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FUZZY FIXED POINTS OF FUZZY MAPPINGS VIA A RATIONAL INEQUALITY

Akbar Azam*

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Abstract

We establish the existence of common fuzzy fixed points for fuzzy mappings under a rational contractive condition on a metric space in connection with the Hausdorff metric on the family of fuzzy sets, and apply it to obtain common fixed points of fuzzy (multivalued) mappings satisfying a rational contractive condition associated with the d_∞ (Hausdorff) metric.

Keywords: Fixed point, Common fixed point, Contractive type mapping, fuzzy mapping.

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1. Introduction and preliminaries

Fixed point theorems are very important tools for providing evidence of the existence and uniqueness of solutions to various mathematical models. The literature of the last four decades flourishes with results which discover fixed points of self and nonself nonlinear operators in a metric space. The Banach contraction theorem plays a fundamental role in fixed point theory and has become even more important because being based on iteration, it can be easily implemented on a computer. On the other hand Fisher [15] obtained common fixed points of a pair $S, T : X \to X$ of single mappings satisfying a rational inequality by demonstrating the convergence of sequence of iterates of S, T in a complete metric space X.

Among various developments of fuzzy sets theory, a progressive development has been made to find the fuzzy analogues of fixed point results of the classical fixed point theorems. In this paper, we use a generalized contractive condition involving a rational expression to study common fuzzy fixed point theorems for fuzzy set valued mappings. In the following we always suppose that (X, d) is a complete metric space and (V, d) a complete metric

*Department of Mathematics, COMSATS Institute of Information Technology, Chak Shahzad, 44000, Islamabad, Pakistan.
E-mail: akbarazam@yahoo.com akbar_azam@comsats.edu.pk

linear space. Moreover, we shall use the following notations which have been recorded from [1, 2, 3, 4, 16, 17, 19, 20, 22, 23].

$$2^X = \{A : A \text{ is a subset of } X\},$$
$$C(2^X) = \left\{A \in 2^X : A \text{ is nonempty and compact}\right\},$$
$$CB(2^X) = \left\{A \in 2^X : A \text{ is nonempty closed and bounded}\right\}.$$

For $A, B \in CB\left(2^X\right)$,

$$d(x, A) = \inf_{y \in A} d(x, y),$$
$$d(A, B) = \inf_{x \in A, y \in B} d(x, y).$$

Then the Hausdorff metric H on $CB\left(2^X\right)$ induced by d is defined as:

$$H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\right\}.$$

A fuzzy set in X is a function with domain X and values in $[0, 1]$, I^X is the collection of all fuzzy sets in X. If A is a fuzzy set and $x \in X$, then the function values $A(x)$ is called the grade of membership of x in A. The α-level set of a fuzzy set A, is denoted by $[A]_\alpha$, and defined as

$$[A]_\alpha = \{x : A(x) \geqslant \alpha\} \text{ if } \alpha \in (0, 1],$$
$$[A]_0 = \overline{\{x : A(x) > 0\}}.$$
$$\widehat{A} = \{x : A(x) = \max_{y \in X} A(y)\}.$$

For $x \in X$, we denote the fuzzy set $\chi_{\{x\}}$ by $\{x\}$ unless and until it is stated, where χ_A is the characteristic function of the crisp set A. A fuzzy set A in a metric linear space V is said to be an approximate quantity if and only if $[A]_\alpha$ is compact and convex in V for each $\alpha \in [0, 1]$ and $\sup_{x \in V} A(x) = 1$.

Define some sub-collections of I^X and I^V as follows:

$$W(V) = \left\{A \in I^V : A \text{ is an approximate quantity in } V\right\},$$
$$K(X) = \left\{A \in I^X : \widehat{A} \in C\left(2^X\right)\right\},$$
$$\mathfrak{C}(X) = \left\{A \in I^X : [A]_\alpha \in C\left(2^X\right), \text{ for each } \alpha \in [0, 1]\right\},$$
$$E(X) = \left\{A \in I^X : [A]_\alpha \in CB\left(2^X\right), \text{ for each } \alpha \in [0, 1]\right\},$$
$$\mathcal{F}(X) = \left\{A \in I^X : [A]_\alpha \in C\left(2^X\right), \text{ for some } \alpha \in [0, 1]\right\},$$
$$\mathcal{G}(X) = \left\{A \in I^X : [A]_\alpha \in CB\left(2^X\right), \text{ for some } \alpha \in [0, 1]\right\}.$$

For $A, B \in I^X$, $A \subset B$ means $A(x) \leqslant B(x)$ for each $x \in X$. If there exists an $\alpha \in [0, 1]$ such that $[A]_\alpha, [B]_\alpha \in C\left(2^X\right)$, then define

$$p_\alpha(A, B) = \inf_{x \in [A]_\alpha, y \in [B]_\alpha} d(x, y),$$
$$D_\alpha(A, B) = H([A]_\alpha, [B]_\alpha).$$

