Hacettepe Journal of Mathematics and Statistics
Volume 40 (3) (2011), 453 – 460

AN ITERATIVE ALGORITHM COMBINING THE VISCOSITY METHOD WITH THE PARALLEL METHOD FOR A FINITE FAMILY OF UNIFORMLY L-LIPSCHITZIAN MAPPINGS[†]

Feng Gu[*]

Received 29 : 05 : 2010 : Accepted 23 : 08 : 2010

Abstract

The purpose of this paper is to prove a strong convergence theorem for a finite family of uniformly L-Lipschitzian mappings through the viscosity parallel iterative algorithm in Banach spaces. The results presented in the paper improve and extend some recent results announced by Chang, Ofoedu, Schu and Zeng, and many others.

Keywords: Asymptotically pseudo-contractive mapping, Uniformly L-Lipschitzian mapping, Viscosity iterative algorithm, Parallel iterative algorithm.

2000 AMS Classification: Primary 47 H 09; Secondary 47 H 10, 47 J 25.

1. Introduction and preliminaries

Let E be a real Banach space with norm $||\cdot||$ and J the normalized duality mapping from E into E^* give by

$$J(x) = \{f \in E^* : \langle x,\ f\rangle = ||x||^2 = ||f||^2,\ ||f|| = ||x||\}$$

for $x \in E$, where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ is the generalized duality pairing between E and E^*. The single-valued normalized duality mapping is denoted by j.

1.1. Definition. Let K be a nonempty closed convex subset of E, $T : K \to K$ a mapping.

[†]The present studies were supported by the National Natural Science Foundation of China (11071169), and the Natural Science Foundation of Zhejiang Province (Y6110287)

[*]Institute of Applied Mathematics and Department of Mathematics, Hangzhou Normal University, Hangzhou, Zhejiang 310036, China. E-mail: gufeng99@sohu.com

(1) T is said to be *uniformly L-Lipschitzian* if there exists $L > 0$ such that for any $x, y \in K$

$$||T^n x - T^n y|| \le ||x - y||,\ \forall n \ge 1;$$

(2) T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for any given $x,\ y \in K$,

$$||T^n x - T^n y|| \le k_n ||x - y||,\ \forall n \ge 1;$$

(3) T is said to be *asymptotically pseudo-contractive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for any given $x,\ y \in K$, there exists $j(x-y) \in J(x-y)$,

$$\langle T^n x - T^n y,\ j(x-y)\rangle \le k_n ||x-y||^2,\ \forall n \ge 1.$$

1.2. Remark. (1) It is easy to see that if T is an asymptotically nonexpansive mapping, then T is a uniformly L-Lipschitzian mapping, where $L = \sup_{n\ge 1} k_n$. And every asymptotically nonexpansive mapping is asymptotically pseudo-contractive, but the inverse is not true, in general.

(2) The concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [3], while the concept of asymptotically pseudo-contractive mappings was introduced by Schu [6], who proved the following theorem:

1.3. Theorem. (Schu [6]) *Let H be a Hilbert space, K a nonempty bounded closed convex subset of H and $T : K \to K$ a completely continuous, uniformly L-Lipschitzian, and asymptotically pseudo-contractive mapping with a sequence $\{k_n\} \subset [1, \infty)$ satisfying the following conditions:*

(i) $k_n \to 1$ *as* $n \to \infty$*;*
(ii) $\sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$, *where* $q_n = 2k_n - 1$.

Suppose further that $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0, 1] such that $\varepsilon < \alpha_n < \beta_n \le b\ \forall n \ge 1$, where $\varepsilon > 0$ and $b \in (0,\ L^{-2}[(1+L^2)^{\frac{1}{2}} - 1])$ are some positive numbers. For any given $x_1 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n.$$

Then $\{x_n\}$ converges strongly to some fixed point of T in K. □

In [1] the author extended Theorem 1.3 to a real uniformly smooth Banach space and proved the following theorem:

1.4. Theorem. (Chang [1]) *Let E be a uniformly smooth Banach space, K be a nonempty bounded closed convex subset of E, $T : K \to K$ an asymptotically pseudo-contractive mapping with a sequence $\{k_n\} \subset [1,\ \infty)$, $k_n \to 1$ and $F(T) \ne \emptyset$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}$ be a sequence in $[0,\ 1]$ satisfying the following conditions:*

