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A NEW EXAMPLE OF STRONGLY π-INVERSE MONOIDS

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Abstract

In [1], Ateş defined the semidirect product version of the Schützenberger product for any two monoids, and examined its regularity. Since this is a new product and there are so many algebraic properties that need to be checked for it, in this paper we determine necessary and sufficient conditions for this new version to be strongly π-inverse, and then give some results.

Keywords: Schützenberger product, π-inverse monoid, Regular monoid.

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1. Introduction

The semidirect product has a venerable history in semigroup theory. It has played a central role not only in the decomposition theory of finite semigroups but also in many algebraic and geometric properties (see, for example, in [2]). It is therefore of interest to improve this product for some other constructions. In this direction one step has been taken by Ateş ([1]). In [1], the author defined a new monoid construction under the semidirect product (which is called the semidirect product version of the Schützenberger product) and then gave necessary and sufficient conditions for this product to be regular. In [7], the author determined necessary and sufficient conditions for the semidirect product of two monoids to be regular. (As far as we can see, one of the starting points for the paper [1] is the main result of the paper [7]). After that work, in [8], the author investigated inverse and orthodox properties of semidirect and wreath products of monoids (see [6] for details of wreath products). Then, in [9], the authors determined necessary and

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sufficient conditions for the semidirect and wreath products of two monoids to be strongly π-inverse. So as a next step of [1], in this paper, by combining some material from [1] and [9], we study the property of being strongly π-inverse for the semidirect product version of the Schützenberger product of any two monoids, and give some results.

1.1. Definition. Let A, B be monoids and θ a monoid homomorphism

$$\theta : B \to \mathrm{End}(A), \quad b \mapsto \theta_b, \quad 1 \mapsto id_{\mathrm{End}(A)},$$

where End(A) denotes the collection of endomorphisms of A, which is itself a monoid with identity $id : A \mapsto A$. Then the *semidirect product* of these two monoids, denoted by $A \rtimes_\theta B$, is the set of ordered pairs (a, b) (where $a \in A$ and $b \in B$) with the multiplication given by

$$(a_1, b_1)(a_2, b_2) = (a_1(a_2)\theta_{b_1}, b_1b_2).$$

We note that, for every $a \in A$ and $b_1, b_2 \in B$,

$$(a)\theta_{b_1b_2} = ((a)\theta_{b_2})\theta_{b_1}, \tag{1.1}$$

and the monoids A and B are identified with the submonoids of $A \rtimes_\theta B$ having elements $(a, 1_B)$ and $(1_A, b)$.

1.2. Definition. For a subset P of $A \times B$ and for $a \in A$, $b \in B$, we let define

$$Pb = \{(c, db); (c, d) \in P\} \quad \text{and} \quad aP = \{(ac, d); (c, d) \in P\}.$$

Then the *Schützenberger product* of the monoids A and B, denoted by $A \diamond B$, is the set $A \times \wp(A \times B) \times B$, (where $\wp(\cdot)$ denotes the power set) with the multiplication given by

$$(a_1, P_1, b_1)(a_2, P_2, b_2) = (a_1a_2, P_1b_2 \cup a_1P_2, b_1b_2).$$

$A \diamond B$ is a monoid with identity $(1_A, \emptyset, 1_B)$ ([5]).

By considering the above two definitions, the following definition has recently been given in [1].

1.3. Definition. [1] The semidirect product version of the Schützenberger product of the monoids A by B, denoted by $A \diamond_{sv} B$, is the set $A \times \wp(A \times B) \times B$ with the multiplication

$$(a_1, P_1, b_1)(a_2, P_2, b_2) = (a_1(a_2)\theta_{b_1}, P_1b_2 \cup P_2, b_1b_2). \tag{1.2}$$

$A \diamond_{sv} B$ is a monoid with identity $(1_A, \emptyset, 1_B)$, where the homomorphism θ is defined as in the semidirect product construction.

Now let us recall the following material that will be needed for this paper. The reader is referred to [3, 4] for more details on this material.