If $[A]_\alpha, [B]_\alpha \in C\left(2^X\right)$ for each $\alpha \in [0,1]$ then define $P(A,B), d_\infty(A,B) : \mathfrak{C}(X)\times\mathfrak{C}(X) \to \mathbb{R}$, (induced by the Hausdorff metric H) as follows:

$$p(A,B) = \sup_\alpha p_\alpha(A,B),$$

$$d_\infty(A,B) = \sup_\alpha D_\alpha(A,B).$$

We note that [16, 17], p_α is a nondecreasing function of α, d_∞ is a metric on $C\left(2^X\right)$ and the completeness of (X,d) implies that $(C\left(2^X\right), H)$ and $(\mathfrak{C}(X), d_\infty)$ are complete. Moreover

$$(X,d) \mapsto \left(CB\left(2^X\right), H\right) \mapsto (E(X), d_\infty),$$

are isometric embeddings by means of $x \mapsto \{x\}$ (crisp set) and $A \mapsto \chi_A$ respectively. Let X be an arbitrary set, Y a metric space. A mapping T is called a fuzzy mapping if T is a mapping from X into I^Y. A fuzzy mapping T is a fuzzy subset on $X \times Y$ with membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$. The family of all mappings from X into I^Y is denoted by $\left(I^Y\right)^X$. For convenience, we denote the α-level set of $T(x)$ by $[Tx]_\alpha$ instead of $[T(x)]_\alpha$.

A point $x^* \in X$ is called a fuzzy fixed point of a fuzzy mapping $T : X \to I^X$ if there exists $\alpha \in (0,1]$ such that $x^* \in [Tx^*]_\alpha$ (see [3, 4]). The point x^* is known as a fixed point of T if $T(x^*)(x^*) \geqslant T(x^*)(x)$ for all $x \in X$ (see [1]). Moreover, we say that x^* is an Heilpern fixed point of T if $\{x^*\} \subset Tx^*$ (see [17]).

1.1. Lemma. [21] *Let A and B be nonempty closed and bounded subsets of a metric space (X,d) If $a \in A$, then $d(a,B) \leqslant H(A,B)$.* □

1.2. Lemma. [21] *Let A and B be nonempty closed and bounded subsets of a metric space (X,d) and $0 < \xi \in \mathbb{R}$. Then for $a \in A$, there exists $b \in B$ such that $d(a,b) \leqslant H(A,B) + \xi$.* □

1.3. Lemma. [2] *Let (V,d) be a complete metric linear space, $T : V \longrightarrow W(V)$ a fuzzy mapping and $x_o \in V$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset T(x_0)$.* □

In the following we always suppose that $\widehat{T}$ is the mapping induced by a fuzzy mapping T i.e.,

$$\widehat{T}(x)(t) = \{y \in X : T(x)(y) = \max_{t\in X} T(x)(t)\}.$$

1.4. Lemma. [1] *Let (X,d) be a metric space, $x^* \in X$ and $T : X \to I^X$ be fuzzy mapping such that $\widehat{T}x \in C\left(2^X\right)$ for all $x \in X$. Then $x^* \in \widehat{T}(x^*)$ iff $T(x^*)(x^*) \geqslant T(x^*)(x)$ for all $x \in X$.*

2. Fuzzy fixed points of fuzzy mappings

Fixed point theorems in fuzzy mathematics are emerging with varying hope and vital trust. Weiss [23] and Butnariu [9] initiated the study of fixed point theorems in fuzzy mathematics. Heilpern [17] first used the concept of fuzzy mappings to prove the Banach contraction principle for fuzzy (approximate quantity-valued) mappings on complete metric linear spaces. Subsequently more than a few authors (e.g. [1, 2, 5, 6, 8, 16, 18, 19, 20, 22]) studied Heilpern fixed point results of fuzzy mappings satisfying a contractive type condition by using the d_∞-metric for fuzzy sets.

In this section we obtain common fuzzy fixed points of a pair of fuzzy mappings satisfying a rational inequality by using the Hasudorff metric for fuzzy sets. These results are free from the conditions of approximate quantity for $T(x)$ and linearity for X.