(i) $\alpha_n \to 0$*;*
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For any given $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n,\ \forall n \ge 0.$$

If there exists a strict increasing function $\phi : [0,\ \infty) \to [0,\ \infty)$, $\phi(0) = 0$, such that

$$\langle T^n x_n - x^*,\ j(x_n - x^*)\rangle \le k_n ||x_n - x^*||^2 - \phi(||x_n - x^*||),\ \forall n \ge 1,$$

where $x^* \in F(T)$ is some fixed point of T in K, then $x_n \to x^*$ as $n \to \infty$. □

Recently, in [5] Ofoedu proved the following theorem.

1.5. Theorem. (Ofoedu [5]) *Let E be a real Banach space, K a nonempty closed convex subset of E, $T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudo-contractive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that $x^* \in F(T)$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$;
(iii) $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$.

For a given $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \ \ \forall n \geq 0.$$

If there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$, such that

$$\langle T^n x - x^*, \ j(x - x^*)\rangle \leq k_n ||x - x^*||^2 - \phi(||x - x^*||), \ \forall\, x \in K,$$

then $\{x_n\}$ converges strongly to x^*. □

1.6. Remark. It should be pointed out that although Theorem 1.5 extends Theorem 1.4 from real uniformly smooth Banach spaces to arbitrary real Banach spaces and removes the boundedness condition imposed on K, we have the following questions:

1.7. Question. Can Theorems 1.3–1.5 be extended from one L-Lipschitzian mapping to a finite family of uniformly L-Lipschitzian mappings?

1.8. Question. What happens if algorithm (1.1) is replaced by the parallel iterative algorithm?

1.9. Question. What happens if algorithm (1.1) is replaced by the viscosity iterative algorithm?

The purpose of this paper is to give affirmative answers to the questions mentioned above. We introduce a new approximation scheme combining the viscosity method with the parallel method for finding a common fixed point of a family of finitely uniformly L-Lipschitzian mappings in a Banach space. We use a simple and quite different method to prove the same conclusion as given in Theorem 1.5 without the assumption that T is an asymptotically pseudo-contractive mapping. Based on this result, we also get some new and interesting results. The results in this paper extend and improve some well-known results in the literature.

For this purpose we first give the following lemmas which will be needed in proving our main results.

1.10. Lemma. (Chang [2]) *Let E be a real Banach space and $J : E \to 2^{E^*}$ the normalized duality mapping. Then for any $x, y \in E$ we have*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, \ j(x + y)\rangle, \ \forall\, j(x + y) \in J(x + y).$$ □

1.11. Lemma. (Moore and Nnoli [4]) *Let $\{\theta_n\}$ be a sequence of nonnegative real numbers and $\{\lambda_n\}$ a real sequence satisfying the following conditions:*

$$0 \leq \lambda_n \leq 1, \ \sum_{n=0}^{\infty} \lambda_n = \infty.$$

If there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ such that

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n \ \forall\, n \geq n_0,$$

where n_0 is some nonnegative integer and $\{\sigma_n\}$ is a sequence of nonnegative numbers such that $\sigma_n = 0(\lambda_n)$. Then $\theta_n \to 0$ (as $n \to \infty$). □

1.12. Lemma. (Tan and Xu [7]) *Let $\{a_n\}, \{b_n\}$, be two nonnegative real sequences satisfying:*

$$a_{n+1} \le (1+\lambda_n)a_n + b_n,$$

where $\{\lambda_n\}$ is a sequence in $(0,1)$ with $\sum_{n=0}^{\infty} \lambda_n < \infty$. If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. □

2. Main results

In this section, we shall prove our main theorems.