For a monoid M, the set of inverses for an element $a \in M$ is defined by

$$a^{-1} = \{b \in M : aba = a \text{ and } bab = b\}.$$

Then M is called a *regular monoid* if and only if the set $a^{-1} \neq \emptyset$ for all $a \in M$. To have an inverse element can also be important in a semigroup. Therefore, for a semigroup S, we call S an *inverse semigroup* if every element has exactly one inverse. The well known examples of inverse semigroups are groups and semilattices. In addition, let $E(S)$ and Reg$S$ be the set of *idempotent* and *regular* elements, respectively, for a semigroup S. Here, S is called π-*regular* if, for every $s \in S$, there is an $m \in \mathbb{N}$ such that $s^m \in$ Reg$S$. Moreover, if S is π-regular and the set $E(S)$ is a commutative subsemigroup of S, then S is called a *strongly π-inverse semigroup*. We recall that Reg$S$ is an inverse subsemigroup of a strongly π-inverse semigroup S.

2. Main Results

In the paper [9, Lemma 1], the authors determined necessary conditions for the semidirect product of two monoids to be strongly π-inverse. Using a similar idea as with this lemma, we present the following preliminary result as a lemma which actually constructs the necessity part of the main result (see Theorem 2.2 below).

2.1. Lemma. *Let $A \diamond_{sv} B$ be a strongly π-inverse monoid. Then we have the following:*

(i) *Both A and B are strongly π-inverse monoids.*
(ii) *For every $e \in E(A)$ and $f \in E(B)$, $(e)\theta_f = e$.*
(iii) *For $a \in A$ and $f \in E(B)$, if $a(a)\theta_f = a$ then $(a)\theta_f = a$.*
(iv) *For every $a \in RegA$ and $f \in E(B)$, $(a)\theta_f = a$.*
(v) *For every $a \in A$ and $b \in B$, there exists $m \in \mathbb{N}$ such that $b^m \in RegB$ and $a^{(m)} \in RegA$, where $a^{(m)} = a\,((a)\theta_b)\,((a)\theta_{b^2})\,\cdots\,((a)\theta_{b^{m-1}})$.*
(vi) *For every $(a, P, b) \in A \diamond_{sv} B$, there exist $P_1 \subseteq A \times B$ and $m \in \mathbb{N}$ such that either*

$$P = P_1 b = \bigcup_{(a_1,b_1)\in P_1} \{(a_1, b_1 b)\} \text{ or } P = P_1 b^2 = \bigcup_{(a_1,b_1)\in P_1} \{(a_1, b_1 b^2)\}$$
$$\text{or } \cdots \text{ or } P = P_1 b^{m-1} \text{ or } P = P_1 b^m.$$

Proof. (i) For arbitrary $a \in A$ and for some $P_1 \subseteq A \times B$, there exist $m \in \mathbb{N}$ and $(a_1, P_1, b_1) \in A \diamond_{sv} B$ such that

$$(a, \emptyset, 1_B)^m (a_1, P_1, b_1)(a, \emptyset, 1_B)^m = (a, \emptyset, 1_B)^m.$$

By applying (1.2), we obtain $(a^m a_1 (a^m)\theta_{b_1}, P_1, b_1) = (a^m, \emptyset, 1_B)$. Then we get $b_1 = 1_B$, and so $a^m a_1 (a^m)\theta_{b_1} = a^m a_1 a^m$. Therefore $a^m a_1 a^m = a^m$, so A is π-regular. Also, for $e, f \in E(A)$, since $(e, \emptyset, 1_B)$, $(f, \emptyset, 1_B) \in E(A \diamond_{sv} B)$, we have $(e, \emptyset, 1_B)(f, \emptyset, 1_B) = (f, \emptyset, 1_B)(e, \emptyset, 1_B)$, and so $ef = fe$. Hence A is a strongly π-inverse monoid.

