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COUPLED FIXED POINT THEOREMS IN GENERALIZED METRIC SPACES

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Abstract

T. G. Bhaskar and V. Lakshmikantham (*Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Analysis **65**, 1379–1393, 2006), V. Lakshmikantham and Lj. B. Ćirić (*Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Analysis **70**, (2009) 4341–4349, 2009) introduced the concept of a coupled coincidence point of a mapping F from $X \times X$ into X and a mapping g from X into X. In the present paper, we prove a coupled coincidence fixed point theorem in the setting of a generalized metric space in the sense of Z. Mustafa and B. Sims.

Keywords: Common fixed Point, Coupled coincidence fixed point, Generalized metric space.

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1. Introduction

The study of common fixed points of mappings satisfying certain contractive conditions has been researched extensively by many mathematicians since fixed point theory plays a major role in mathematics and applied sciences. For a survey of coincidence point theory in metric and cone metric spaces, we refer the reader (as examples) to [1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 17, 18, 19, 24]. Mustafa and Sims [14] introduced a new notion of generalized metric space called a G-metric space. Mustafa, Sims and others studied fixed point theorems for mappings satisfying different contractive conditions (see [13, 15, 16, 21, 22, 23]). Abbas and Rhoades [2] obtained some common fixed point theorems for non commuting maps without continuity, satisfying different contractive conditions in the setting of generalized metric spaces. While V. Lakshmikantham *et al.* in [5, 11] introduced the concept of a coupled coincidence point of a mapping F from $X \times X$ into X and a mapping g from X into X, and studied fixed point theorems in

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partially ordered metric spaces. In [20] S. Sedghi *et al.* proved a coupled fixed point theorem for contractive mappings in complete fuzzy metric spaces. The aim of the present paper is to prove a coupled coincidence fixed point theorem in the setting of a generalized metric space in the sense of Z. Mustafa and B. Sims.

2. Basic Concepts.

The following definition was introduced by Mustafa and Sims [14].

2.1. Definition. see [14]. Let X be a nonempty set and $G : X \times X \times X \to \mathbf{R}^+$ a function satisfying the following properties:

(G_1) $G(x, y, z) = 0$ if $x = y = z$,
(G_2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,
(G_3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, symmetry in all three variables,
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

2.2. Definition. [14]. Let (X, G) be a G-metric space and (x_n) a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence (x_n), if $\lim_{n,m\to+\infty} G(x, x_n, x_m) = 0$, and we say that the sequence (x_n) is G-convergent to x or that (x_n) G-converges to x.

Thus, $x_n \to x$ in a G-metric space (X, G) if for any $\varepsilon > 0$, there exists $k \in \mathbf{N}$ such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq k$.

2.3. Proposition. [14] *Let (X, G) be a G-metric space. Then the following are equivalent:*

(1) *(x_n) is G-convergent to x.*
(2) *$G(x_n, x_n, x) \to 0$ as $n \to +\infty$.*
(3) *$G(x_n, x, x) \to 0$ as $n \to +\infty$.*
(4) *$G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.*

2.4. Definition. [12] Let (X, G) be a G-metric space. A sequence (x_n) is called G-Cauchy if for every $\varepsilon > 0$, there is $k \in \mathbf{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq k$; that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$

2.5. Proposition. [14] *Let (X, G) be a G-metric space. Then the following are equivalent:*

(1) *The sequence (x_n) is G-Cauchy.*
(2) *For every $\epsilon > 0$, there is $k \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq k$.*

2.6. Definition. [14] Let (X, G) and (X', G') be G-metric spaces and $f : (X, G) \to (X', G')$ a function. Then f is said to be G-continuous at a point $a \in X$ if and only if for every $\varepsilon > 0$, there is $\delta > 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

2.7. Proposition. [14] *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

The following are examples of G-metric spaces.

2.8. Example. [14] Let $(\mathbf{R}, d)$ be the usual metric space. Define G_s by

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z)$$

for all $x, y, z \in \mathbf{R}$. Then it is clear that $(\mathbf{R}, G_s)$ is a G-metric space.