2.1. Theorem. *Let* $S, T : X \to I^X$ *and for* $x \in X$, *there exist* $\alpha_S(x), \alpha_T(x) \in (0,1]$ *such that* $[Sx]_{\alpha_S(x)}, [Tx]_{\alpha_T(x)} \in CB\left(2^X\right)$. *If for all* $x, y \in X$

$$H\big([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}\big) \leqslant \alpha d(x,y) + \beta d\big(x, [Sx]_{\alpha_S(x)}\big) + \gamma d\big(y, [Ty]_{\alpha_T(y)}\big) + \frac{\delta d\big(x, [Sx]_{\alpha_S(x)}\big) d\big(y, [Ty]_{\alpha_T(y)}\big)}{1 + d(x,y)} \tag{1}$$

and

$$\gamma + \frac{\delta d\big(x, [Sx]_{\alpha_S(x)}\big)}{1 + d(x,y)} < 1, \;\; \beta + \frac{\delta d\big(y, [Ty]_{\alpha_T(y)}\big)}{1 + d(x,y)} < 1, \tag{2}$$

where $\alpha, \beta, \gamma, \delta$ *are non negative real numbers with* $\alpha + \beta + \gamma + \delta < 1$. *Then there exists* $u \in X$ *such that* $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)}$.

Proof. We consider the following three possible cases:

(i) $\alpha + \beta = 0$;
(ii) $\alpha + \gamma = 0$;
(iii) $\alpha + \beta \neq 0$, $\alpha + \gamma \neq 0$.

Case (i): $\alpha + \beta = 0$. For $x \in X$, there exists $\alpha_S(x) \in (0,1]$ such that $[Sx]_{\alpha_S(x)}$ is a nonempty closed bounded subset of X. Take $y \in [Sx]_{\alpha_S(x)}$ and similarly $z \in [Ty]_{\alpha_T(y)}$. Then, by Lemma 1.1 we obtain

$$d\big(y, [Ty]_{\alpha_T(y)}\big) \leqslant H\big([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}\big).$$

Now, inequality (1) implies that

$$d\big(y, [Ty]_{\alpha_T(y)}\big) \leqslant \alpha d(x,y) + \beta d\big(x, [Sx]_{\alpha_S(x)}\big) + \gamma d\big(y, [Ty]_{\alpha_T(y)}\big) + \frac{\delta d\big(x, [Sx]_{\alpha_S(x)}\big) d\big(y, [Ty]_{\alpha_T(y)}\big)}{1 + d(x,y)}$$

Using $\alpha + \beta = 0$ we obtain

$$\left[1 - \gamma - \frac{\delta d\big(x, [Sx]_{\alpha_S(x)}\big)}{1 + d(x,y)}\right] d\big(y, [Ty]_{\alpha_T(y)}\big) \leqslant 0.$$

Then, one of the inequalities (2) yields

$$d\big(y, [Ty]_{\alpha_T(y)}\big) \leqslant 0.$$

It follows that $y \in [Ty]_{\alpha_T(y)}$. Again inequality (1) implies that

$$(1 - \beta)\, d\big(y, [Sy]_{\alpha_S(y)}\big) \leqslant \gamma d\big(y, [Ty]_{\alpha_T(y)}\big) + \frac{\delta d\big(y, [Sy]_{\alpha_S(y)}\big) d\big(y, [Ty]_{\alpha_T(y)}\big)}{1 + d(y,y)} = 0.$$

It follows that

$$y \in [Sy]_{\alpha_S(y)} \cap [Ty]_{\alpha_T(y)}.$$

Case (ii): $\alpha + \gamma = 0$. For $x \in X$, as in case(i), take $y \in [Sx]_{\alpha_S(x)}$, $z \in [Ty]_{\alpha_T(y)}$ and

$$\begin{aligned} d\big(z, [Sz]_{\alpha_T(z)}\big) &= H\big([Ty]_{\alpha_T(y)}, [Sz]_{\alpha_T(z)}\big) \\ &\leqslant \alpha d(z,y) + \beta d\big(z, [Sz]_{\alpha_S(z)}\big) + \gamma d\big(y, [Ty]_{\alpha_T(y)}\big) \\ &\quad + \frac{\delta d\big(z, [Sz]_{\alpha_S(z)}\big) d\big(y, [Ty]_{\alpha_T(y)}\big)}{1 + d(z,y)} \end{aligned}$$

Now, $\alpha+\gamma=0$ yields

$$\left[1-\beta-\frac{\delta d\big(y,[Ty]_{\alpha_T(y)}\big)}{1+d\,(z,y)}\right]d\big(z,[Sz]_{\alpha_S(z)}\big)\leqslant 0.$$

Thus, $z\in[Sz]_{\alpha_S(z)}$, moreover,

$$(1-\gamma)\,d\big(z,[Tz]_{\alpha_T(z)}\big)\leqslant\beta d\big(z,[Sz]_{\alpha_S(z)}\big)+\frac{\delta d\big(z,[Sz]_{\alpha_S(z)}\big)d\big(z,[Tz]_{\alpha_T(z)}\big)}{1+d(z,z)}=0.$$

It follows that

$$z\in[Sz]_{\alpha_S(z)}\cap[Tz]_{\alpha_T(z)}\,.$$

Case (iii): Let,

$$\max\left\{\left(\frac{\alpha+\gamma}{1-\beta-\delta}\right),\left(\frac{\alpha+\beta}{1-\gamma-\delta}\right)\right\}=\lambda.$$