Similarly, for arbitrary $b \in B$ and for some $P_2 \subseteq A \times B$, there exist $m \in \mathbb{N}$ and $(a_2, P_2, b_2) \in A \diamond_{sv} B$ such that

$$(1_A, \emptyset, b)^m (a_2, P_2, b_2)(1_A, \emptyset, b)^m = (1_A, \emptyset, b)^m,$$

that is, $(1_A, \emptyset, b^m)(a_2, P_2, b_2)(1_A, \emptyset, b^m) = (1_A, \emptyset, b^m)$ which implies $b^m b_2 b^m = b^m$. Thus B is π-regular. In addition, for $g, h \in E(B)$, since $(1_A, \emptyset, g)$, $(1_A, \emptyset, h) \in E(A \diamond_{sv} B)$ and $A \diamond_{sv} B$ is a strongly π-inverse monoid, we have $(1_A, \emptyset, g)(1_A, \emptyset, h) = (1_A, \emptyset, h)(1_A, \emptyset, g)$, and so $(1_A, \emptyset, gh) = (1_A, \emptyset, hg)$. Hence, we get $gh = hg$ which implies that B is a strongly π-inverse monoid.

(ii) Let $e \in E(A)$ and $f \in E(B)$. Then $(e, \emptyset, 1_B)$, $(1_A, \emptyset, f) \in E(A \diamond_{sv} B)$ and

$$(e, \emptyset, 1_B)(1_A, \emptyset, f) = (1_A, \emptyset, f)(e, \emptyset, 1_B),$$

which implies $(e(1_A)\theta_{1_B}, \emptyset, f) = (1_A(e)\theta_f, \emptyset, f)$. Thus $(e, \emptyset, f) = ((e)\theta_f, \emptyset, f)$, and so $(e)\theta_f = e$.

(iii) If $a((a)\theta_f) = a$, then $(a, \emptyset, f) \in E(A \diamond_{sv} B)$ and $(a, \emptyset, f)(1_A, \emptyset, f) = (1_A, \emptyset, f)(a, \emptyset, f)$ since $(1_A, \emptyset, f) \in E(A \diamond_{sv} B)$ and $A \diamond_{sv} B$ is strongly π-inverse. Therefore we obtain $(a)\theta_f = a$.

(iv) From (i), for every $a \in \mathrm{Reg}A$, there exists a unique $a_1 \in A$ (and a unique $(a_1)\theta_f \in A$) such that

$$a((a_1)\theta_f)a = a \text{ and } ((a_1)\theta_f)a((a_1)\theta_f) = (a_1)\theta_f.$$

Then $(a((a_1)\theta_f)a)\theta_f = (a)\theta_f$. Further, $((a)\theta_f)(((a_1)\theta_f)\theta_f)((a)\theta_f) = (a)\theta_f$ and so $((a)\theta_f)((a_1)\theta_{f^2})((a)\theta_f) = (a)\theta_f$. Since $f \in E(B)$, we obtain

$$((a)\theta_f)((a_1)\theta_f)((a)\theta_f) = (a)\theta_f.$$

Additionally, for every $f \in E(B)$, since $a((a_1)\theta_f) \in E(A)$, we have

$$(a(\underbrace{(a_1)\theta_f}_{a_1}))\theta_f = a((a_1)\theta_f) = (aa_1)\theta_f$$

by (ii), and then we get

$$\begin{aligned}(a_1)\theta_f a(a_1)\theta_f &= (a_1)\theta_f(aa_1)\theta_f \\ &= (a_1)\theta_f(a)\theta_f(a_1)\theta_f = (a_1)\theta_f.\end{aligned}$$

Hence both a and $(a)\theta_f$ are inverses of $(a_1)\theta_f$, and so $(a)\theta_f = a$.