2.9. Example. [14] Let $X = \{a, b\}$. Define G on $X \times X \times X$ by

$$G(a, a, a) = G(b, b, b) = 0,$$
$$G(a, a, b) = 1, \; G(a, b, b) = 2$$

and extend G to $X \times X \times X$ by using the symmetry in the variables. Then it is clear that (X, G) is a G-metric space.

2.10. Definition. [14] A G-metric space (X, G) is called G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

2.11. Definition. [5] An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

2.12. Definition. [11] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

2.13. Definition. [11] Let X be a nonempty set. Then we say that the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if $gF(x, y) = F(gx, gy)$.

3. Main Results

We start our work by proving the following crucial lemma.

3.1. Lemma. *Let (X, G) be a G-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that*

$$G(F(x, y), F(u, v), F(z, w)) \leq k(G(gx, gu, gz) + G(gy, gv, gw)) \tag{3.1}$$

for all $x, y, z, w, u, v \in X$. Assume that (x, y) is a coupled coincidence point of the mappings F and g. If $k \in [0, \frac{1}{2})$, then

$$F(x, y) = gx = gy = F(y, x).$$

Proof. Since (x, y) is a coupled coincidence point of the mappings F and g, we have $gx = F(x, y)$ and $gy = F(y, x)$. Assume $gx \neq gy$. Then by (3.1), we get

$$\begin{aligned} G(gx, gy, gy) &= G(F(x, y), F(y, x), F(y, x)) \\ &\leq k(G(gx, gy, gy) + G(gy, gx, gx)). \end{aligned}$$

Also by (3.1), we have

$$\begin{aligned} G(gy, gx, gx) &= G(F(y, x), F(x, y), F(x, y)) \\ &\leq k(G(gy, gx, gx) + G(gx, gy, gy)). \end{aligned}$$

Therefore

$$G(gx, gy, gy) + G(gy, gx, gx) \leq 2k(G(gx, gy, gy) + G(gy, gx, gx)).$$

Since $2k < 1$, we get

$$G(gx, gy, gy) + G(gy, gx, gx) < G(gx, gy, gy) + G(gy, gx, gx),$$

which is a contradiction. So $gx = gy$, and hence

$$F(x, y) = gx = gy = F(y, x). \qquad \square$$

3.2. Theorem. *Let (X, G) be a G-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that*

$$G(F(x, y), F(u, v), F(z, w)) \leq k(G(gx, gu, gz) + G(gy, gv, gw)) \tag{3.2}$$

for all $x, y, z, w, u, v \in X$. Assume that F and g satisfy the following conditions:

(1) $F(X \times X) \subseteq g(X)$,

(2) *$g(X)$ is G-complete, and*
(3) *g is G-continuous and commutes with F.*

If $k \in \left(0, \frac{1}{2}\right)$, then there is a unique x in X such that $gx = F(x,x) = x$.

Proof. Let $x_0, y_0 \in X$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Again since $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing this process, we can construct two sequences (x_n) and (y_n) in X such that $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$. For $n \in \mathbf{N}$, by (3.2) we have

$$G(gx_n, gx_{n+1}, gx_{n+1}) = G(F(x_{n-1}, y_{n-1}), F(x_n, y_n), F(x_n, y_n)) \tag{3.3}$$

$$\leq k(G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n)). \tag{3.4}$$

From

$$\begin{aligned} G(gx_{n-1}, gx_n, gx_n) &= G(F(x_{n-2}, y_{n-2}), F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1})) \\ &\leq k(G(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G(gy_{n-2}, gy_{n-1}, gy_{n-1})), \end{aligned}$$

and

$$\begin{aligned} G(gy_{n-1}, gy_n, gy_n) &= G(F(y_{n-2}, x_{n-2}), F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1})) \\ &\leq k(G(gy_{n-2}, gy_{n-1}, gy_{n-1}) + G(gx_{n-2}, gx_{n-1}, gx_{n-1})), \end{aligned}$$

we have

$$\begin{aligned} G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n) \leq 2k(&G(gx_{n-2}, gx_{n-1}, gx_{n-1}) \\ &+ G(gy_{n-2}, gy_{n-1}, gy_{n-1})) \end{aligned}$$

holds for all $n \in \mathbf{N}$. Thus, we get that

$$\begin{aligned} G(gx_n, gx_{n+1}, gx_{n+1}) &\leq k(G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n)) \\ &\leq 2k^2(G(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G(gy_{n-2}, gy_{n-1}, gy_{n-1})) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\leq \frac{1}{2}(2k)^n(G(gx_0, gx_1, gx_1) + G(gy_0, gy_1, gy_1)). \end{aligned}$$

