_\beta(y_\beta)]), \end{aligned}$$

where

$$s_\alpha = s_{i_r} \ldots s_{i_1} : NL_{n-\#\alpha} \to L_n, s_\beta = s_{j_s} \ldots s_{j_1} : NL_{n-\#\beta} \to L_n,$$

$p : L_n \to NL_n$ is defined by composite projections $p = p_{n-1} \ldots p_0$ with $p_j = 1 - s_j d_j$ for $j = 0, 1, \ldots, n-1$ and $[-,-] : L_n \times L_n \to L_n$ denotes the Lie bracket.

From [1], we will now consider the ideal I_n in L_n which is generated by all elements of the form;

$$M_{\alpha,\beta}(x_\alpha, y_\beta),$$

where $x_\alpha \in NL_{n-\#\alpha}$ and $y_\beta \in NL_{n-\#\beta}$ and for all $(\alpha,\beta) \in P(n)$.

Consider $M_{\alpha,\beta}(x_\alpha, y_\beta)$ and $M_{\beta,\alpha}(y_\beta, x_\alpha)$, here one uses $[s_\alpha(x_\alpha), s_\beta(y_\beta)]$, the other giving

$$[s_\alpha(x_\alpha), s_\beta(y_\beta)] = -[s_\beta(y_\beta), s_\alpha(x_\alpha)],$$

so changing α and β only gives a minus sign.

3.1. Proposition. [1] *Let* $\mathbf{L}$ *be a simplicial Lie algebra,* $n > 0$ *and* D_n *the ideal in* L_n *generated by degenerate elements. We suppose* $L_n = D_n$*, and let* I_n *be the ideal generated by elements of the form* $M_{\alpha,\beta}(x_\alpha, y_\beta)$ *with* $(\alpha,\beta) \in P(n)$*, where* $x_\alpha \in NL_{n-\#\alpha}$*,* $y_\beta \in NL_{n-\#\beta}$ *with* $1 \leqslant r, s \leqslant n$*. Then* $NL_n = I_n$*, and as a corollary* $\partial_n(NL_n) = \partial_n(I_n)$*.* □

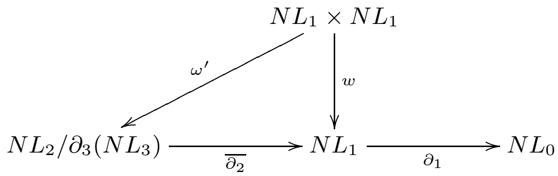
Using the above proposition for $n = 2$ and 3, Akça and Arvasi have show what the image of I_n looks like by using ∂_n. The image of I_2 using ∂_2 is $[\ker d_0, \ker d_1]$. For $n = 3$, the ideal I_3 in NL_3 is generated by the elements; for $x \in NL_1$ and $y \in NL_2$,

$$\begin{aligned} M_{(1,0),(2)}(x,y) &= [s_1 s_0 x - s_2 s_0 x, s_2 y], \\ M_{(2,0),(1)}(x,y) &= [s_2 s_0 x - s_2 s_1 x, s_1 y - s_2 y], \\ M_{(1),(0)}(x,y) &= [s_1 x, s_0 y - s_1 y] + [s_2 x, s_2 y], \\ M_{(2),(0)}(x,y) &= [s_2 x, s_0 y], \\ M_{(1),(0)}(x,y) &= [s_2 x, s_1 y - s_2 y]. \end{aligned}$$

For the images of these elements, see [1].

Now, we construct a functor from the simplicial Lie algebras to the quadratic modules of Lie algebras by using the functions $M_{\alpha,\beta}$. Suppose that $E_n = D_n$ for all $n \geqslant 0$.

Let **L** be a simplicial Lie algebra with Moore complex **NL**. We will obtain a quadratic module of Lie algebras by using the following diagram:



where the map w is given by

$$w(x,y) = [s_0d_1x, y] - [x,y]$$

for $x, y \in NL_1$, and the map ω' by

$$\omega'(x,y) = ([s_0x, s_1y] - [s_1x, s_1y]) + \partial_3(NL_3)$$

for $x, y \in NL_1$. We see that

$$d_2([s_0x, s_1y] - [s_1x, s_1y]) = [s_0d_1x, y] - [x,y] = w(x,y).$$

Let $P_3(\partial_1)$ be the ideal of NL_1 generated by elements of the form

$$w(x, w(y,z)) \text{ and } w(w(x,y), z)$$

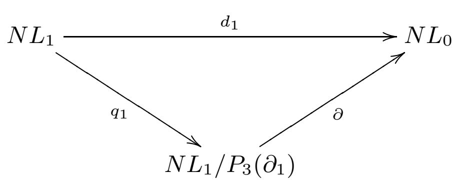
for $x, y, z \in NL_1$. Since

$$\begin{aligned} d_1(w(x, w(y,z))) &= d_1([s_0d_1x, w(y,z)] - [x, w(y,z)]) \\ &= [d_1x, d_1w(y,z)] - [d_1x, d_1w(y,z)] \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} d_1(w(w(x,y), z)) &= d_1([s_0d_1w(x,y), z] - [w(x,y), z]) \\ &= [d_1w(x,y), d_1z] - [d_1w(x,y), d_1z] \\ &= 0 \end{aligned}$$