(v) Since $A \diamond_{sv} B$ is a strongly π-inverse monoid, for every $(a,P,b) \in A \diamond_{sv} B$ there exist $m \in \mathbb{N}$ and $(a_1,P_1,b_1) \in A \diamond_{sv} B$ such that

$$(a,P,b)^m(a_1,P_1,b_1)(a,P,b)^m = (a,P,b)^m.$$

Then, by (1.2), we get

$$\begin{aligned}\big(a((a)\theta_b)((a)\theta_{b^2})\cdots((a)\theta_{b^{m-1}}), Pb^{m-1} \cup \cdots \cup Pb \cup P, b^m\big)\,(a_1,P_1,b_1)(a,P,b)^m \\ = (a,P,b)^m.\end{aligned}$$

By processing the left hand side, we have

$$\begin{aligned}&\big(a((a)\theta_b)((a)\theta_{b^2})\cdots((a)\theta_{b^{m-1}})((a_1)\theta_{b^m}), \\ &\qquad (Pb^{m-1} \cup \cdots \cup Pb \cup P)b_1 \cup P_1, b^m b_1\big)(a,P,b)^m = (a,P,b)^m,\end{aligned}$$

and then, by iterating this process, we obtain

$$\begin{aligned}&\big(a((a)\theta_b)((a)\theta_{b^2})\cdots((a)\theta_{b^{m-1}})((a_1)\theta_{b^m})(a((a)\theta_b)((a)\theta_{b^2})\cdots((a)\theta_{b^{m-1}}))\theta_{b^m b_1}, \\ &\quad ((Pb^{m-1} \cup \cdots \cup Pb \cup P)b_1 \cup P_1)b^m \cup (Pb^{m-1} \cup \cdots \cup Pb \cup P),\ b^m b_1 b^m\big) \\ &\qquad = \big(a((a)\theta_b)((a)\theta_{b^2})\cdots((a)\theta_{b^{m-1}}), Pb^{m-1} \cup \cdots \cup Pb \cup P, b^m\big).\end{aligned} \tag{2.1}$$

So, $b^m b_1 b^m = b^m$ and thus $b^m \in \mathrm{Reg}B$. First components of the equality in (2.1) give that

$$\begin{aligned}a((a)\theta_b)((a)\theta_{b^2})\cdots((a)\theta_{b^{m-1}})((a_1)\theta_{b^m})\big(a((a)\theta_b)((a)\theta_{b^2})\cdots((a)\theta_{b^{m-1}})\big)\theta_{b^m b_1} \\ = a((a)\theta_b)((a)\theta_{b^2})\cdots((a)\theta_{b^{m-1}}).\end{aligned}$$

For simplicity, let us label $a((a)\theta_b)((a)\theta_{b^2})\cdots((a)\theta_{b^{m-1}})$ by $a^{(m)}$. Thus we have

$$a^{(m)}((a_1)\theta_{b^m})((a^{(m)})\theta_{b^m b_1}) = a^{(m)}.$$

By (iii) we get $(a^{(m)})\theta_{b^m b_1} = a^{(m)}$, since $b^m b_1 \in E(B)$. Indeed

$$(b^m b_1)^2 = \underbrace{b^m b_1 b^m}_{b^m} b_1 = b^m b_1.$$

Therefore

$$a^{(m)}((a_1)\theta_{b^m})a^{(m)} = a^{(m)},$$

which gives $a^{(m)} \in \mathrm{Reg}A$, as required.

(vi) By Equality (2.1), we have

$$((Pb^{m-1} \cup \cdots \cup Pb \cup P)b_1 \cup P_1)b^m \cup (Pb^{m-1} \cup \cdots \cup Pb \cup P) = Pb^{m-1} \cup \cdots \cup Pb \cup P$$

and then

$$\begin{aligned}(Pb^{m-1}b_1b^m \cup \cdots \cup Pbb_1b^m \cup Pb_1b^m \cup P_1b^m) \cup (Pb^{m-1} \cup \cdots \cup Pb \cup P) \\ = Pb^{m-1} \cup \cdots \cup Pb \cup P.\end{aligned}$$

Moreover, again by (2.1), since $b^m b_1 b^m = b^m$, for every $(a, P, b) \in A \diamond_{sv} B$, either $P = P_1 b$ or $P = P_1 b^2$ or $\cdots$ or $P = P_1 b^{m-1}$ or $P = P_1 b^m$, where $P_1 \subseteq A \times B$. Otherwise, for $P' = Pb^{m-1} \cup \cdots \cup Pb \cup P$, there would not be an equality between P' and $(P' b_1 \cup P_1) b^m \cup P'$, which gives a contradiction with $A \diamond_{sv} B$ being a strongly π-inverse monoid.