Then by $\alpha+\gamma,\ \alpha+\beta\neq 0,\ \alpha+\beta+\gamma+\delta<1$, it follows that $0<\lambda<1$. Choose $x_0\in X$. Then, by hypotheses, there exists $\alpha_S(x_0)\in(0,1]$ such that $[Sx_0]_{\alpha_S(x_0)}$ is a nonempty closed, bounded subset of X. For convenience, we denote $\alpha_S(x_0)$ by α_1. Let $x_1\in[Sx_0]_{\alpha_1}$. For this x_1 there exists $\alpha_T(x_1)\in(0,1]$ such that $[Tx_1]_{\alpha_T(x_1)}\in CB\left(2^X\right)$. Denote $\alpha_T\,(x_1)$ by α_2. By Lemma 1.2, there exists $x_2\in[Tx_1]_{\alpha_2}$ such that

$$d(x_1,x_2)\leqslant H\big([Sx_0]_{\alpha_1},[Tx_1]_{\alpha_2}\big)+\lambda\,(1-\gamma-\delta)\,. \tag{3}$$

By the same argument we can find $\alpha_3\in(0,1]$ and $x_3\in[Sx_2]_{\alpha_3}$ such that

$$d(x_2,x_3)\leqslant H\big([Sx_2]_{\alpha_3},[Tx_1]_{\alpha_2}\big)+\lambda^2\,(1-\beta-\delta)\,. \tag{4}$$

Using induction we produce a sequence $\{x_n\}$ of points of X,

$$x_{2k+1}=[Sx_{2k}]_{\alpha_{2k+1}}\,,$$
$$x_{2k+2}=[Tx_{2k+1}]_{\alpha_{2k+2}}\,,\ k=0,1,2,\ldots,$$

such that,

$$d(x_{2k+1},x_{2k+2})\leqslant H\big([Sx_{2k}]_{\alpha_{2k+1}},[Tx_{2k+1}]_{\alpha_{2k+2}}\big)+\lambda^{2k+1}\,(1-\gamma-\delta)\,,$$
$$d(x_{2k+2},x_{2k+3})\leqslant H\big([Sx_{2k+2}]_{\alpha_{2k+3}},[Tx_{2k+1}]_{\alpha_{2k+2}}\big)+\lambda^{2k+2}\,(1-\beta-\delta)\,.$$

Using inequalities (1) and (3) we have

$$d(x_1,x_2)\leqslant\alpha d(x_0,x_1)+\beta d\big(x_0,[Sx_0]_{\alpha_1}\big)+\gamma d\big(x_1,[Tx_1]_{\alpha_2}\big)$$
$$+\frac{\delta d(x_0,[Sx_0]_{\alpha_1})d(x_1,[Tx_1]_{\alpha_2})}{1+d(x_0,x_1)}+\lambda\,(1-\gamma-\delta)\,.$$

The above inequality implies

$$d(x_1,x_2)\leqslant\frac{(\alpha+\beta)}{(1-\gamma-\delta)}d(x_0,x_1)+\lambda.$$

Using inequalities (1) and (4) we obtain

$$d(x_2,x_3)\leqslant\alpha d(x_2,x_1)+\beta d\big(x_2,[Sx_2]_{\alpha_3}\big)+\gamma d(x_1,[Tx_1]_{\alpha_2})$$
$$+\frac{\delta d\big(x_2,[Sx_2]_{\alpha_3}\big)d\big(x_1,[Tx_1]_{\alpha_2}\big)}{1+d(x_2,x_1)}+\lambda^2\,(1-\beta-\delta)\,.$$

Thus,

$$\begin{aligned} d(x_2,x_3) &\leqslant \left(\frac{\alpha+\gamma}{1-\beta-\delta}\right) d(x_1,x_2)+\lambda^2 \\ &\leqslant \lambda d(x_1,x_2)+\lambda^2. \end{aligned}$$

This implies that

$$\begin{aligned} d(x_n,x_{n+!}) &\leqslant \lambda d(x_{n-1},x_n)+\lambda^n. \\ &\leqslant \lambda\left[\lambda d(x_{n-2},x_{n-1})+\lambda^{n-1}\right]+\lambda^n \\ &\leqslant \lambda^2 d(x_{n-2},x_{n-1})+2\lambda^n \\ &\leqslant \lambda^3 d(x_{n-3},x_{n-2})+3\lambda^n. \end{aligned}$$