we can write $d_1(P_3(\partial_1)) = 0$. Then, $\partial : NL_1/P_3(\partial_1) \longrightarrow NL_0$ given by $\partial(x + P_3(\partial_1)) = d_1x$ for $x \in NL_1$ is a well defined homomorphism. Thus, we obtain the following commutative diagram



where q_1 is the quotient map and $\partial : NL_1/P_3(\partial_1) \longrightarrow NL_0$ becomes a nil(2)-module.

Let $P_3'(\partial_1)$ be an ideal of $NL_2/\partial_3(NL_3)$ generated by the formal Peiffer elements of x, y, z, i.e., generated by elements of the form

$$\omega'(w(x,y), z) \text{ and } \omega'(x, w(y,z))$$

for $x, y, z \in NL_1$. Since

$$\begin{aligned} \overline{\partial_2}\omega'(x, w(y,z)) &= d_2([s_0x, s_1w(y,z)] - [s_1x, s_1w(y,z)]) \\ &= [s_0d_1x, w(y,z)] - [x, w(y,z)] \\ &= w(x, w(y,z)) \in P_3(\partial_1) \end{aligned}$$

$$\begin{aligned}\overline{\partial_2}\omega'(w(x,y),z) &= d_2([s_0w(x,y),s_1z]-[s_1w(x,y),s_1z]) \\ &= [s_0d_1w(x,y),z]-[w(x,y),z] \\ &= w(w(x,y),z)\in P_3(\partial_1),\end{aligned}$$

we obtain $\overline{\partial_2}(P_3'(\partial_1))\subseteq P_3(\partial_1)$. Let

$$M = NL_1/P_3(\partial_1)$$

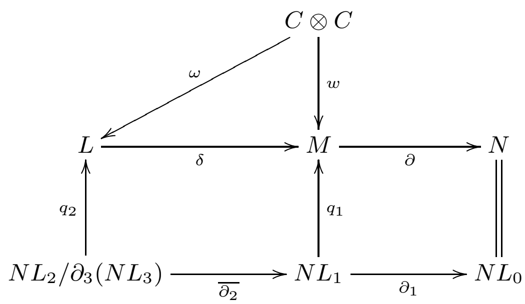
and

$$L = (NL_2/\partial_3(NL_3))/P_3'(\partial_1).$$

We thus see that the map

$$\delta : (NL_2/\partial_3 NL_3))/P_3'(\partial_1) \longrightarrow NL_1/P_3(\partial_1)$$

given by $\delta(a+P_3'(\partial_1))=\overline{\partial_2}(a)+P_3(\partial_1)$ is a well defined homomorphism since $\overline{\partial_2}(P_3'(\partial_1))\subseteq P_3(\partial_1)$. Therefore, we obtain the following commutative diagram,



where $C = M^{cr}/[M^{cr},M^{cr}]$ and q_1,q_2 are the quotient maps, and the quadratic map ω can be given by

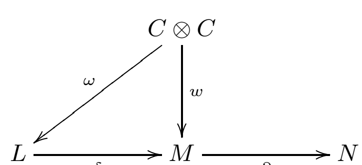
$$\omega([q_1x]\otimes[q_1y]) = q_2\omega'(x,y)$$

and

$$w([q_1x]\otimes[q_1y]) = \partial(q_1(x))\cdot q_1(y) - [q_1(x),q_1(y)]$$

for $q_1x, q_1y\in M$ and $[q_1x]\otimes[q_1y]\in C\otimes C$. Thus we have:

3.2. Proposition. *The diagram*



is a quadratic module of Lie algebras.

Proof. We show that all the axioms of a quadratic module are verified by using the images of the $M_{\alpha,\beta}$ functions from [1].