The above arguments complete the proof of the lemma. □

The main result of this paper is the following.

2.2. Theorem. *Let A and B be any two monoids. Then $A \diamond_{sv} B$ is a strongly π-inverse monoid if and only if*

(i) *Both A and B are strongly π-inverse monoids.*
(ii) *For every $a \in RegA$ and $f \in E(B)$, $(a)\theta_f = a$.*
(iii) *For every $a \in A$ and $b \in B$, there exists $m \in \mathbb{N}$ such that $b^m \in RegB$ and $a^{(m)} \in RegA$, where $a^{(m)} = a((a)\theta_b)((a)\theta_{b^2}) \cdots ((a)\theta_{b^{m-1}})$.*
(iv) *For every $(a, P, b) \in A \diamond_{sv} B$, either*

$$P = P_1 b = \bigcup_{(a_1, b_1) \in P_1} \{(a_1, b_1 b)\} \textit{ or } P = P_1 b^2 = \bigcup_{(a_1, b_1) \in P_1} \{(a_1, b_1 b^2)\}$$
$$\textit{or } \cdots \textit{ or } P = P_1 b^{m-1} \textit{ or } P = P_1 b^m,$$

where $P_1 \subseteq A \times B$.

Proof. Necessity is obvious by Lemma 2.1, so it just remains to prove sufficiency. Therefore, let us suppose that the conditions (i)–(iv) of the theorem hold.

By (iii), for every $(a, P, b) \in A \diamond_{sv} B$, there exist $m \in \mathbb{N}$, $a_1 \in A$ and $b_1 \in B$ such that

$$b^m b_1 b^m = b^m \text{ and } a^{(m)} a_1 a^{(m)} = a^{(m)}.$$

Also, by (ii), we have $a^{(m)} (a_1)\theta_{b^m} a^{(m)} = a^{(m)}$, for $b^m b_1 \in E(B)$. By using (iv), we further obtain the equality

$$(a, P, b)^m (a_1, P_1, b_1)(a, P, b)^m = (a, P, b)^m,$$

which gives the π-regularity of $A \diamond_{sv} B$.

Now we need to show that $E(A \diamond_{sv} B)$ is commutative. Firstly, for arbitrary $(e, P, f) \in E(A \diamond_{sv} B)$, we must prove that $e \in E(A)$ and $f \in E(B)$. In fact, if $(e, P, f)^2 = (e, P, f)$, then $f^2 = f$, $Pf \cup P = P$ and $e(e)\theta_f = e$. Thus $(e)\theta_f \in E(A)$ (since $f^2 = f$ and so $f \in E(B)$), and then by (iii) there exists $m \in \mathbb{N}$ such that

$$\underbrace{e((e)\theta_f)}_{e}((e)\theta_{f^2}) \cdots ((e)\theta_{f^{m-1}}) = e^{(m)} \in \mathrm{Reg}A$$
$$\Longrightarrow \underbrace{e((e)\theta_{f^2})}_{e} \cdots ((e)\theta_{f^{m-1}}) = e^{(m)} \in \mathrm{Reg}A$$
$$\Longrightarrow \cdots \Longrightarrow e((e)\theta_{f^{m-1}}) = e \in \mathrm{Reg}A.$$

By considering (ii), for every $e \in RegA$ and $f \in E(B)$ we have $(e)\theta_f = e$. If we multiply both sides of this equation by $e \in A$, then we obtain $\underbrace{e((e)\theta_f)}_{e} = e^2$. Hence $e^2 = e$, which means that $e \in E(A)$.