2.1. Theorem. *Let E be a real Banach space, K a nonempty closed convex subset of E, $T_i : K \to K$, $i = 1, 2, \dots, m$ be m uniformly L_i-Lipschitzian mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, where $F(T_i)$ is the set of fixed points of T_i in K and x^* be a given point in F. Let $\{k_n\} \subset [1, \infty)$ be a sequence with $k_n \to 1$.*

Let f be a contraction of K into itself, $\{u_n\}$ a bounded sequence in K, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ four sequences in [0, 1] satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1,\ \forall\, n \ge 0$;
(ii) $\sum_{n=0}^{\infty} \gamma_n^2 < \infty$;
(iii) $\sum_{n=0}^{\infty} \alpha_n < \infty$ *and* $\alpha_n = o(\gamma_n)$;
(iv) $\sum_{n=0}^{\infty} \delta_n < \infty$ *and* $\delta_n = o(\gamma_n)$;
(v) $\sum_{n=0}^{\infty} \gamma_n(k_n - 1) < \infty$.

For any given $x_1 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{m} t_i T_i^n x_n + \delta_n u_n,\ \forall\ n \ge 0, \tag{2.1}$$

where $\{t_i\}_{i=1}^{m}$ is a finite sequence of positive number such that $\sum_{i=1}^{m} t_i = 1$. If there exists a strict increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ such that

$$\langle T_i^n x - x^*,\ j(x - x^*)\rangle \le k_n ||x - x^*||^2 - \phi(||x - x^*||)$$

for all $j(x - x^*) \in J(x - x^*)$ and $x \in K$, $i = 1, 2, \dots, m$, then $\{x_n\}$ converges strongly to x^*.

Proof. The proof is divided into two steps.

(I) Denote $L = \max\{L_1, L_2, \dots, L_N\}$ and $M = \max\{\sup_{n\ge1}\{\|u_n - x^*||, ||f(x^*) - x^*||\}$. First, we prove that the sequence $\{x_n\}$ defined by (2.1) is bounded.

In fact, since f be a contraction of K into itself, there exists $\alpha \in [0, 1)$ such that

$$||f(x) - f(y)|| \le \alpha ||x - y||$$

for all $x, y \in K$. It follows from (2.1) and Lemma 1.10 that

$$\begin{aligned}
&||x_{n+1} - x^*||^2 \\
&\quad = \left\| \alpha_n (f(x_n) - x^*) + \beta_n (x_n - x^*) + \gamma_n \left(\sum_{i=1}^{m} t_i T_i^n x_n - \sum_{i=1}^{m} t_i x^* \right) + \delta_n (u_n - x^*) \right\|^2 \\
&\quad \le \beta_n^2 ||x_n - x^*||^2 + 2\alpha_n \big\langle f(x_n) - x^*,\ j(x_{n+1} - x^*)\big\rangle \\
&\qquad + 2\gamma_n \sum_{i=1}^{m} t_i \big\langle T_i^n x_n - x^*,\ j(x_{n+1} - x^*)\big\rangle + 2\delta_n \big\langle u_n - x^*,\ j(x_{n+1} - x^*)\big\rangle
\end{aligned}$$

$$
\begin{aligned}
&\le \beta_n^2||x_n - x^*||^2 + 2\alpha_n\langle f(x_n) - f(x^*),\ j(x_{n+1} - x^*)\rangle \\
&\qquad + 2\alpha_n\langle f(x^*) - x^*,\ j(x_{n+1} - x^*)\rangle \\
&\qquad\qquad + 2\gamma_n \sum_{i=1}^{m} t_i\big\langle T_i^n x_{n+1} - x^*,\ j(x_{n+1} - x^*)\big\rangle \\
&\qquad\qquad\qquad + 2\gamma_n \sum_{i=1}^{m} t_i\big\langle T_i^n x_n - T_i^n x_{n+1}, j(x_{n+1} - x^*)\big\rangle \\
&\qquad\qquad\qquad\qquad + 2\delta_n||u_n - x^*|| \cdot ||x_{n+1} - x^*|| \\
&\le \beta_n^2||x_n - x^*||^2 + 2\alpha_n\alpha||x_n - x^*|| \cdot ||x_{n+1} - x^*|| \\
&\qquad + 2\alpha_n||f(x^*) - x^*|| \cdot ||x_{n+1} - x^*|| \\
&\qquad\qquad + 2\gamma_n\big\{k_n||x_{n+1} - x^*||^2 - \phi(||x_{n+1} - x^*||)\big\} \\
&\qquad\qquad\qquad + 2\gamma_n \sum_{i=1}^{m} t_i\big|\big|T_i^n x_n - T_i^n x_{n+1}\big|\big| \cdot ||x_{n+1} - x^*|| \\
&\qquad\qquad\qquad\qquad + 2\delta_n M||x_{n+1} - x^*|| \\
&\le (1-\gamma_n)^2||x_n - x^*||^2 + 2\alpha_n\alpha||x_n - x^*|| \cdot ||x_{n+1} - x^*|| \\
&\qquad + 2\alpha_n M \cdot ||x_{n+1} - x^*|| + 2\gamma_n\big\{k_n||x_{n+1} - x^*||^2 - \phi(||x_{n+1} - x^*||)\big\} \\
&\qquad\qquad + 2\gamma_n L||x_n - x_{n+1}|| \cdot ||x_{n+1} - x^*|| + 2\delta_n M||x_{n+1} - x^*||.
\end{aligned} \tag{2.2}
$$