$\mathfrak{QM}1$)- Clearly, $\partial : M \to N$ is a nil(2)-module. Because, for $x + P_3(\partial_1)$, $y + P_3(\partial_1)$, $z + P_3(\partial_1) \in M = NL_1/P_3(\partial_1)$,

$$\begin{aligned}\langle x + P_3(\partial_1), \langle y + P_3(\partial_1), z + P_3(\partial_1)\rangle\rangle &= \langle x, \langle y, z\rangle\rangle + P_3(\partial_1)\\ &= 0 + P_3(\partial_1) \text{ (by } \langle x, \langle y, z\rangle\rangle \in P_3(\partial_1))\\ &= P_3(\partial_1).\end{aligned}$$

$\mathfrak{QM}2$)- For $x, y \in NL_1$, $q_1x, q_1y \in M$ and $[q_1x] \otimes [q_1y] \in C \otimes C$, we have

$$\begin{aligned}\delta\omega([q_1x] \otimes [q_1y]) &= \delta q_2\omega'(x \otimes y)\\ &= q_1(d_2([s_0x, s_1y] - [s_1x, s_1y]) \quad (\because \ \delta q_2 = q_1\partial_2)\\ &= q_1([s_0d_1x, y] - [x, y])\\ &= d_1q_1(x) \cdot y - [q_1x, q_1y]\\ &= w([q_1x] \otimes [q_1y]).\end{aligned}$$

$\mathfrak{QM}3$)- For $q_2a \in L$ and $q_1x \in M$, we have

$$\begin{aligned}\omega\left([\delta q_2a] \otimes [q_1x]\right) &= \omega\left([q_1\partial_2a] \otimes [q_1x]\right)\\ &= q_2\omega'(\partial_2a \otimes x)\\ &= q_2([s_1d_2a, (s_1x - s_0x)]).\end{aligned}$$

We have from [1], for $x \in NL_1$ and $a \in NL_2$,

$$\partial_3(M_{(1,0)(2)}(x, a)) = [s_1d_2a, (s_1x - s_0x)] - [s_1x, a] \in \partial_3(NL_3),$$

and then we get

$$\begin{aligned}\omega\left([\delta q_2a] \otimes [q_1x]\right) &\equiv q_2([s_1x, a]) \quad \text{mod } (\partial_3(NL_3))\\ &= q_2(x \cdot a).\end{aligned}$$

On the other hand,

$$\begin{aligned}\omega([q_1x] \otimes [\delta q_2a]) &= \omega([q_1x] \otimes [q_1\partial_2a])\\ &= q_2\omega'(x \otimes \partial_2a)\\ &= q_2([s_0x, s_1d_2a] - [s_1x, s_1d_2a]).\end{aligned}$$

Again from [1] we have for $x \in NL_1$ and $a \in NL_2$,

$$\partial_3(M_{(2,0)(1)}(x, a)) = [s_0x, s_1d_2a] - [s_1x, s_1d_2a] - [s_0x, a] + [s_1x, a] \in \partial_3(NL_3),$$

and then

$$\begin{aligned}\omega([q_1x] \otimes [\delta q_2a]) &\equiv q_2([s_0x, a] - [s_1x, a]) \quad \text{mod } \partial_3(NL_3)\\ &= q_2([s_1s_0d_1x, a] - [s_1x, a])\\ &= \partial_1q_1(x) \cdot q_2(a) - q_2(x \cdot a).\end{aligned}$$

Thus we obtain

$$\omega\left([\delta q_2a] \otimes [q_1x] + [q_1x] \otimes [\delta q_2a]\right) = \partial_1q_1(x) \cdot q_2(a).$$

$\mathfrak{QM}4$)- For $q_2a, q_2b \in L$, we have

$$\begin{aligned}\omega([\delta q_2a] \otimes [\delta q_2b]) &= \omega([q_1\partial_2a] \otimes [q_1\partial_2b])\\ &= q_2\omega'(\partial_2a \otimes \partial_2b)\\ &= q_2([s_1d_2a, s_1d_2b] - [s_1d_2a, s_0d_2a]).\end{aligned}$$

Similarly from [1] we have for $a, b \in NL_2$,

$$\partial_3(M_{(1)(0)}(a, b)) = [s_1d_2a, s_1d_2b] - [s_1d_2a, s_0d_2a] - [a, b] \in \partial_3(NL_3),$$

and then we get

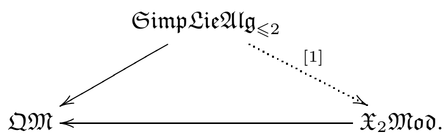
$$\begin{aligned}\omega([\delta q_2 a] \otimes [\delta q_2 b]) &\equiv q_2([a,b]) \quad \mathrm{mod}\ \partial_3(NL_3) \\ &= [q_2 a, q_2 b].\end{aligned}$$

□

If the Moore complex of the simplicial Lie algebra is of length 2, we have $\partial_3(NL_3) = 0$ and thus we have a functor from the category of simplicial Lie algebras with Moore complex of length 2 to that of the quadratic modules

$$\mathfrak{SimpLieAlg}_{\leqslant 2} \longrightarrow \mathfrak{QM}.$$

Therefore, the situations observed in this paper can be summarized in the following diagram of unbroken arrows



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