Thus for each $n \in \mathbf{N}$, we have

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq \frac{1}{2}(2k)^n(G(gx_0, gx_1, gx_1) + G(gy_0, gy_1, gy_1)). \tag{3.5}$$

Let $m, n \in \mathbf{N}$ with $m > n$. By Axiom G_5 of the definition of G-metric spaces, we have

$$\begin{aligned} G(gx_n, gx_m, gx_m) \leq G(gx_n, gx_{n+1}, gx_{n+1}) &+ G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \\ &+ \cdots + G(gx_{m-1}, gx_m, gx_m). \end{aligned}$$

Since $2k < 1$, by (3.5) we get that

$$\begin{aligned} G(gx_n, gx_m, gx_m) &\leq \frac{1}{2}\left(\sum_{i=n}^{m-1}(2k)^i\right)(G(gx_0, gx_1, gx_1) + G(gy_0, gy_1, gy_1)) \\ &\leq \frac{(2k)^n}{2(1-2k)}(G(gx_0, gx_1, gx_1) + G(gy_0, gy_1, gy_1)). \end{aligned}$$

Letting $n, m \to +\infty$, we have

$$\lim_{n,m\to+\infty} G(x_n, gx_m, gx_m) = 0.$$

Thus (gx_n) is G-Cauchy in $g(X)$. Similarly, we may show that (gy_n) is G-Cauchy in $g(X)$. Since $g(X)$ is G-complete, we get that (gx_n) and (gy_n) are G-convergent to some $x \in X$ and $y \in X$ respectively. Since g is G-continuous, we have (ggx_n) is G-convergent to gx and (ggy_n) is G-convergent to gy. Also, since g and F commute, we have $ggx_{n+1} = g(F(x_n, y_n)) = F(gx_n, gy_n)$, and

$$ggy_{n+1} = g(F(y_n, x_n)) = F(gy_n, gx_n).$$

Thus

$$\begin{aligned} G(ggx_{n+1}, F(x,y), F(x,y)) &= G(F(gx_n, gy_n), F(x,y), F(x,y)) \\ &\leq k(G(ggx_n, gx, gx) + G(ggy_n, gy, gy)). \end{aligned}$$

Letting $n \to +\infty$, and using the fact that G is continuous on its variables, we get that

$$G(gx, F(x,y), F(x,y)) \leq k(G(gx, gx, gx) + G(gy, gy, gy)) = 0.$$

Hence $gx = F(x,y)$. Similarly, we may show that $gy = F(y,x)$. By Lemma 3.1, (x,y) is a coupled fixed point of the mappings F and g. So

$$gx = F(x,y) = F(y,x) = gy.$$

Since (gx_{n+1}) is subsequence of (gx_n) we have that (gx_{n+1}) is G convergent to x. Thus

$$\begin{aligned} G(gx_{n+1}, gx, gx) &= G(gx_{n+1}, F(x,y), F(x,y)) \\ &= G(F(x_n, y_n), F(x,y), F(x,y)) \\ &\leq k(G(gx_n, gx, gx) + G(gy_n, gy, gy)). \end{aligned}$$

Letting $n \to +\infty$, and using the fact that G is continuous on its variables, we get that

$$G(x, gx, gx) \leq k(G(x, gx, gx) + G(y, gy, gy)).$$

Similarly, we may show that

$$G(y, gy, gy) \leq k(G(x, gx, gx) + G(y, gy, gy)).$$

Thus

$$G(x, gx, gx) + G(y, gy, gy) \leq 2k(G(x, gx, gx) + G(y, gy, gy)).$$

Since $2k < 1$, the last inequality happens only if $G(x, gx, gx) = 0$ and $G(y, gy, gy) = 0$. Hence $x = gx$ and $y = gy$. Thus we get

$$gx = F(x,x) = x.$$

To prove the uniqueness, let $z \in X$ with $z \neq x$ such that

$$z = gz = F(z,z).$$

Then

$$\begin{aligned} G(x,z,z) &= G(F(x,x), F(z,z), F(z,z)) \\ &\leq 2kG(gx, gz, gz) \\ &= 2kG(x,z,z). \end{aligned}$$