Since

$$
\begin{aligned}
&||x_{n+1} - x_n|| \\
&= \left\|\alpha_n(f(x_n) - x_n) + \gamma_n\left(\sum_{i=1}^{m} t_i T_i^n x_n - \sum_{i=1}^{m} t_i x_n\right) + \delta_n(u_n - x_n)\right\| \\
&\le \alpha_n||f(x_n) - f(x^*) + f(x^*) - x^* + x^* - x_n|| \\
&\qquad + \gamma_n \sum_{i=1}^{m} t_i||T_i^n x_n - x^* + x^* - x_n|| + \delta_n||u_n - x^* + x^* - x_n|| \\
&\le \alpha_n(\alpha+1)||x_n - x^*|| + \alpha_n||f(x^*) - x^*|| \\
&\qquad + \gamma_n \sum_{i=1}^{m} t_i(||T_i^n x_n - x^*|| + ||x_n - x^*||) + \delta_n||u_n - x^*|| + \delta_n||x_n - x^*|| \\
&\le \{\alpha_n(\alpha+1) + \gamma_n(L+1) + \delta_n\}||x_n - x^*|| + \alpha_n M + \delta_n M \\
&\le (1 + \alpha_n\alpha + \gamma_n L)||x_n - x^*|| + \alpha_n M + \delta_n M
\end{aligned} \tag{2.3}
$$

Substituting (2.3) into (2.2) we have

$$
\begin{aligned}
&||x_{n+1} - x^*||^2 \\
&\le (1-\gamma_n)^2||x_n - x^*||^2 + 2\gamma_n k_n||x_{n+1} - x^*||^2 - 2\gamma_n\phi(||x_{n+1} - x^*||) \\
&\qquad + 2\{\alpha_n\alpha + \gamma_n L(\alpha_n(\alpha+1) + \gamma_n(L+1) + \delta_n)\}||x_n - x^*|| \cdot ||x_{n+1} - x^*|| \\
&\qquad\qquad + 2(\alpha_n + \delta_n)(1 + \gamma_n L)M \cdot ||x_{n+1} - x^*|| \\
&\le (1-\gamma_n)^2||x_n - x^*||^2 + 2\gamma_n k_n||x_{n+1} - x^*||^2 - 2\gamma_n\phi(||x_{n+1} - x^*||) \\
&\qquad + \{\alpha_n\alpha + \gamma_n L(\alpha_n(\alpha+1) + \gamma_n(L+1) + \delta_n)\}\big\{||x_n - x^*||^2 \\
&\qquad\qquad + ||x_{n+1} - x^*||^2\big\} + (\alpha_n + \delta_n)(1 + \gamma_n L)\big(M^2 + ||x_{n+1} - x^*||^2\big).
\end{aligned} \tag{2.4}
$$

Simplifying this, we have

$$\begin{aligned}
&||x_{n+1}-x^*||^2\\
&\le \frac{A_n}{B_n}||x_n-x^*||^2-\frac{2\gamma_n}{B_n}\phi(||x_{n+1}-x^*||)+\frac{(\alpha_n+\delta_n)(1+\gamma_n L)M^2}{B_n}\\
&=\Big\{1+\frac{2\gamma_n(k_n-1)+\gamma_n^2+2\{\alpha_n\alpha+\gamma_n L(\alpha_n(\alpha+1)+\gamma_n(L+1)+\delta_n)\}}{B_n}\\
&\qquad+\frac{(\alpha_n+\delta_n)(1+\gamma_n L)}{B_n}\Big\}||x_n-x^*||^2\\
&\qquad-\frac{2\gamma_n}{B_n}\phi(||x_{n+1}-x^*||)+\frac{(\alpha_n+\delta_n)(1+\gamma_n L)M^2}{B_n}.
\end{aligned}\tag{2.5}$$