It follows that for each $n=1,2,\ldots$,

$$d(x_n,x_{n+!}) \leqslant \lambda^n d(x_0,x_1)+n\lambda^n$$

Now for each positive integer m,n $(n>m)$, we have

$$\begin{aligned} d(x_m,\ x_n) &\leqslant d(x_m,\ x_{m+1})+d(x_{m+1},\ x_{m+2})+\cdots+d(x_{n-1},\ x_n) \\ &\leqslant \lambda^m d(x_0,x_1)+m\lambda^m+\lambda^{m+1}d(x_0,x_1)+(m+1)\,\lambda^{m+1} \\ &\qquad\qquad +\cdots+\lambda^{n-1}d(x_0,x_1)+(n-1)\,\lambda^{n-1} \\ &\leqslant \sum_{i=m}^{n-1}\lambda^i d(x_0,x_1)+\sum_{i=m}^{n-1} i\lambda^i \\ &\leqslant \frac{\lambda^m}{1-\lambda}d(x_0,x_1)+S_{n-1}-S_{m-1},\ \text{where } S_n=\sum_{i=1}^{n} i\lambda^i. \end{aligned}$$

Since $\lambda<1$, it follows from Cauchy's root test that $\sum i\lambda^i$ is convergent, and hence $\{x_n\}$ is a Cauchy sequence in X. As X is complete, there exists $u\in X$ such that $x_n\to u$. Now, Lemma 1.1 implies that

$$\begin{aligned} d\big(u,[Su]_{\alpha_S(u)}\big) &\leqslant d(u,x_{2n})+d\big(x_{2n},[Su]_{\alpha_S(u)}\big) \\ &\leqslant d(u,x_{2n})+H\big([Tx_{2n-1}]_{\alpha_{2n}},[Su]_{\alpha_S(u)}\big). \end{aligned}$$

The above inequality implies that

$$\begin{aligned} &d\big(u,[Su]_{\alpha_S(u)}\big) \\ &\leqslant \left(1-\beta-\delta\frac{d(x_{2n-1},x_{2n})}{1+d(u,x_{2n-1})}\right)^{-1}\big(d(u,x_{2n})+\alpha d(u,x_{2n-1})+\gamma d(x_{2n-1},x_{2n})\big). \end{aligned}$$

Letting $n\to\infty$, we have

$$d\big(u,[Su]_{\alpha_S(u)}\big)\leqslant 0.$$

This implies that $u\in[Su]_{\alpha_S(u)}$. Similarly, by using

$$d\big(u,[Tu]_{\alpha_T(u)}\big)\leqslant d\big(u,x_{2n+1}\big)+d\big(x_{2n+1},[Tu]_{\alpha_T(u)}\big),$$

we can show that $u\in[Tu]_{\alpha_T(u)}$, which implies that $u\in[Su]_{\alpha_S(u)}\cap[Tu]_{\alpha_T(u)}$. □

2.2. Example. Let $X=[0,1]$, $d(x,y)=|x-y|$, whenever $x,y\in X$, $\lambda,\mu\in(0,1]$ and $S,T:X\longrightarrow I^X$ are fuzzy mappings such that $T(x)$, $S(x)\in I^X$ are defined as follows:

$$S(0)(t)=\begin{cases}1 & \text{if } t=0\\ \frac{1}{2} & \text{if } 0<t\leqslant\frac{1}{100}\\ 0 & t>\frac{1}{100},\end{cases}\qquad T(0)(t)=\begin{cases}1 & \text{if } t=0\\ \frac{1}{3} & \text{if } 0<t\leqslant\frac{1}{150}\\ 0 & t>\frac{1}{150},\end{cases}$$

if $x \neq 0$,

$$S(x)(t)=\begin{cases}\lambda & \text{if } 0\leqslant t<\frac{x}{14}\\ \frac{\lambda}{2} & \text{if } \frac{x}{14}\leqslant t\leqslant\frac{x}{10}\\ \frac{\lambda}{3} & \text{if } \frac{x}{12}<t<x\\ 0 & \text{if } x\leqslant t<\infty,\end{cases}\qquad T(x)(t)=\begin{cases}\mu & \text{if } 0\leqslant t<\frac{x}{16}\\ \frac{\mu}{4} & \text{if } \frac{x}{16}\leqslant t\leqslant\frac{x}{10}\\ \frac{\mu}{10} & \text{if } \frac{x}{12}<t<x\\ 0 & \text{if } x\leqslant t<\infty.\end{cases}$$

Note that

$$[S0]_{\lambda_S(0)}\ [T0]_{\lambda_T(0)}=\{0\},\ \text{if } \lambda_S(0)=\lambda_T(0)=1$$

for $x \neq 0$,

$$\widehat{S}x=[Sx]_{\lambda}=\left[0,\frac{x}{14}\right) \text{ and } \widehat{T}x=[Tx]_{\mu}=\left[0,\frac{x}{16}\right),$$

thus, $Sx,\ Tx \notin K(X)$. However,

$$[Sx]_{\frac{\lambda}{2}}=\left[0,\frac{x}{10}\right] \text{ and } [Tx]_{\frac{\mu}{4}}=\left[0,\frac{x}{10}\right],$$

hence, $Sx,\ Tx \in \mathcal{F}(X) \subseteq G(X)$. Moreover,

$$\gamma+\delta\frac{d\big(x,[Sx]_{\lambda_S(x)}\big)}{1+d(x,y)}\leqslant\frac{1}{30}+\frac{1}{40}\frac{\left|\frac{9x}{10}\right|}{1+|x-y|}\leqslant 1.$$