Since $2k < 1$, we get $G(x,z,z) < G(x,z,z)$, which is a contradiction. Thus F and g have a unique common fixed point. □

3.3. Corollary. *Let (X, G) be a G-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that*

$$G(F(x,y), F(u,v), F(u,v)) \leq k(G(gx, gu, gu) + G(gy, gv, gv)) \tag{3.6}$$

for all $x, y, u, v \in X$. Assume F and g satisfy the following conditions:

(1) *$F(X \times X) \subseteq g(X)$,*
(2) *$g(X)$ is G-complete, and*
(3) *g is G-continuous and commutes with F.*

If $k \in \left(0, \frac{1}{2}\right)$, *then there is a unique* x *in* X *such that* $gx = F(x,x) = x$.

Proof. Follows from Theorem 3.1 by taking $z = u$ and $v = w$. □

3.4. Corollary. *Let* (X, G) *be a complete* G*-metric space. Let* $F : X \times X \to X$ *be a mapping such that*

(3.7) $$G(F(x,y), F(u,v), F(u,v)) \le k(G(x,u,u) + G(y,v,v))$$

for all $x, y, u, v \in X$. *If* $k \in \left[0, \frac{1}{2}\right)$, *then there is a unique* x *in* X *such that* $F(x,x) = x$.

Proof. Define $g : X \to X$ by $gx = x$. Then F and g satisfy all the hypothesis of Corollary 3.1. Hence the result follows. □

Now, we introduce some examples of our theorem.

3.5. Example. Let $X = [0,1]$. Define $G : X \times X \times X \to \mathbf{R}^+$ by

$$G(x,y,z) = |x-y| + |x-z| + |y-z|$$

for all $x, y, z \in X$. Then (X, G) is a complete G-metric space. Define a map

$$F : X \times X \to X$$

by $F(x,y) = \frac{1}{6}xy$ for $x, y \in X$. Also, define $g : X \to X$ by $g(x) = \frac{1}{2}x$ for $x \in X$. Since

$$|xy - uv| \le |x-u| + |y-v|$$

holds for all $x, y, u, v \in X$, we have

$$\begin{aligned} G(F(x,y), F(u,v), F(z,w)) &= \frac{1}{6}|xy-uv| + \frac{1}{6}|xy-zw| + \frac{1}{6}|uv-zw| \\ &\le \frac{1}{6}\big(|x-u| + |y-v|\big) + \frac{1}{6}\big(|x-z| + |y-w|\big) \\ &\qquad + \frac{1}{6}\big(|u-z| + |v-w|\big) \\ &\le \frac{1}{3}\big(G(gx, gu, gz) + G(gy, gv, gw)\big) \end{aligned}$$

holds for all $x, y.u, v, z, w \in X$. It is an easy matter to see that F and g satisfy all the hypothesis of Theorem 3.1. Thus F and g have a unique common fixed point. Here $F(0,0) = g(0) = 0$.

3.6. Example. Let $X = [-1,1]$. Define $G : X \times X \times X \to \mathbf{R}^+$ by

$$G(x,y,z) = |x-y| + |x-z| + |y-z|$$

for all $x, y, z \in X$. Then (X, G) is a complete G-metric space. Define a map

$$F : X \times X \to X$$

by

$$F(x,y) = \frac{1}{8}x^2 + \frac{1}{8}y^2 - 1$$

for $x, y \in X$. Then $F(X \times X) = \left[-1, -\frac{3}{4}\right]$. Also,

$$\begin{aligned} G(F(x,y), F(u,v), F(u,v)) &= \frac{1}{4}(|x^2 - u^2 + y^2 - v^2|) \\ &\le \frac{1}{4}(2|x-u| + 2|y-v|) \\ &= \frac{1}{4}(G(x,u,u) + G(y,v,v)). \end{aligned}$$

Then by Corollary 3.2, F has a unique fixed point. Here $x = 2 - 2\sqrt{2}$ is the unique fixed point of F; that is, $F(x,x) = x$.

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