where

$$\begin{aligned}
A_n&=(1-\gamma_n)^2+\{\alpha_n\alpha+\gamma_n L(\alpha_n(\alpha+1)+\gamma_n(L+1)+\delta_n)\},\\
B_n&=1-2\gamma_n k_n-\{\alpha_n\alpha+\gamma_n L(\alpha_n(\alpha+1)+\gamma_n(L+1)+\delta_n)\}\\
&\qquad-(\alpha_n+\delta_n)(1+\gamma_n L).
\end{aligned}$$

Since $\alpha_n\to 0,\ \gamma_n\to 0,\ \delta_n\to 0,\ k_n\to 1\ (as\ n\to\infty)$, there exists a positive integer n_0 such that $\frac{1}{2}<B_n\le 1, \forall\, n\ge n_0$. Therefore it follows from (2.5) that

$$\begin{aligned}
&||x_{n+1}-x^*||^2\\
&\le \big\{1+2[2\gamma_n(k_n-1)+\gamma_n^2+2(\alpha_n\alpha+\gamma_n L(\alpha_n(\alpha+1)+\gamma_n(L+1)+\delta_n))\\
&\qquad+(\alpha_n+\delta_n)(1+\gamma_n L)]\big\}||x_n-x^*||^2-2\gamma_n\phi(||x_{n+1}-x^*||)\\
&\qquad+2(\alpha_n+\delta_n)(1+\gamma_n L)M^2,\ \forall\, n\ge n_0.
\end{aligned}\tag{2.6}$$

Therefore we have

$$\begin{aligned}
&||x_{n+1}-x^*||^2\\
&\le \big\{1+2[2\gamma_n(k_n-1)+\gamma_n^2+2(\alpha_n\alpha+\gamma_n L(\alpha_n(\alpha+1)+\gamma_n(L+1)+\delta_n))\\
&\qquad+(\alpha_n+\delta_n)(1+\gamma_n L)]\big\}||x_n-x^*||^2+2(\alpha_n+\delta_n)(1+\gamma_n L)M^2,\ \forall\, n\ge n_0.
\end{aligned}\tag{2.7}$$

By conditions (ii)–(v), we have

$$\begin{aligned}
2\sum_{n=0}^{\infty}\big\{&\gamma_n(k_n-1)+\gamma_n^2+2\big(\alpha_n\alpha+\gamma_n L(\alpha_n(\alpha+1)+\gamma_n(L+1)+\delta_n)\big)\\
&+(\alpha_n+\delta_n)(1+\gamma_n L)\big\}<\infty,
\end{aligned}$$

and

$$2\sum_{n=0}^{\infty}\big\{(\alpha_n+\delta_n)(1+\gamma_n L)M^2\big\}<\infty.$$

It follows from Lemma 1.12 that the limit $\lim_{n\to\infty}||x_n-x^*||$ exists. Therefore the sequence $\{||x_n-x^*||\}$ is bounded. Without loss of generality, we can assume that

$$||x_n-x^*||^2\le M^*,$$

where M^* is some positive constant.

(II) Now we consider (2.6) and prove that $x_n\to x^*$ (as $n\to\infty$).

Taking $\theta_n=||x_n-x^*||$, $\lambda_n=2\gamma_n$ and

$$\begin{aligned}
\sigma_n&=2[2\gamma_n(k_n-1)+\gamma_n^2+2(\alpha_n\alpha+\gamma_n L(\alpha_n(\alpha+1)+\gamma_n(L+1)+\delta_n))\\
&\quad+(\alpha_n+\delta_n)(1+\gamma_n L)]M^*+2(\alpha_n+\delta_n)(1+\gamma_n L)M^2,
\end{aligned}$$

inequality (2.6) can be written as:

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n.$$

By conditions (i)–(v) we know that all the conditions in Lemma 1.11 are satisfied. Therefore

$$||x_n - x^*|| \to 0 \ \ (as\ n \to \infty),$$

that is, $x_n \to x^*$ (as $n \to \infty$). This completes the proof of Theorem 2.1. □

2.2. Remark. (1) Theorem 2.1 extends and improves the corresponding results in Chang [1], Ofoedu [5], Schu [6] and Zeng [8, 9];

(2) The method used in the proof of Theorem 2.1 is quite different the method given in Ofoedu [5].