Similarly,

$$\beta+\delta\frac{d\big(y,[Ty]_{\lambda_T(y)}\big)}{1+d(x,y)}\leqslant 1.$$

Hence, if $x=y, H\big([Sx]_{\lambda_S(x)},[Ty]_{\lambda_T(y)}\big)=0$ and if $x\neq y$,

$$H\big([Sx]_{\lambda_S(x)},[Ty]_{\lambda_T(y)}\big)<\frac{1}{10}|x-y|+\frac{1}{20}\left|x-\frac{x}{10}\right|+\frac{1}{30}\left|y-\frac{y}{10}\right| +\frac{1}{40}\left[\frac{\left|x-\frac{x}{10}\right|\left|y-\frac{y}{10}\right|}{1+|x-y|}\right].$$

Since X is not linear and $[Sx]_\lambda,\ [Tx]_\lambda$ are not compact for each λ, all previous relevant fixed point results [1, 2, 5, 6, 8, 16, 18, 19, 20, 22] for fuzzy (approximate quantity-valued) mappings on complete metric linear spaces are not applicable even if $\delta=0$. However, S and T satisfy all conditions of Theorem 2.1 for $\alpha=\frac{1}{10},\ \beta=\frac{1}{20},\ \gamma=\frac{1}{30},\ \delta=\frac{1}{40}$.

3. Fixed points of fuzzy and multivalued mappings

As an application of the fuzzy fixed point result of the previous section we obtain common fixed points of fuzzy mappings and multivalued mappings (see, [7, 11, 12, 21]).

3.1. Theorem. *Let $S,T:X\to I^X$ be such that $\widehat{S}x,\widehat{T}x\in CB\left(2^X\right)$ and for all $x,y\in X$,*

$$H(\widehat{S}x,\widehat{T}y)\leqslant\alpha d(x,y)+\beta d(x,\widehat{S}x)+\gamma d(y,\widehat{T}y)+\frac{\delta d(x,\widehat{S}x)d(y,\widehat{T}y)}{1+d(x,y)}.$$

where $\alpha,\beta,\gamma,\delta$ are non negative real numbers with $\alpha+\beta+\gamma+\delta<1$ and

$$\gamma+\frac{\delta d\left(x,\widehat{S}x\right)}{1+d(x,y)}<1,\quad \beta+\frac{\delta d\left(y,\widehat{T}y\right)}{1+d(x,y)}<1.$$

Then there exists a point $x^*\in X$ such that $T(x^*)(x^*)\geqslant T(x^*)(x)$ and $S(x^*)(x^*)\geqslant S(x^*)(x)$ for all $x\in X$.

Proof. Let

$$\max_{t\in X} S(x)(t) = \mu, \quad \max_{t\in X} T(x)(t) = \nu.$$

Then for all $x, y \in X$,

$$\widehat{S}x = [Sx]_\mu, \quad \widehat{T}y = [Ty]_\nu$$

and

$$H\big([Sx]_\mu, [Ty]_\nu\big) \leqslant \alpha d(x,y) + \beta d\big(x, [Sx]_\mu\big) + \gamma d\big(y, [Ty]_\nu\big) + \frac{\delta d\big(x,[Sx]_\mu\big) d\big(y,[Ty]_\nu\big)}{1 + d(x,y)}. \tag{5}$$

By Theorem 2.1 there exists $x^* \in X$ such that $x^* \in [Sx^*]_\mu \cap [Tx^*]_\nu = \widehat{S}x^* \cap \widehat{T}x^*$. Now Lemma 1.4 implies that

$$T(x^*)(x^*) \geqslant T(x^*)(x), \quad S(x^*)(x^*) \geqslant S(x^*)(x)$$

for all $x \in X$. □

3.2. Theorem. *Let $A, B : X \to CB\left(2^X\right)$ be multivalued mappings and for all $x, y \in X$,*

$$H(Ax, By) \leqslant \alpha d(x,y) + \beta d(x, Ax) + \gamma d(y, By) + \frac{\delta d(x,Ax) d(y,By)}{1 + d(x,y)},$$

where $\alpha, \beta, \gamma, \delta$ are non negative real numbers with $\alpha + \beta + \gamma + \delta < 1$ and

$$\gamma + \delta \frac{d(x, Sx)}{1 + d(x,y)} < 1, \quad \beta + \frac{\delta d(y, Ty)}{1 + d(x,y)} < 1. \tag{6}$$

Then there exists $u \in X$ such that $u \in Au \cap Bu$.