Now, for (e,P,f), $(e_1,P_1,f_1) \in E(A \diamond_{sv} B)$, we have $e, e_1 \in E(A)$ and $f, f_1 \in E(B)$. Actually, by (i) and (ii), we get

$$\begin{aligned}(e,P,f)(e_1,P_1,f_1) &= (e\underbrace{((e_1)\theta_f)}_{e_1}, Pf_1 \cup P_1, ff_1)\\ &= (ee_1, \{\text{Case}\,1, \text{Case}\,2\}, ff_1) \text{ (see below for Cases 1 and 2)}\\ &= (e_1 e, \text{Case}\,1 \text{ - Case}\,2, f_1 f)\\ &= (e_1((e)\theta_{f_1}), P_1 f \cup P, f_1 f) = (e_1, P_1, f_1)(e, P, f).\end{aligned}$$

Case 1: If $f = f_1$, then $Pf_1 \cup P_1 = Pf \cup P_1$. Since $f \in E(B)$, P must be equal to $P_2 f$, where $P_2 \subseteq A \times B$. Indeed, if we take $P = P_2 f$, then we get $Pf \cup P_1 = P_2 f^2 \cup P_2 f = P_2 f \cup P_2 f = P_2 f = P_1 f \cup P$.

Case 2: If $P_1 = P$, then $Pf_1 \cup P_1 = Pf_1 \cup P$. Now we can take $P = P_2 f$ or $P = P_2 f^2$, so we obtain

$$\begin{aligned}Pf_1 \cup P = P_2 f^2 f_1 \cup P_2 f^2 &= P_2 f f_1 f \cup P_2 f \text{ (since } E(B) \text{ is commutative)}\\ &= P_2 f \cup P_2 f \text{ (since } B \text{ is } \pi\text{-regular)}\\ &= P_2 f = P.\end{aligned}$$

From the other side, we also have

$$\begin{aligned}P_1 f \cup P = Pf \cup P = P_2 ff \cup P_2 f &= P_2 f \cup P_2 f \text{ (since } f \in E(B))\\ &= P_2 f = P.\end{aligned}$$

We note that Case 2 coincides with the general case of P. In other words, for $P_1 \neq P$, we can also take $P = P_2 f$ or $P = P_2 f^2$.

Hence the result. □

In the following we present some consequences of Theorem 2.2 concerning when the "inverse" and "strongly π-inverse" properties hold. Before stating the first and immediate consequence of Theorem 2.2 (and also of [1, Theorem 2.5]), let us recall the following important result.

2.3. Theorem. [4, Theorem 5.1.1] *The following conditions on a semigroup S are equivalent:*

(i) *S is an inverse semigroup.*
(ii) *Every R-class of S contains exactly one idempotent and every L-class of S contains exactly one idempotent.*
(iii) *S is regular and the idempotents of S commute with one another.* □

Now, by considering [1, Theorem 2.5] and Theorem 2.3, we can present the following corollary which states necessary and sufficient conditions for $A \diamond_{sv} B$ to be an inverse monoid.

2.4. Corollary. *$A \diamond_{sv} B$ is an inverse monoid if and only if the following conditions hold:*

(i) *Both A and B are inverse monoids.*
(ii) *For every $a \in A$ and $b \in B$, there exists an idempotent $f^2 = f \in B$ such that $bB = fB$ and $a \in A(a)\theta_f$, where $\theta : B \to \mathrm{End}(A)$ is a homomorphism as in (1.1).*
(iii) *For every $(a,P,b) \in A \diamond_{sv} B$, either*

$$P = P_1 b = \bigcup_{(a_1,b_1)\in P_1} \{(a_1, b_1 b)\} \textit{ or } P = P_1 bd = \bigcup_{(a_1,b_1)\in P_1} \{(a_1, b_1 bd)\},$$

where $P_1 \subseteq A \times B$ and $d \in b^{-1}$.

(iv) $(a)\theta_f = a$*, for every* $a \in RegA$ *and* $f \in E(B)$.

Proof. Let us suppose that $A \diamond_{sv} B$ is an inverse monoid. By [1, Theorem 2.5], A and B are regular and so the conditions (ii) and (iii) hold. In addition, since $E(A \diamond_{sv} B)$ is a commutative subsemigroup of $A \diamond_{sv} B$, for $e, f \in E(A)$ and $g, h \in E(B)$, we have $(e, \emptyset, 1_B)(f, \emptyset, 1_B) = (f, \emptyset, 1_B)(e, \emptyset, 1_B)$ and $(1_A, \emptyset, g)(1_A, \emptyset, h) = (1_A, \emptyset, h)(1_A, \emptyset, g)$. Thus we get $ef = fe$ and $gh = hg$. Hence (i) holds. Finally, by Lemma 2.1, one can see that (iv) holds.