(3) Theorem 2.1 extends and improves Theorem 3.1 of Ofoedu [5], it abolishes the assumption that T is an asymptotically pseudo-contractive mapping.

2.3. Remark. If $\alpha_n = 0\ \forall\, n \geq 1$ in Theorem 2.1, then we can obtain corresponding results for the parallel iterative algorithm, but these are omitted here.

The following theorem can be immediately obtained from Theorem 2.1:

2.4. Theorem. *Let E be a real Banach space, K a nonempty closed convex subset of E, $T : K \to K$ a uniformly L-Lipschitzian mapping with $F(T) \neq \emptyset$, where $F(T)$ is the set of fixed points of T in K, and x^* is a point in $F(T)$. Let $\{k_n\} \subset [1,\ \infty)$ be a sequence with $k_n \to 1$. Let f be a contraction of K into itself, $\{u_n\}$ a bounded sequence in K, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ four sequences in [0, 1] satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1,\ \forall\, n \geq 0$;
(ii) $\sum_{n=0}^{\infty} \gamma_n^2 < \infty$;
(iii) $\sum_{n=0}^{\infty} \alpha_n < \infty$ *and* $\alpha_n = o(\gamma_n)$;
(iv) $\sum_{n=0}^{\infty} \delta_n < \infty$ *and* $\delta_n = o(\gamma_n)$;
(v) $\sum_{n=0}^{\infty} \gamma_n (k_n - 1) < \infty$.

For any given $x_1 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n x_n + \delta_n u_n,\ \forall\, n \geq 0. \tag{2.8}$$

If there exists a strictly increasing function $\phi : [0,\ \infty) \to [0,\ \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - x^*,\ j(x - x^*)\rangle \leq k_n ||x - x^*||^2 - \phi(||x - x^*||)$$

for all $j(x - x^*) \in J(x - x^*)$ and $x \in K$, then $\{x_n\}$ converges strongly to x^*. □

2.5. Remark. In Theorem 2.4 it is not assumed that T is an asymptotically pseudo-contractive mapping. Thus, Theorem 2.4 is a generalization and improvement of Ofoedu [5, Theorem 3.2].

2.6. Remark. If $\delta_n = 0,\ \forall\, n \geq 1$ in Theorems 2.1 and 2.4, then we can obtain corresponding results for the viscosity iterative process without errors, but this is omitted here.

References

[1] Chang, S. S. *Some results for asymptotically pseudo-contractive mappings and asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **129**, 845–853, 2001.

[2] Chang, S. S. *On Chidume's open questions and approximation solutions of multi-valued strongly accretive mapping equation in Banach spaces*, J. Math. Anal. Appl. **216**, 94–111, 1997.

[3] Gobel, K and Kirk, W. A. *A fixed points theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35**, 171–174, 1972.

[4] Moore, C. and Nnoli, B. V. *Iterative solution of nonlinear equations involving set-valued uniformly accretive operators*, Computers Math. Applic. **42**, 131–140, 2001.

[5] Ofoedu, E. U. *Strong convergence theorem for uniformly L-Lipschitzian asymptotically pseudocontractive mapping in a real Banach space*, J. Math. Anal. Appl. **321**, 722–728, 2006.

[6] Schu, J. *Iterative construction of fixed point of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **158**, 407–413, 1991.

[7] Tan, K. K. and Xu, H. K. *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. **178**,301–308, 1993.

[8] Zeng, L. C. *On the iterative approximation for asymptotically pseudo-contractive mappings in uniformly smooth Banach spaces* (in Chinese), Chinese Math. Ann. **26A:2**, 283–290, 2005.

[9] Zeng, L. C. *On the approximation of fixed points for asymptotically nonexpansive mappings in Banach spaces* (in Chinese), Acta Math. Sci. **23A:1**, 31–37, 2003.