Proof. Consider a pair of arbitrary mappings $F, G : X \to (0,1]$ and a pair of fuzzy mappings $S, T : X \to I^X$ defined by

$$S(x)(t) = \begin{cases} Fx & t \in Ax \\ 0 & t \notin Ax, \end{cases} \qquad T(x) = \begin{cases} Gx & t \in Bx \\ 0 & t \notin Bx. \end{cases}$$

Then, for $x \in X$,

$$[Sx]_{\alpha_S(x)} = \{t : S(x)(t) \geqslant \alpha_S(x)\} = Ax, \quad [Tx]_{\alpha_T(x)} = Bx.$$

Therefore, Theorem 2.1 can be applied to obtain $u \in X$ such that

$$u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)} = Au \cap Bu.$$ □

4. Heilpern fixed points of fuzzy mappings

In [1, 2, 5, 6, 8, 16, 18, 19, 20, 22] the authors obtained Heilpern fixed points of fuzzy contractive and fuzzy locally contractive mappings on a metric linear space with the d_∞-metric for fuzzy sets, and established fuzzy extensions of some Banach type and Edelstein fixed point theorems [11, 12].

In the following we establish common fixed point theorems for fuzzy mappings under a rational contractive condition on a metric space with the d_∞-metric for fuzzy sets. The study of fixed points of fuzzy set-valued mappings related to the d_∞-metric is useful for computing Hausdorff dimensions. These dimensions may help us to understand ε^∞-space, which are used in high energy physics [13, 14]. This is because events in this case are mostly fuzzy sets.

4.1. Theorem. *Let $S,T : X \to E(X)$, and for all $x, y \in X$,*

$$d_\infty(S(x), T(y)) \leqslant \alpha d(x,y) + \beta p(x, S(x)) + \gamma p(y, T(y)) + \frac{\delta p(x, S(x)) p(y, T(y))}{1 + d(x,y)},$$

where $\alpha, \beta, \gamma, \delta$ are non negative real numbers with $\alpha + \beta + \gamma + \delta < 1$ and

$$\gamma + \delta \frac{p(x, S(x))}{1 + d(x,y)} < 1, \quad \beta + \frac{\delta p(y, T(y))}{1 + d(x,y)} < 1. \tag{7}$$

Then there exists a point $u \in X$ such that $\{u\} \subset Su$, $\{u\} \subset Tu$.

Proof. Pick $x \in X$. Then by the assumptions, $[Sx]_1, [Tx]_1$ are nonempty closed and bounded subsets of X. Now, for all $x, y \in X$,

$$\begin{aligned} &D_1(S(x), T(y)) \\ &\qquad \leqslant d_\infty(S(x), T(y)) \\ &\qquad \leqslant \alpha d(x,y) + \beta p(x, S(x)) + \gamma p(y, T(y)) + \frac{\delta p(x, S(x)) p(y, T(y))}{1 + d(x,y)}. \end{aligned}$$

Since $[Sx]_1 \subseteq [Sx]_\alpha \in CB(2^X)$ for each $\alpha \in [0,1]$, therefore $d(x, [Sx]_\alpha) \leqslant d(x, [Sx]_1)$ for each $\alpha \in [0,1]$ and it implies that $p(x, S(x)) \leqslant d(x, [Sx]_1)$. This further implies that

$$\begin{aligned} H\big([Sx]_1, [Ty]_1\big) &\leqslant \alpha d(x,y) + \beta d(x, [Sx]_1) \\ &\qquad + \gamma d\big(y, [Ty]_1\big) + \frac{\delta d\big(x, [Sx]_1\big) d\big(y, [Ty]_1\big)}{1 + d(x,y)}. \end{aligned}$$

Now, by Theorem 2.1 there exists $u \in X$ such that $\{u\} \subset Su$, $\{u\} \subset Tu$. □

4.2. Theorem. *Let $S,T : V \to W(V)$ and for all $x, y \in V$,*

$$d_\infty(S(x), T(y)) \leqslant \alpha d(x,y) + \beta p(x, S(x)) + \gamma p(y, T(y)) + \frac{\delta p(x, S(x)) p(y, T(y))}{1 + d(x,y)},$$

where $\alpha, \beta, \gamma, \delta$ are non negative real numbers with $\alpha + \beta + \gamma + \delta < 1$ and

$$\gamma + \frac{\delta p(x, S(x))}{1 + d(x,y)} < 1, \quad \beta + \frac{\delta p(y, T(y))}{1 + d(x,y)} < 1. \tag{8}$$

Then there exists a point $u \in V$ such that $\{u\} \subset Su$, $\{u\} \subset Tu$.