Conversely suppose that the monoids A and B both satisfy conditions (i)-(iv). It is obvious that, by the sufficiency part of [1, Theorem 2.5], we get the regularity of $A \diamond_{sv} B$. Moreover, by considering (iii) and (iv), it can be easily shown that $E(A \diamond_{sv} B)$ is commutative. But, as in the proof of Theorem 2.2, we need to emphasize that one can consider $P = P_1 f$ or $P = P_1 f^2$ (where $f \in E(B)$) in (iii). □

2.5. Corollary. *Let A and B be two monoids and let $A \diamond_{sv} B$ be a strongly π-inverse monoid. Then*

(i) $(e, P, f) \in E(A \diamond_{sv} B)$ *if and only if* $e \in E(A)$ *and* $f \in E(B)$*, for some* $P \subseteq A \times B$.

(ii) $E(A \diamond_{sv} B) \cong E(A) \diamond_{sv} E(B)$.

(iii) $(a, P, b) \in Reg(A \diamond_{sv} B)$ *if and only if* $a \in RegA$ *and* $b \in RegB$*, for some* $P \subseteq A \times B$.

(iv) $Reg(A \diamond_{sv} B) \cong RegA \diamond_{sv} RegB$.

Proof. In the proof, we first note that conditions (i) and (ii) are immediate consequences of Lemma 2.1 and Theorem 2.2.

The proof of (iii) can be done as follows: For an element (a, P, b) in $\mathrm{Reg}(A \diamond_{sv} B)$, there exists $(a_1, P_1, b_1) \in A \diamond_{sv} B$ such that $(a, P, b)(a_1, P_1, b_1)(a, P, b) = (a, P, b)$. Then, by applying the equality in (1.2), we get

$$bb_1b = b \text{ and } a((a_1)\theta_b)((a)\theta_{bb_1}) = a.$$

Meanwhile, since $bb_1b = b$ we have $b \in RegB$. Now let us consider $a((a_1)\theta_b)((a)\theta_{bb_1}) = a$ and let us take $(a((a_1)\theta_b)((a)\theta_{bb_1}))\theta_b = (a)\theta_b$. This actually gives us

$$((a)\theta_b)((a_1)\theta_{b^2})((a)\theta_{bb_1b}) = (a)\theta_b.$$

Again, by using $bb_1b = b$, we obtain $((a)\theta_b)((a_1)\theta_{b^2})((a)\theta_b) = (a)\theta_b$. Thus we have $(a((a_1)\theta_b)a)\theta_b = (a)\theta_b$. Hence $a \in \mathrm{Reg}A$.

Conversely, for $b \in \mathrm{Reg}B$ and $a \in \mathrm{Reg}A$, there exist $b_1 \in B$ and $a_1 \in A$ such that $bb_1b = b$ and $aa_1a = a$. Hence

$$\begin{aligned}(a, \emptyset, b)((a_1)\theta_{b_1}, \emptyset, b_1)(a, \emptyset, b) &= (a(((a_1)\theta_{b_1})\theta_b)((a)\theta_{bb_1}), \emptyset, bb_1b)\\ &= (a((a_1a)\theta_{b_1})\theta_b, \emptyset, bb_1b)\\ &= (a((a_1a)\theta_{bb_1}), \emptyset, bb_1b)\\ &= (aa_1a, \emptyset, bb_1b) \text{ (by Lemma 2.1(ii))}\\ &= (a, \emptyset, b).\end{aligned}$$

Finally, condition (iv) is quite obvious. □

Following on from this paper, some other algebraic properties (e.g., orthodox and periodicity) can be studied for the semidirect product version of the Schützenberger product of any two monoids (semigroups) as a future project.

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