Proof. Let $x \in V$. Then by Lemma 1.3 there exists $y \in V$ such that $\{y\} \subset S(x)$. This implies that $p_\alpha(y, S(x) = 0$ for all $\alpha \in [0,1]$, which is possible if and only if $y \in [Sx]_1$. Similarly, we can find $z \in V$ such that $z \in [Tx]_1$. It follows that for each $x \in V$, $[Sx]_1$, $[Tx]_1 \in C(2^V)$ for all $x, y \in X$. The remaining part of the proof is similar as that of the previous theorem. □

4.3. Corollary. [17] *Let (V, d) be a complete metric linear space and $T : V \to W(V)$ a fuzzy mapping such that for all $x, y \in V$,*

$$d_\infty(T(x), T(y)) \leqslant \alpha d(x,y),$$

where $0 \leqslant \lambda < 1$. Then there exists $u \in V$ such that $\{u\} \subset Tu$. □

4.4. Example. Let $X = \{1,2,3\}$, $\{1\}, \{2\}, \{3\}$ be crisp sets. Define $d : X \times X \to R$ as follows:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{5}{7} & \text{if } x \neq y \text{ and } x,y \in X \setminus \{2\}, \\ 1 & \text{if } x \neq y \text{ and } x,y \in X \setminus \{3\}, \\ \frac{4}{7} & \text{if } x \neq y \text{ and } x,y \in X \setminus \{1\}. \end{cases}$$

Define fuzzy mappings $T : X \to I^X$ as follows:

$$T(1)\,(t) = T(3)\,(t) = \begin{cases} 1 & \text{if } t = 3, \\ \frac{1}{2} & \text{if } t = 2, \\ 0 & \text{if } t = 1, \end{cases} \quad \textit{and } T(2)\,(t) = \begin{cases} 1 & \text{if } t = 1, \\ \frac{1}{3} & \text{if } t = 2, \\ 0 & \text{if } t = 3. \end{cases}$$

Then,

$$[Tx]_1 = \{t : T(x)(t) = 1\} = \begin{cases} \{3\} & \text{if } x \neq 2, \\ \{1\} & \text{if } x = 2. \end{cases}$$

$$\begin{aligned} H\big(\{1\}, \{3\}\big) &= \max\left\{ \sup_{\alpha\in\{1\}} d\,(a, \{3\})\,, \sup_{b\in\{3\}} d\,(\{1\}\,, b) \right\} \\ &= \max\{d\,(1, 3)\,, d\,(1, 3)\} = \frac{5}{7} \end{aligned}$$

Now,

$$d_\infty\big(T\,(3)\,, T\,(2)\big) \geqslant D_1\big(T\,(3)\,, T\,(2)\big) = H\big(\{3\}\,, \{1\}\big) = \frac{5}{7},$$

$$d(3, 2) = \frac{4}{7}.$$

Since, $d_\infty\big(T\,(3)\,, T\,(2)\big) > \alpha d(3, 2)$ for each $\alpha < 1$ and X is not linear, [17, Corollary 4.3] and the results in [2, 5, 6, 8, 19, 20, 22] are cannot be applied to find $3 \in [T3]_1$. In order to apply Theorem 4.1, consider the fuzzy mapping $S : X \to I^X$ given by

$$S\,(1)\,(t) = S\,(2)\,(t) = S\,(3)\,(t) = \begin{cases} 1 & \text{if } t = 3, \\ \frac{1}{20} & \text{if } t = 2, \\ 0 & \text{if } t = 1. \end{cases}$$

Then, $[Sx]_1 = \{t : S(x)(t) = 1\} = \{3\}$, for all $x \in X$,

$$H([Sx]_1\,, [Ty]_1) = D_1(Sx, Ty) = \begin{cases} H(\{3\}\,, \{3\}) = 0 & \text{if } y \neq 2, \\ H(\{3\}\,, \{1\}) = \frac{5}{7} & \text{if } y = 2. \end{cases}$$

For $\alpha = \beta = \frac{1}{20},\ \delta = \frac{1}{7},\ \gamma = \frac{5}{7}$ and $y = 2$,

$$\alpha d(x, y)\ + \beta p(x, S\,(x)) + \gamma p(y, T\,(y)) + \frac{\delta p(x, S\,(x))p(y, T\,(y))}{1 + d(x, y)} > \frac{5}{7}$$

and

$$\frac{5}{7} + \frac{\frac{1}{7}p(x, S\,(x))}{1 + d\,(x, y)} < 1, \quad \frac{1}{20} + \frac{\frac{1}{7}p(y, T\,(y))}{1 + d\,(x, y)} < 1.$$

Hence, all the conditions of Theorem 4.1 are satisfied to obtain $3 \in [S3]_1 \cap [T3]_1$.

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